Game Theory and Economics

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Module No. # 03

Illustrations of Nash Equilibrium

Lecture No. # 09

War of Attrition

Welcome to lecture 9 of module 3 of the course called game theory and economics. So far what we have been doing is that we have been taking several applications of the idea of Nash equilibrium. What we have done so far are the cases of Carnot equilibrium, Bertrand equilibrium and Electoral competition.

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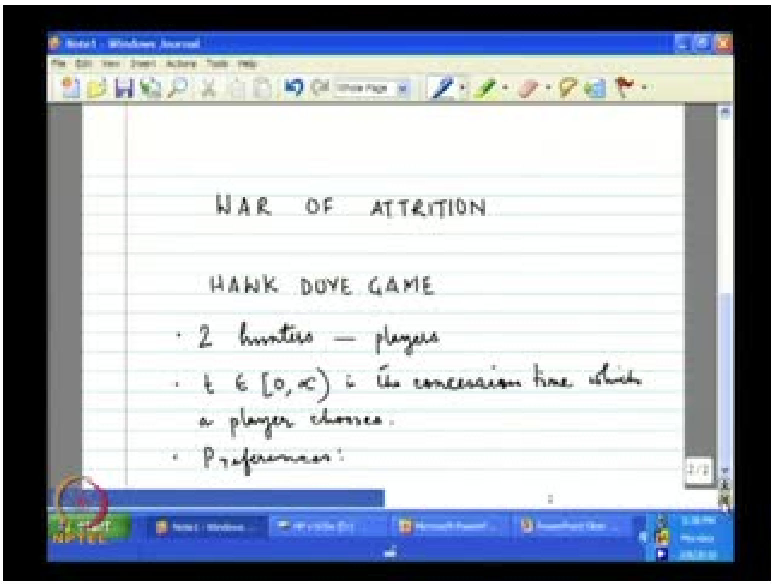

Today, we shall look into another separate field of application of Nash equilibrium which is known as War of Attrition. War of attrition is basically trying to model situations where there are two parties who are fighting with each other over some common prey for example. May be, these two parties are two hunters or it can be more general case of two people fighting over any common resource, while fighting they are inflecting damages to each other, so fighting is costly to both. As soon as I give up fighting - that is I concede - then the other party gets that object for which I have been fighting.

The problem is that right at the time when I give up, all this time I have been fighting that has been costly and at that point when I give up, I lose the object and there is this cost of fighting also, so I get a minus payoff in that case. When the object goes to my rival there is some valuation that he derives from that object but he has been fighting also with me, so that cost of fighting has to be deducted from his valuation to find out what is his final payoff from this entire exercise, this is more or less the setting.

So, this model in the sense it tries to generalize the case of what we have seen before it was the Hawk Dove game. In hawk dove game, there two parties were fighting over some object and each party can a take two sorts of action. It can be aggressive or it can be passive. Depending on the aggression and passivity their payoffs are determined. Here also, we are trying to look at similar situation where this idea of giving up could be thought of as the action of being passive, whereas if you continue to fight you are being aggressive, so this is the case then.

There are 2 hunters let us say, so these are the players, actions what are they choosing here? They are choosing at what point of time they are going to concede. So, t - can vary from 0 to infinity - is the concession time which a player chooses. Thirdly, preferences is given by the following payoff function.

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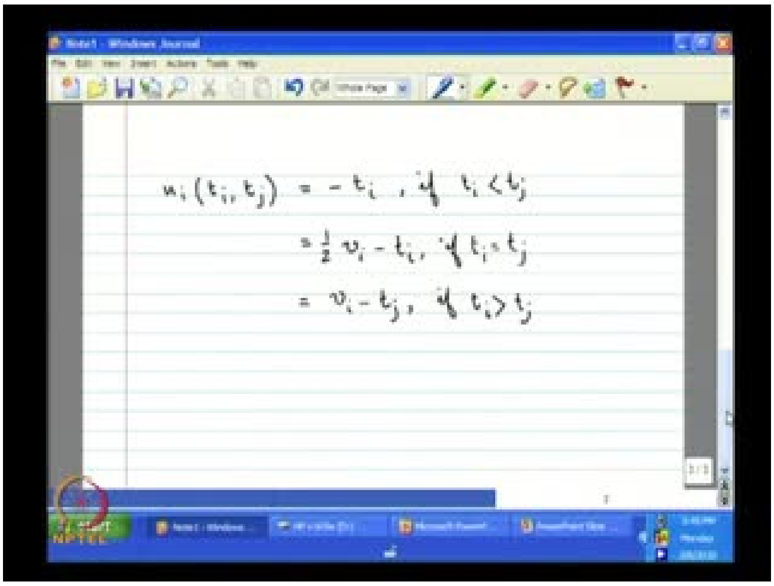

Suppose, I am talking about the payoff of player i then, payoff of player i depends on the concession time it has chosen, which is t i and it also depends on the time that its rival has chosen, which is t j. So, this is given by minus t i, if t i is less than t j which means, if i is the player which concedes first then, whatever time it has chosen to fight which is t i is the total loss that it is accruing and that is why minus t i is the payoff that it is getting.

By the way, in this model we are going to represent this time in the same unit as the valuation of the object. One can think that one minute that I keep on fighting with my rival is worth 1 rupee and rupee is the unit with which I try to denote the value of that object also. What happens if the time that I am devoting to fight is same as the time that my rival is devoting to fight?

So, if t i is equal to t j then what happens? In that case, there is a possibility of half that I will get the object, it is not sure that I will get the object, because he is also given the same amount of time in fighting with me.

In that case, I get half of v i; v i is how I value the object, so v i this index of i is coming because the valuation of the object can vary from person to person. How I value the object might be different from how my rival values the object. So, v i is different from v j in general. Half the probability that I will get the object, half multiplied by v i is the expected value of the object to me from that I subtract the time I spent in fighting that is t i and I get the payoff.

If my rival concedes first, concedes earlier than me, then I get the object. I get v i minus the time that he has spent in fighting with me because at the time when he is giving up I am getting the object. So, when his giving up he has fought for t j - a units of time - and that is the amount of time I also devoted in fighting with him, so that is the time - the value - of which I have to deduct from the valuation of the object, so that is what I am doing here v i minus t j.

If t i is greater than t j, this is how the payoff functions of both the players look like. If i is equal to 1, j is equal to 2; if i is equal to 2, j is equal to 1. Now, before we try to find out what is the Nash equilibrium here, the policy that we are going to take to find the Nash equilibrium is the same strategy that we have been following so far. We are going to construct the best response functions and try to see at what points or point they intersect with each other, those points will be the Nash equilibrium.

Now, before we try to find out what are the best response functions of this 2 players, let us try to figure out what is going on here. Obviously, my payoff from fighting from this war is dependent on the time that he, that is my rival is spending in fighting with me. If I know that my rival is going to spend not a long time - very short period of time - to fight with me in that case, it is worthwhile for me to see him off to fight with him, because I know that he is going to give up after a brief period of time.

If I devote that time in fighting with him, I will get the object. If the valuation of the object is quite a bit then, I am getting some positive payoff out of this fighting. So, what becomes important is how t 2 is; if I am the player 1, how t 2 is compare to my valuation that is v 1, so that is becoming crucial is t 2 greater than v 1 or is t 2 less than v 1. If t 2 is less than v 1 then it is worthwhile for me to fight. On the other hand, if t 2 is very high suppose, in that case it might be greater than v 1 also.

If it is greater than v 1 then, it is no way worthwhile for me to fight with him because longer I fight the more costly it is for me, in that case I should give up immediately, I should not fight. So, these are the basic ideas which will influence the result of this model.

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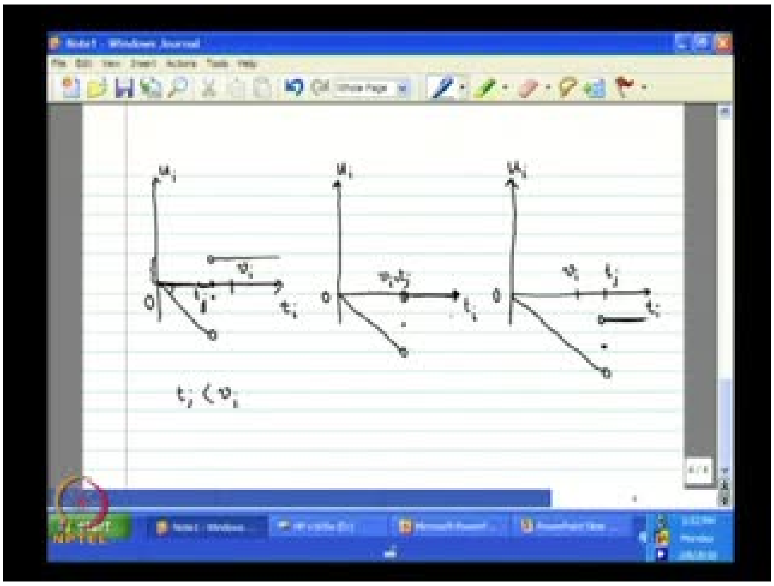

Let us try to look at it in a more systematic way; I can draw the following diagrams. Suppose, in this diagram along the horizontal axis I am taking t i - that is the time devoted by player 1 - and suppose, v i is here it is a parameter. Now, we have seen that what is important is how the time of the other player that is t j compares to v i, so there can be broadly 3 cases, one is t j is less than v i, suppose t j is here.

