Game Theory and Economics

Prof. Dr. Debarshi Das
Department of Humanities and Social Sciences
Indian Institute of Technology, Guwahati

Module No. # 03
Illustrations of Nash Equilibrium
Lecture No. # 10
Auction: Second Price Sealed Bid

Welcome to lecture 10 of module 3 of the course - Game Theory and Economics. So, what we have been discussing in this course and in this module so far is that we have been taking up different applications of Nash equilibrium and the current applications. Application that we have been discussing is the case of war of attrition, where there are two parties and they are fighting with each other over certain objects - may be a prey like two hunters might be fighting over a prey. The more they continue to fight, it is worse for both of them. They are inflicting damages on each other. So, the earlier my rival gives up, it is better for me.

I get the object, but my payoff from getting the object will definitely depend on how much time he has spent in fighting with me. This is the setting and we have found that the main result of this framework is that in the Nash equilibrium. There will be an asymmetry in the sense that one player will give up immediately without fighting, without spending any time in fighting with the rival. The rival will get the object and vice versa in the sense that the rival can give up at the beginning itself and the first player can get the object without fighting. So, both sets of Nash equilibria are possible today. We shall go further and look at different aspects of this same setting. So, one exercise that we want to do in real life application is the following.

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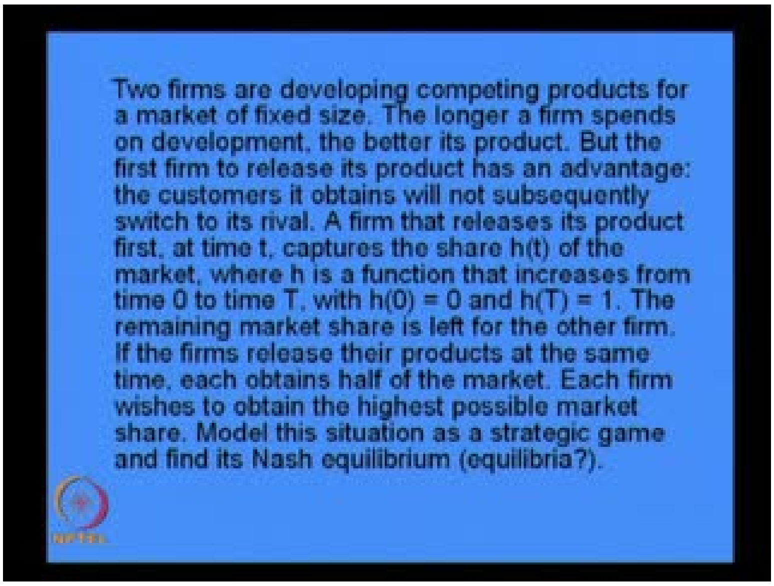

Two firms are developing competing products for a market of fixed size. The longer a firm spends on development, the better its product. The first firm to release its product has an advantage: the customers it obtains will not subsequently switch to its rival. A firm that releases its product first at time t, captures the share h t of the market, where h is a function that increases from 0 to time capital T with h 0 is equal to 0 and h of capital T is equal to 1.

The remaining market share is left for the other firm. If the firms release their products at the same time, each obtains half of the market. Each firm wishes to obtain the highest possible market share. Model this situation as strategic game and find its Nash equilibrium or equilibria.

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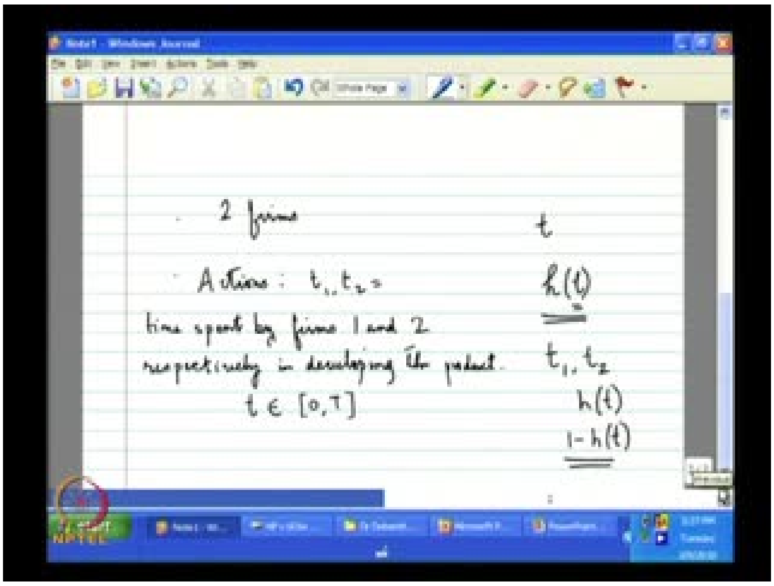

In nutshell, what is happening is that there are two firms and these are set of players. What are the actions that they are undertaking? They are deciding at what time they will spend in developing a product. If I develop a product and release my product before my rival, then the time that I have spent in developing the product before releasing the product into the market will affect my market share in particular. If I spent t time to develop the product and release it before my rival, then I get h t of the market share.

So, t is something, which I shall decide on. In particular, t can take two values t 1 and t 2. Here, t 1 is the time that firm 1 has decided to spend on developing the product and t 2 is the time that the firm 2 decides to develop on the product. So, t 1 and t 2 are the decision variables. Actions t 1, t 2 are time spent by firms 1 and 2, respectively in developing the product.

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