Depending on different values of t i there will be different values for the payoff, if t i is somewhere in this range is less than t j, then I know the payoff is going to be represented by a 45 degree line - this angle is 45 degree - except for the point here, so this point is not included. If t i is equal to t j, the player gets half of v i minus t j, so half of v i is somewhere here minus t j - so it is negative this is what the player gets.

If t i is greater than t j then the player j is conceding first, then player i is getting the object and what it is getting is v i minus t j; v i minus t j remains as it is, as t i goes on rising, so this is the line. Now, one thing which should be clear from this is that, once my rival is giving up and I am getting the object. It does not matter how much t i had committed in the beginning. Even if t is very high, it is not affecting my payoff as long as t j is here, in this which is less than v i and as long as my t i is greater than his t j.

Now, if this is the situation then what is the optimal time that player i shall devote in fighting this war. So, every player i know is going to maximize the payoff here, the maximum value of the payoff function is occurring when t i is strictly greater than t j, in that case this is the value of the payoff which is positive and this cannot be better.

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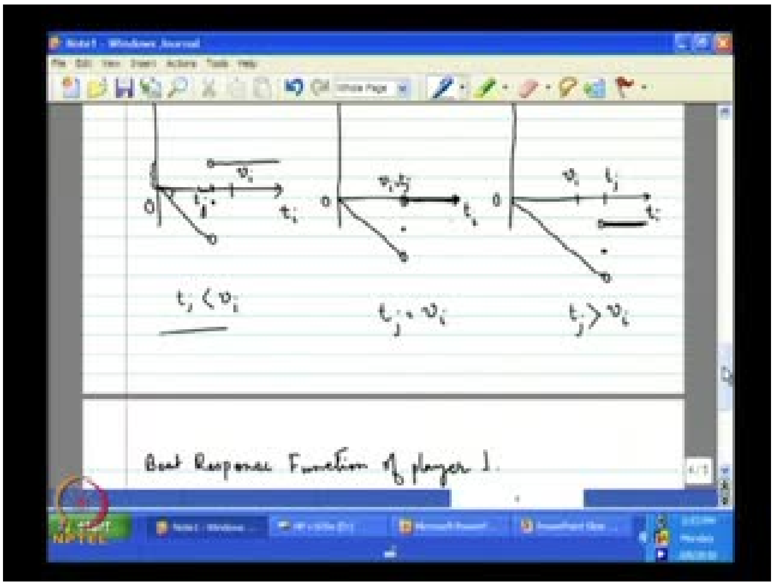

So, that is what the best response function of I should look like, if t j is less than v i then, t i should be strictly greater than t j, this is what we are going to write down little bit later. This is the case of v i equaling t j, like before we have this 45 degree line except this point. At this point he is getting half v i minus t j, this is the point which is representing his payoff, if t i is equal to t j. If t i is greater than t j then his payoff is v i minus t j which is 0, so it is coinciding with this axis here also and this point is not included.

The aim of the player i is to maximize the payoff, if you wants to maximize the payoff, the maximum payoff that he can get is 0, otherwise its negative. 0 is obtained in 2 cases, one is that t i is equal to 0 - this point origin - or if t i is strictly greater than t j in that case the payoff function is coinciding with the horizontal axis and the player is getting 0.

Finally, we shall consider the case of t j greater than v i, here before as long as t i is less than t j is going to be this 45 degree line. At this point it is going to be half of v i minus t j, so I am getting this point - this point is excluded. If t i is greater than t j then, the player getting is v i minus t j which is negative, so I am getting this parallel line to the horizontal axis. So, in this case the best player i can do is to get 0 and that he gets if t i is equal to 0.

So, putting all these together suppose, I want to write down the best response function of player 1 t i. Let us see how it will look like; firstly, if t 2 is less than v i - this case, this is t j is equal to v i, this is t j is greater than v i. If t 2 is less than v 1 and then I know that player 1 is going to choose t 1 strictly greater than t 2.

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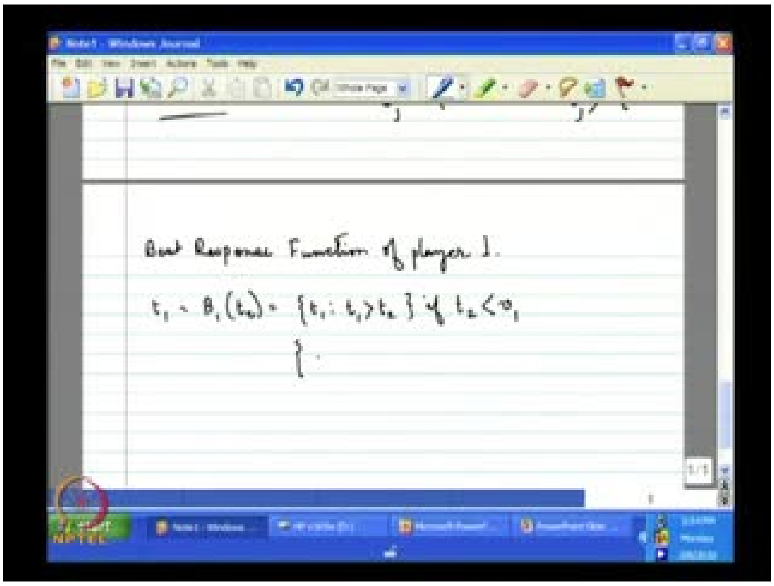

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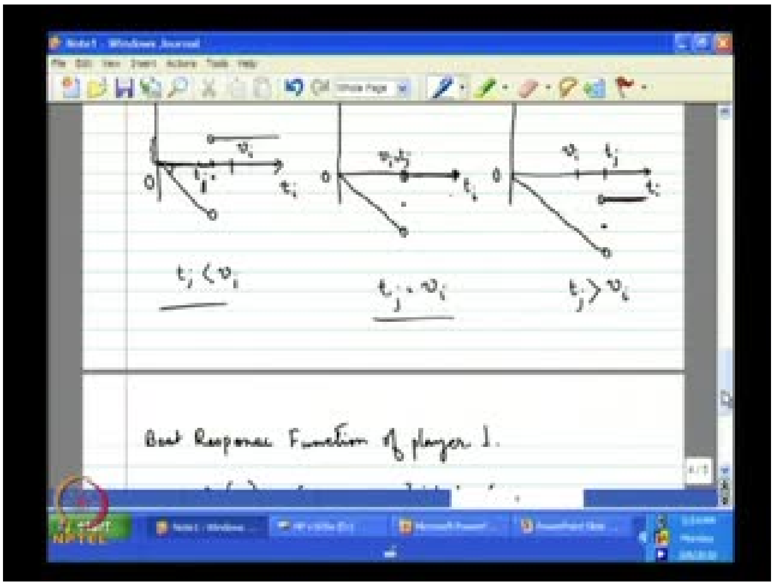

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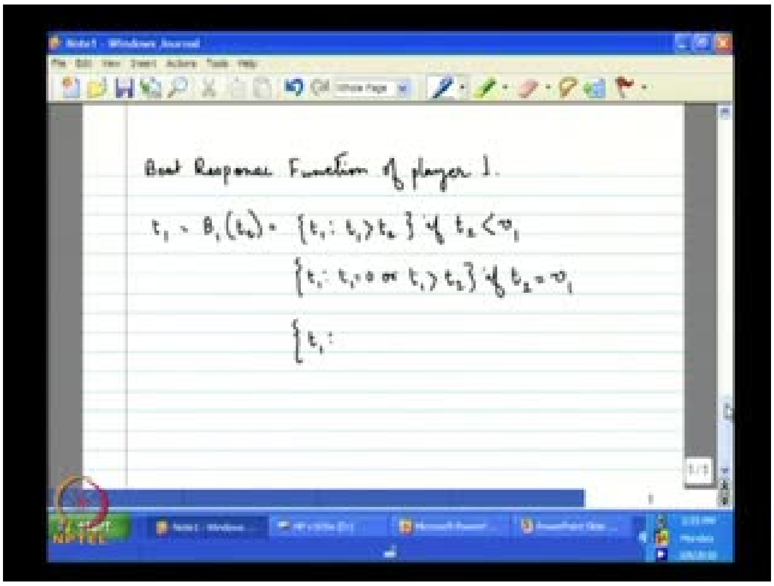

Let us take the second case if t 2 is just equal to v 1, the case in between. In that case there are 2 sets of best responses, one is t 1 is equal to 0 or t 1 is strictly greater than t j t 2. So, this is the range that I am talking about or strictly greater than t 2, if t 2 is equal to v 1.

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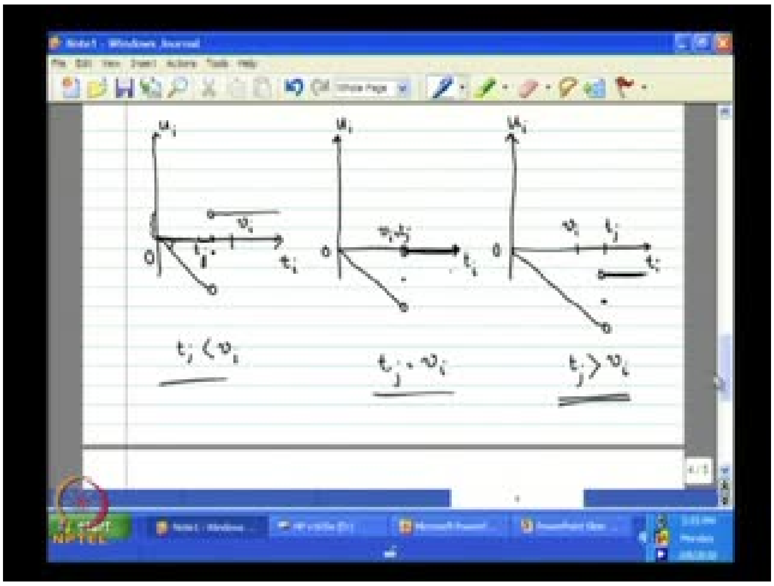

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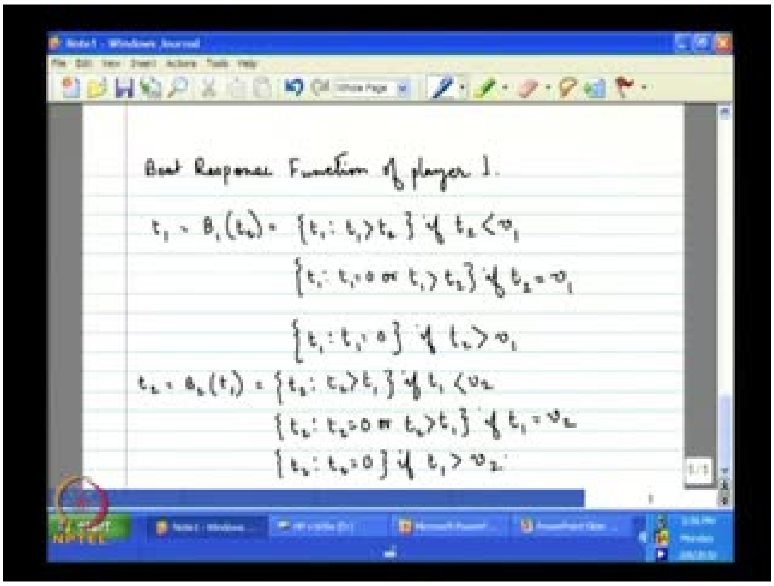

Finally, if t 2 is greater than v 1 then, I know that player 1 will choose only 0, so t 1 is equal to 0, if t 2 is greater than v 1. This is the best response function of player 1; for player 2 it will be similar - let me just write it down, so this is how the best response functions are.

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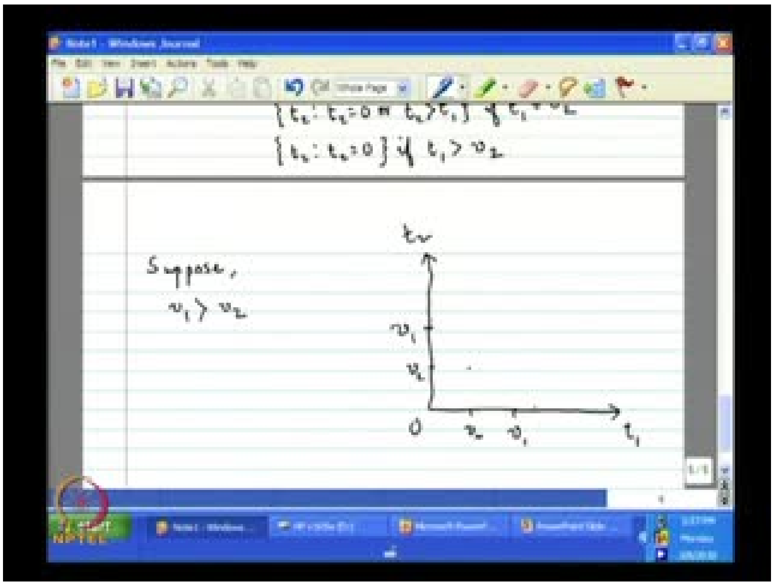

We now can draw the best response functions and find out the intersection points. Suppose, without loss of generality that v 1 is strictly greater than v 2.

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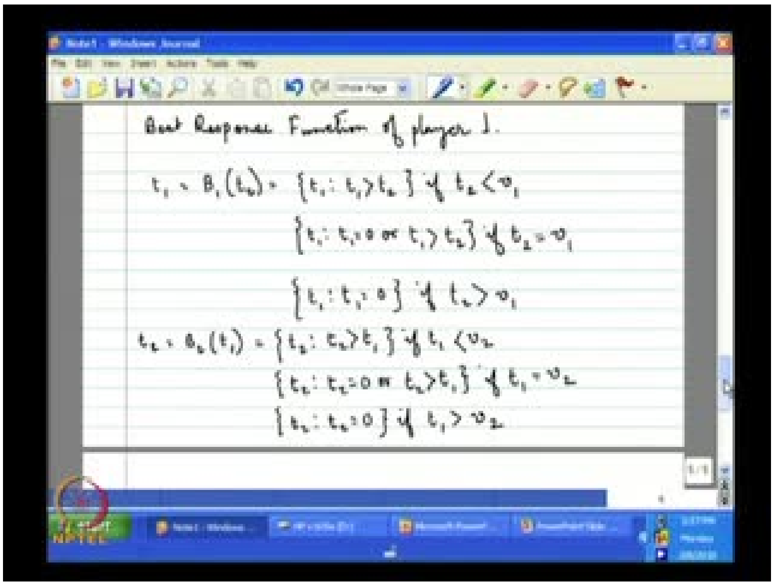

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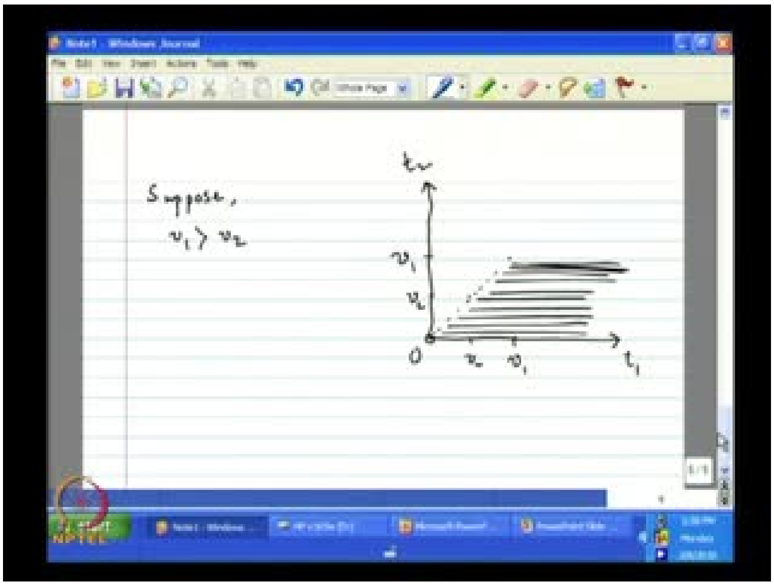

So, v 2 is here and v 1 is here; first, let us draw this best response function of player 1 b 1 t 2. If t 2 is less than v 1 then, t 1 is strictly greater than t 2, so this is the point 45 degree line I can imagine. If t 2 is less than v 1 then, t 1 should be strictly greater than t 2.

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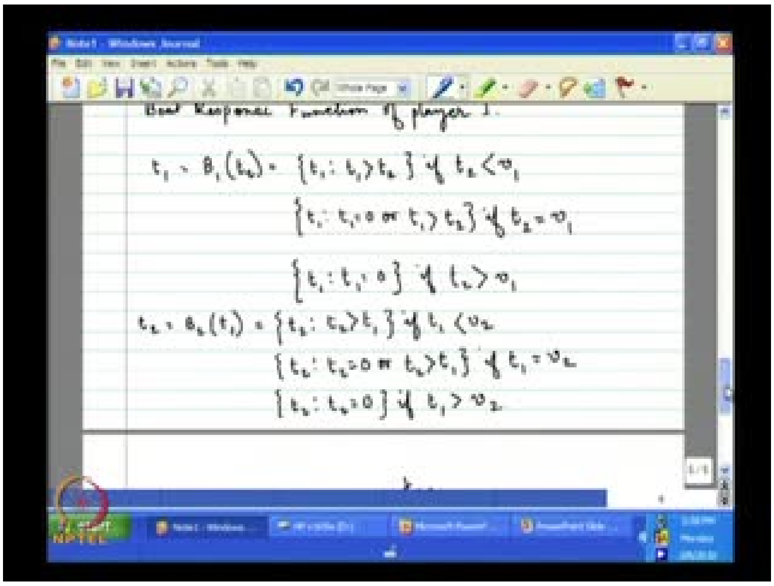

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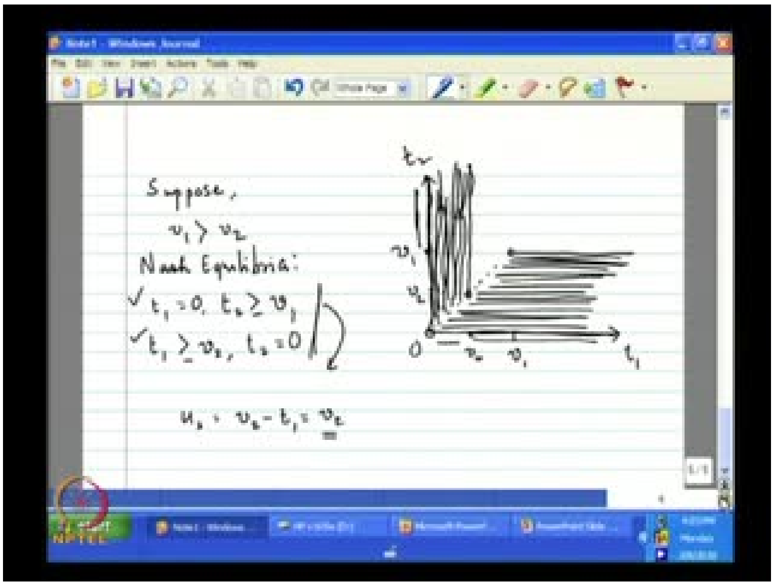

Now, this point is not included because t 1 has to be strictly greater than t 2. If t 2 is equal to v 1, t 1 is either 0 or t 1 is strictly greater than t 2. This point is not included and this point is included, if t 2 is greater than v 1 t 1 is equal to 0, so I am taking this axis.

This is how the best response function of player 1 looks like, whatever the best response function of players 2. If t 1 is less than v 2, I am talking about this, then t 2 is strictly greater than t 1. So, this axis is included entire axis except the origin and all these points are included, the points on the 45 degree are not included. If t 1 is equal to v 2 then, either t 2 is equal to 0, this point or t 2 is strictly greater than t 1, so this line is included. If t 1 is greater than v 2, then t 2 is equal to 0, so this axis itself, this is how it looks like.

Now, putting this two things together what we have found is the Nash equilibria or the points where the 2 best response functions are overlapping with each other are occurring in this part that is, in v 2 and greater than v 2. If t 1 is equal to v 2 or greater than v 2 and t 2 is equal to 0 or in this part that is t 1 is equal to 0, t 2 is equal to v 1 or greater than v 1.

Nash equilibria t 1 is equal to 0 that I am talking about this or t 1 is strictly greater than equal to v 2, t 2 is equal to 0 there are 2 sets of Nash equilibria. In particular, there are infinite number of Nash equilibria. Now, this is the result but what does it mean what is the intuition? The intuition is that let us talk about this first set of equilibria that is, t 1 is equal to 0, t 2 is greater than equal to v 1 here t 2 that is player 2's concession time is either equal to player 1's valuation or greater than player 1's valuation.

Now, in this case obviously if player 1 wants to fight with player 2, either he will get 0 because if t 2 is equal to v 1 and t 1 is also equal to v 1 then, infact player 1 will get negative payoff, so there is no way he can get a 0 payoff either. If t 2 is greater than v 1 player 1 can fight with player 2 and can win the object, but its payoff is going to be negative because v 1 minus t 2 is going to be negative.

By deviating or by choosing any positive time player 1 can only be worse off, so that is why he is taking to 0. From player 2's point of view also is not worthwhile to do something else, because here what is the payoff of player 2, player 2's payoff is in this Nash equilibrium it is getting v 2 minus t j which is t 1 which is equal to v 2 because t 1 is equal to 0.

Player 2 is getting the entire value of the object, he is not losing anything and so this is the best thing he can do, he cannot deviate be better off, so that is why this is a Nash equilibrium from both sides, by similar logic this is also a Nash equilibrium.

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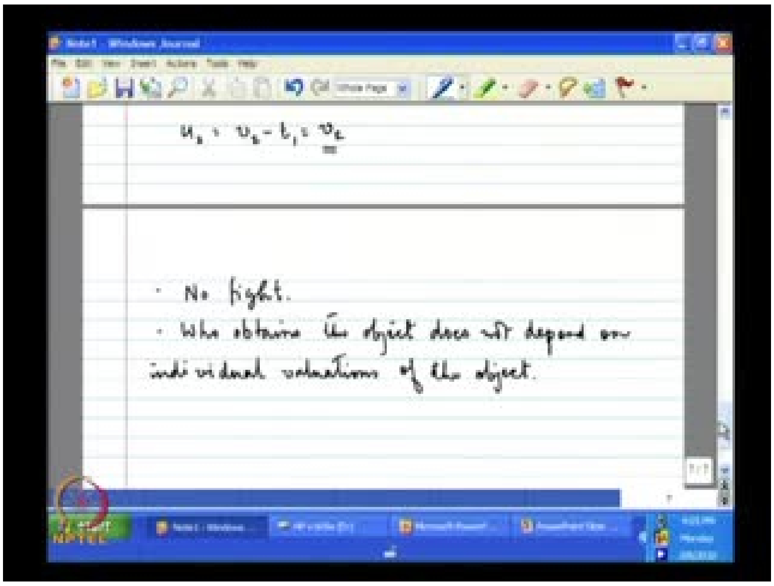

In the second case it is player 1 who is getting the object, player 2 is not getting the object, so that is how it is. Now some quick observations, first is that in this Nash equilibrium there is no fight in equilibrium because you see one of the players is always giving up in the beginning itself, he is not fighting, the other player is announcing some fighting time but since the first player is not fighting which in fact means that in the equilibrium there is no fight.

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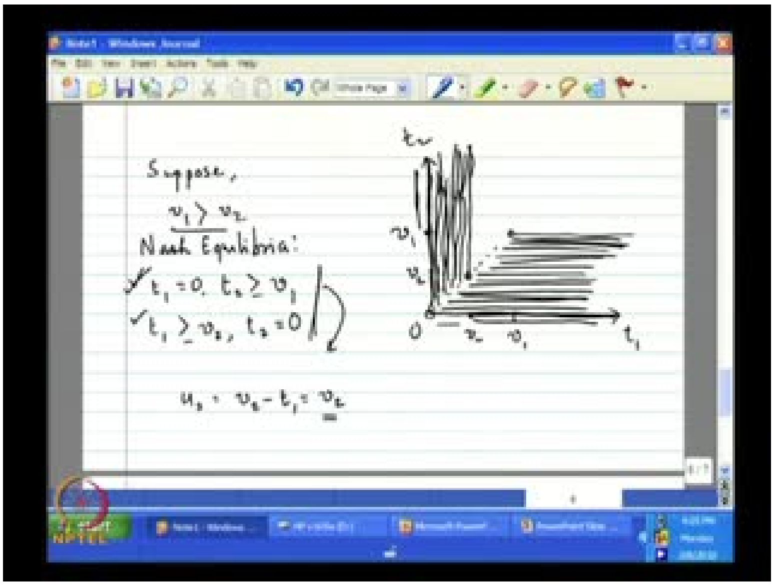

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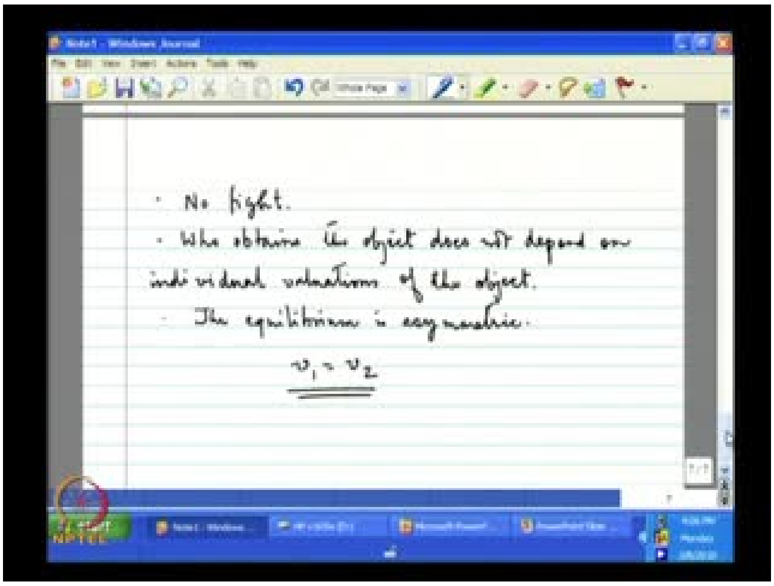

Secondly, who obtains the object which means that in the equilibrium it may very well happen that I value the object less and still I get the object, so it is not necessary that the player who values the object more, he or she will get the object. In this case for example, we had assumed that v 1 is equal to v 2 is greater than v 2 that is, player 1 values the object more than player 2, but we had this equilibrium here player 1 is not getting the object but player 2 is getting the object. It means that even if you value the object less you might still get the object and finally, the equilibrium is asymmetric if one remembers the definition of symmetric game and symmetric equilibrium if we make this game into a symmetric game by assuming v 1 is equal to v 2 then, the game becomes a symmetric game.

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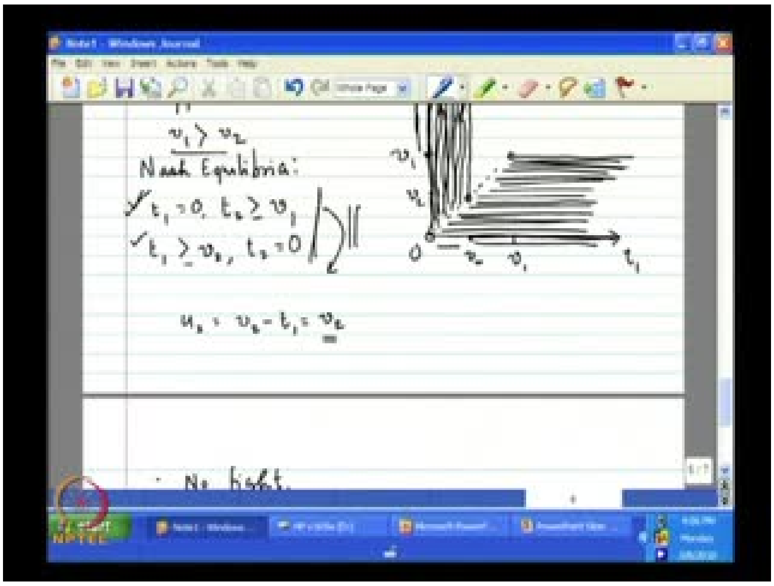

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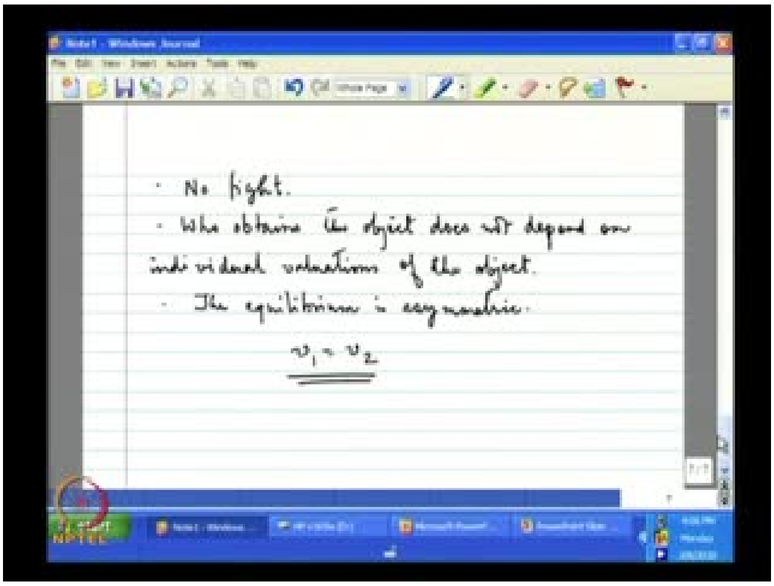

Now, in that case it does not mean that we have an equilibrium which is symmetric, because in the equilibrium the actions of players are different they are not the same and that was the property of the symmetric equilibrium, so that the actions must be the same. So this game does not have any symmetric equilibrium even if we make the game symmetric.

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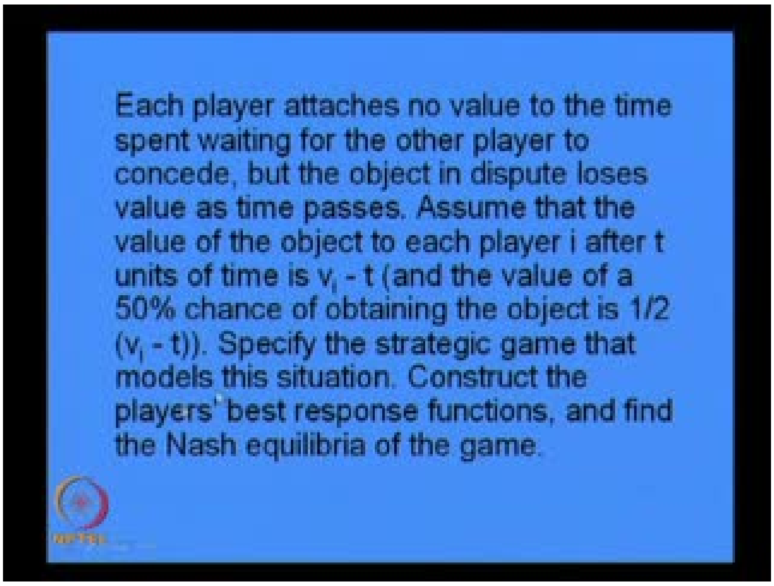

On this idea of war of attrition, we can look some other aspects of this war of attrition. This is a variation of this original game, each player attaches no value to the time spent waiting for the other player to concede, but the object in dispute loses value as time passes. Assume that the value of the object to each player i after t units of time is v i minus t - and the value of a 50 percent chance of obtaining the object is half of v i minus t. Specify the strategic game that models the situation construct the players best response functions, and find the Nash equilibria of the game.

Here, what is important is that I do not attach any value to the time that I am spending in getting the object but the object itself is depleting, so more we fight the more the object depletes. Suppose, what difference does it make to the original model? The difference it makes is that suppose, I concede first and he is getting the object.

In the original game it was the case that I was getting minus t i if t i is less than t j and in this case I do not attach any value to this time that I spent, so in that case when t i is less than t j the payoff of player i will be 0 in this case instead of minus t i. Second difference that we are seeing here is, if the 2 players are conceding at the same point of time then, each is getting half of v i minus t, so that is also something different in the previous case we had half of v i then minus t i, if t i and t j are equal. This is an exercise about war of attrition, let me first write down the exercise the question of the exercise and then, we shall try to solve it.

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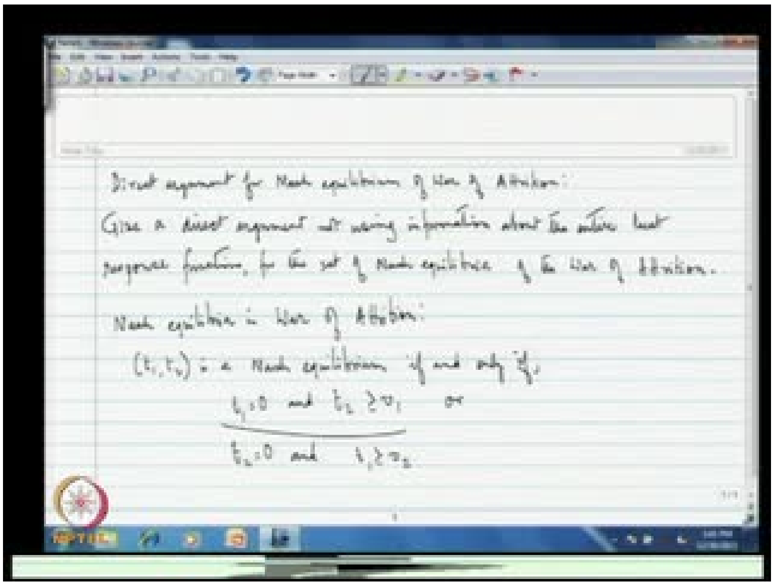

This is the direct argument for Nash equilibrium in war of attrition, so this is the question. Give a direct argument not using information about the entire best response functions, for the set of Nash equilibria of the war of attrition. First, let us remember what was the Nash equilibrium or the set of Nash equilibria in the war of attrition game then, we shall try to give a direct argument why is it so.

There are 2 sets of Nash equilibria one is, this set t 1 is equal to 0 and the t 2's are all greater than or equal to v 1 and v 1 is, remember the valuation of player 1 for the object this is one set of Nash equilibria. The other set of Nash equilibria is that t 2 is equal to 0 that is, player 2 is spending no time and t 1 is greater than or equal to v 2, v 2 is the valuation of player 2 for the object.

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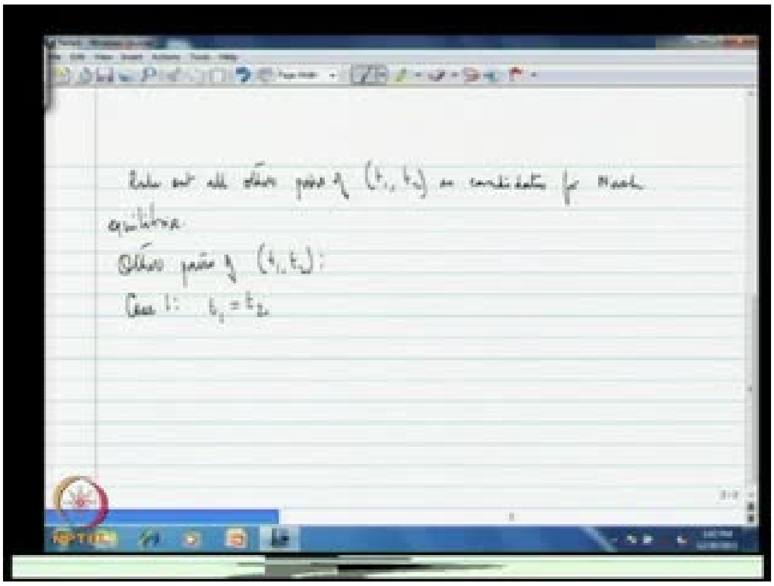

Now, how do we prove that this is indeed the Nash equilibria, what we are going to do is that we are going to rule out all other pairs of t 1 and t 2 as candidates for Nash equilibria.

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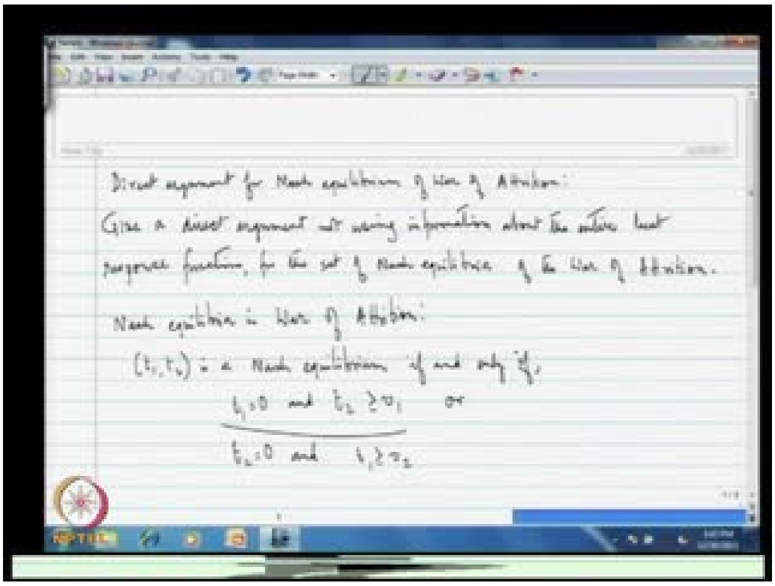

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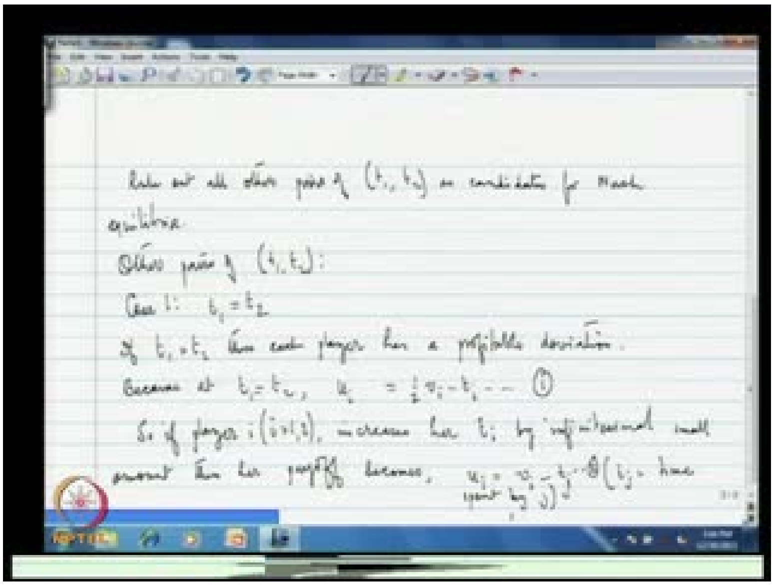

So, what is the complementary set or what are the other possibilities? Others pairs of t 1 t 2 could be the following cases; case 1, t 1 is equal to t 2 remember, in this case in the Nash equilibrium since v 1 and v 2 are always greater than 0, it is never the case in Nash equilibrium that t 1 is equal to t 2. So, if we rule out this case t 1 is equal to t 2 that basically gets rid of lot of possibilities, so why t 1 is equal to t 2 not a Nash equilibrium candidate.

If t 1 is equal to t 2 then each player has profitable deviation how do I know that because at t 1 is equal to t 2 what is u i? u i is equal to half of v i minus t i, this is the payoff function in war of attrition game.

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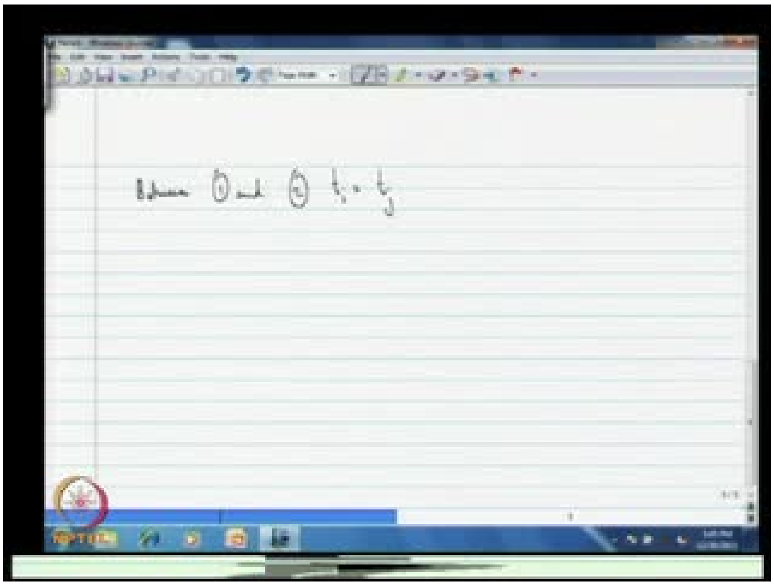

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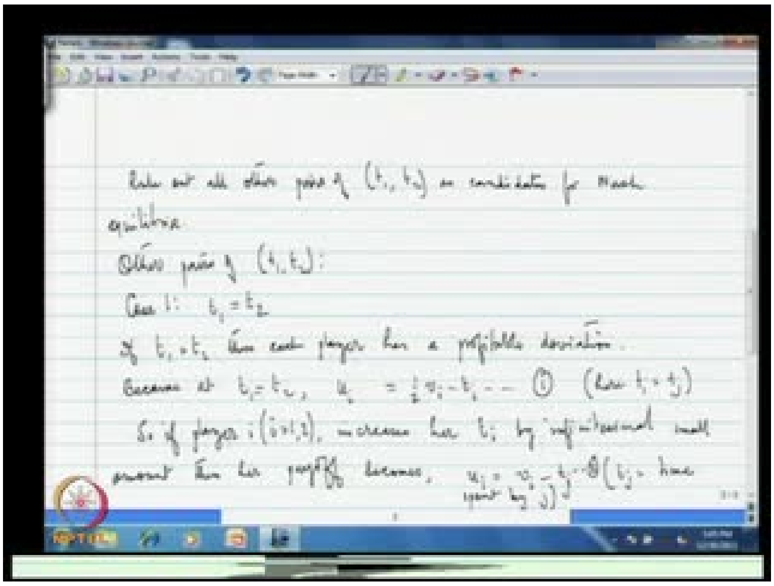

So, if player i - i can be 1 or 2 - increases her t i by infinitesimal small amount then her payoff becomes, uj - he gets in this case the complete - v i minus t j which is time spent by j. Now remember, between 1 and 2 t i is equal to t j because the other players time in case of one was equal to t i, here t i is equal to t j.

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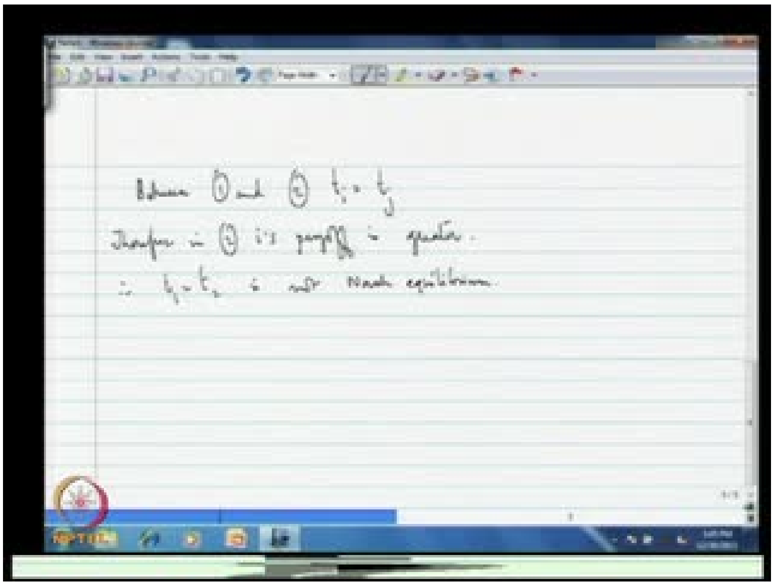

Therefore in 2, i's payoff is greater thus there is a profitable deviation, so t 1 equal to t 2 is not Nash equilibrium, now this is case 2.

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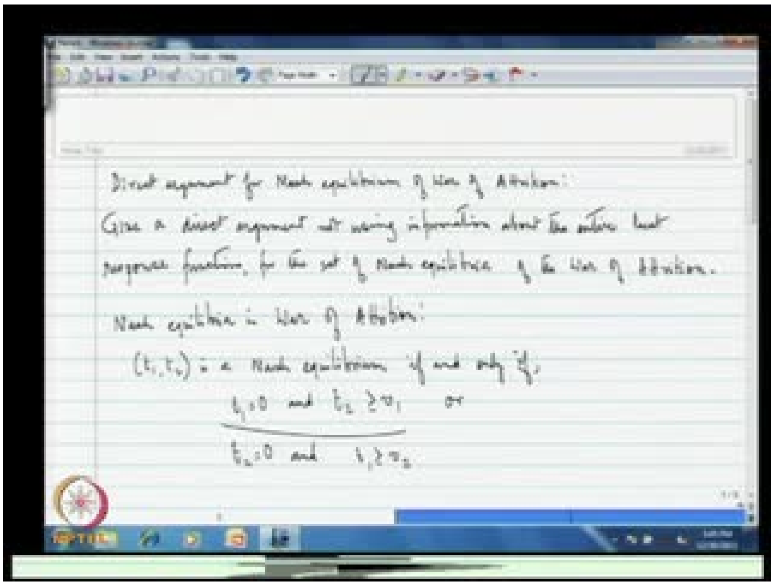

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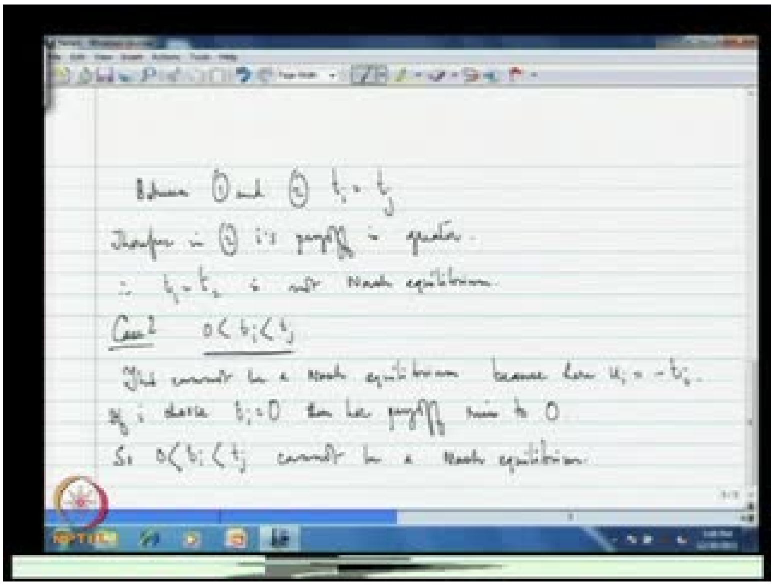

What are the other cases? The other cases are the following; case 2, let us suppose t i and t j are not equal and the t which is less suppose, that is t i is greater than 0. So, this is one possibility that t i and t j are not equal but the lower t is still greater than 0.

Remember, this is ruled out this is not the case in Nash equilibrium because in the Nash equilibrium the lower t is always equal to 0. Now can this be a Nash equilibrium? This cannot be a Nash equilibrium because, here u i is equal to minus t i if i chooses t i is equal to 0 then her payoff rises to 0, so 0 less than t i less than t j cannot be a Nash equilibrium.

So, this is the second case which is being ruled out, what is the third case that can be there and which has to be ruled out, it is the case that suppose, t i is equal to 0 but t j is not greater than v i, t j is strictly less than v i that again we have to ruled out, so this is case 3.

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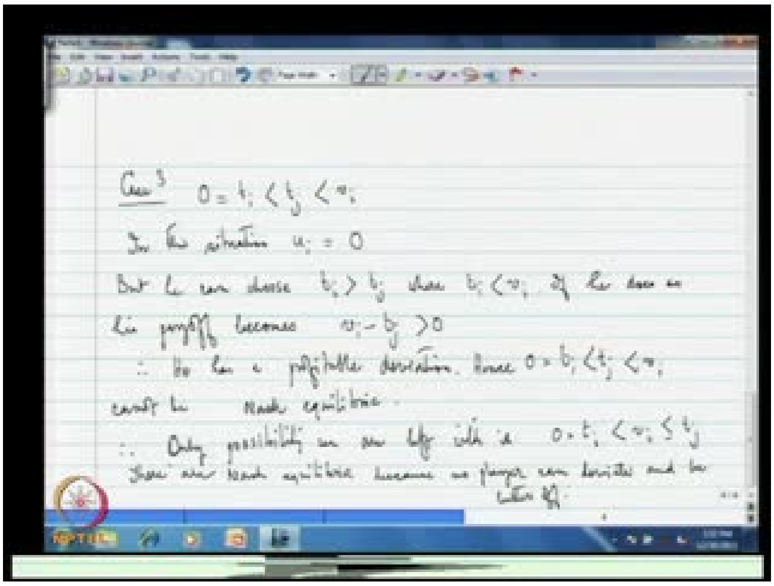

This is equal to t i and this is less than v i, why this is not a Nash equilibrium? This is not a Nash equilibrium because what is player i getting here, u i is equal to 0 because he is spending no time he is giving up in the very beginning, so the payoff that he is receiving is also 0, but can he do better off, he can choose t i strictly greater than t j where t i is less than v i. Remember, t j is less than v i, so there is a gap between t j and v i player i can place his t i in that gap.

If he does so, his payoff becomes v i minus t j and which is greater than 0 therefore, he has a profitable deviation, hence 0 equal to t i strictly less than t j strictly less than v i cannot be a Nash equilibrium or let us say cannot be Nash equilibria because here there are many possibilities. The only possibility we are left with is 0 equal to t i less than t j and this t j could be greater than v i, so let us write it like this.

This is the only possibility that we are left with this is a Nash equilibrium, why this is a Nash equilibrium? or let us say these are Nash equilibria Because no player can deviate and be better off, player j is getting the maximum that he can get, so there is no possibility of improving his payoff, player i also cannot deviate and be better off because the other players t is greater than his valuation.

Before we wrap of this lecture, what we have been doing in this lecture, let me sum it up. We have introduced the new concept - a new application - of Nash equilibrium in this lecture which is called the war of attrition, where the two parties are fighting over some common object and the longer they fight the worse it is for both of them. If the players think that by fighting more for a longer time they can get the object, so that is what they are trying get as much payoff from this fight as possible.

We have seen that in the Nash equilibrium what will happen is that only 1 player will get the object and the other player is not going to fight. So, in the equilibrium there is not going to be any fight either player 1 gets the object or the player 2 gets, the object the player who is not getting the object is going to concede immediately, he is not going to fight. This is a case where the equilibrium is asymmetric in the sense that the time that players say that they are going to fight in the equilibrium there are going to be different, so that is the conclusion of this model.

We have considered one variation of the model and we have seen that in variation also the equilibrium is such that only 1 player is getting the object and the equilibrium is asymmetric in the sense that times are different, thank you.

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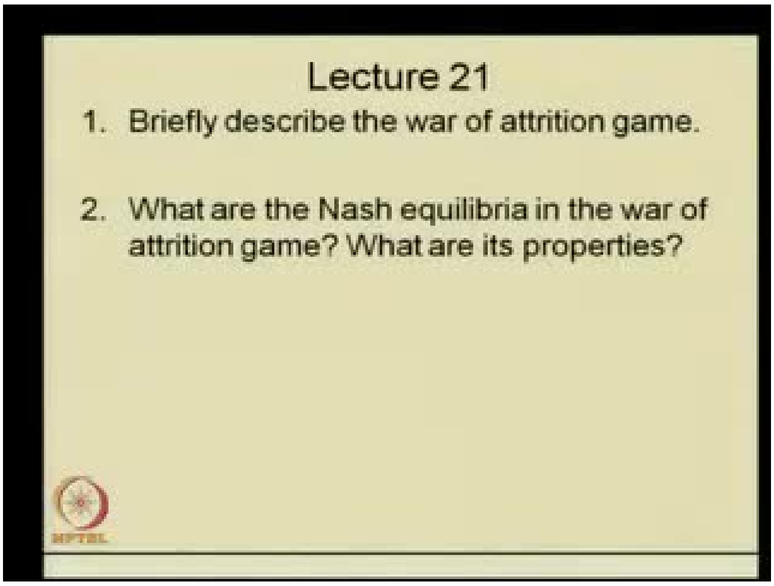

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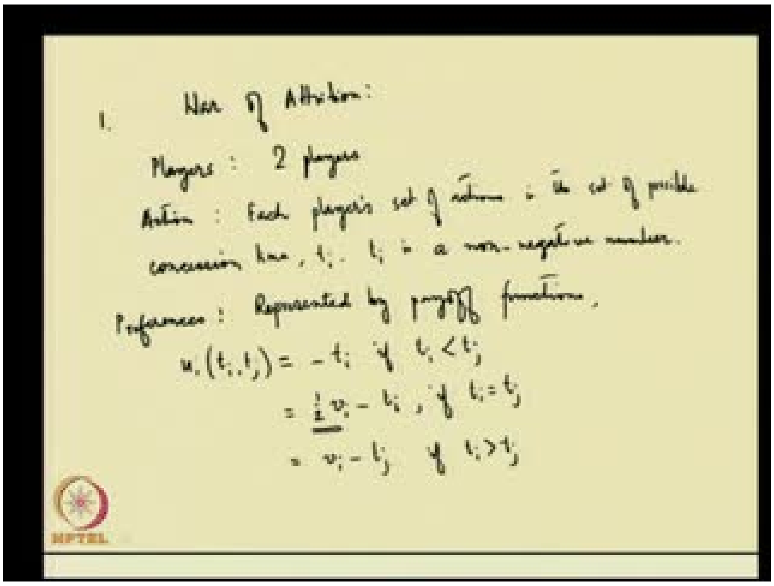

Briefly describe the war of attrition game, let us describe the war of attrition game, 2 players; actions, t i is non-negative number each player is choosing t i and this is what is known as a concession time - the time at which you are conceding the game to the others - in fact the game is a waiting game, 2 players are waiting for the other player to concede. If any player concedes earlier than his rival, then that player loses the object, there is an object. Thereafter, if I concede before my competitor the object goes to the other player and while I was waiting for this object I was incurring some cost also the preferences can be written by this.

This is the payoff of player i given that he has chosen t i and the other player has chosen t j, so if he concedes earlier but he has already waited for t i amount of time and that is costly, so he loses this amount of time which is minus t i. What if they concede at the same time then, there is a half probability - half chance - that each will get the object. In this case, if t i is equal to t j with half probability I will get the object and the valuation of the object to player i is given by v i.

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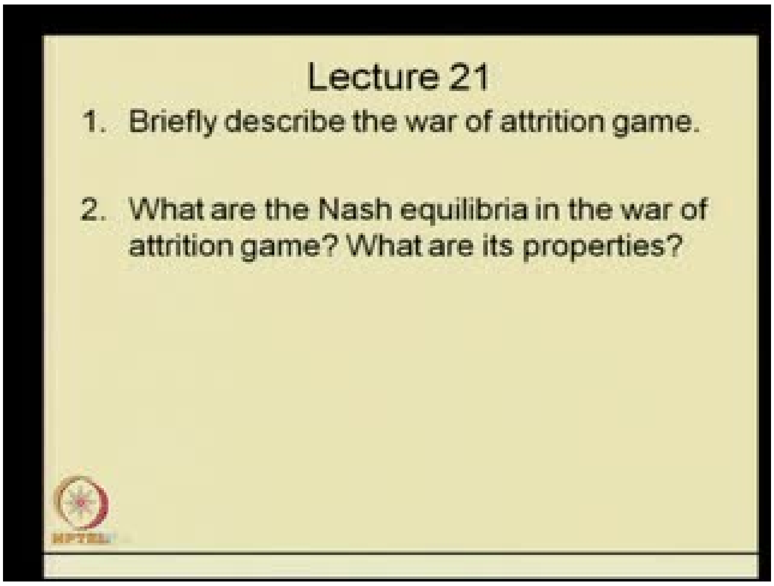

So, half v i minus t i the time that has been spent in waiting and this is equal to v i minus t j if t i is greater than t j. If I waited for longer time than my competitor then I get v i minus t j, t j is the time at which my competitor gives up. What are the Nash equilibria in the war of attrition game what are its properties?

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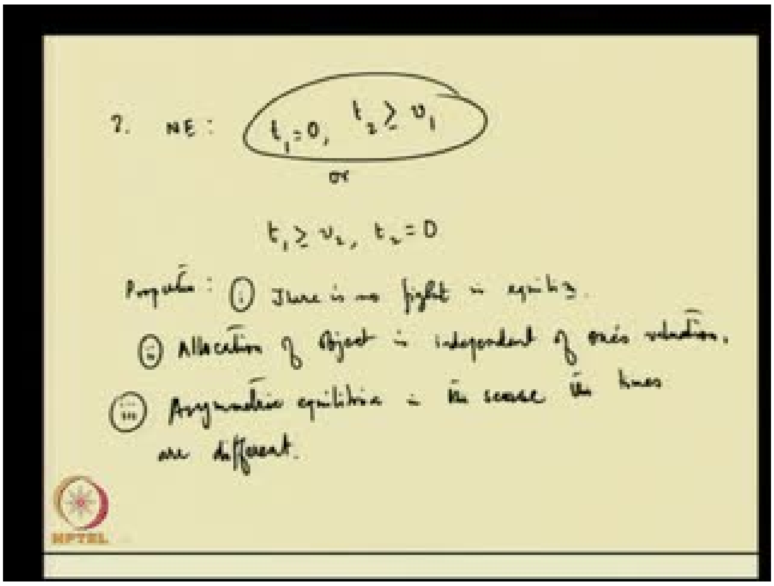

So, Nash equilibrium in war of attrition game is given by the following, there are infinite Nash equilibrium; one set of Nash equilibria is given by this and the other set is given by this. So, in the first case player 1 gives up immediately and player 2 gives up after v 1. In the second place player 2 gives up immediately and player 1 gives up after v 2. Properties, there is no fight because in each case 1 player is giving up immediately nobody is waiting for the other player to concede.

Allocation of object is independent of one's valuation, so it might very well be the case v 1 is greater than v 2 but player 2 could get the object. Thirdly, equilibria are asymmetric, equilibria in the sense this is not the strict definition of I know symmetric equilibrium and asymmetric equilibria, what we mean is that in equilibrium the times are different. This property of a non-equal time will hold even if we make the games sort of symmetric by assuming v 1 is equal to v 2. Even if the valuation of the first player is equal to the valuation of the second player in equilibrium there will be no such outcome where t 1 is equal to t 2, these are the answers to the exercise.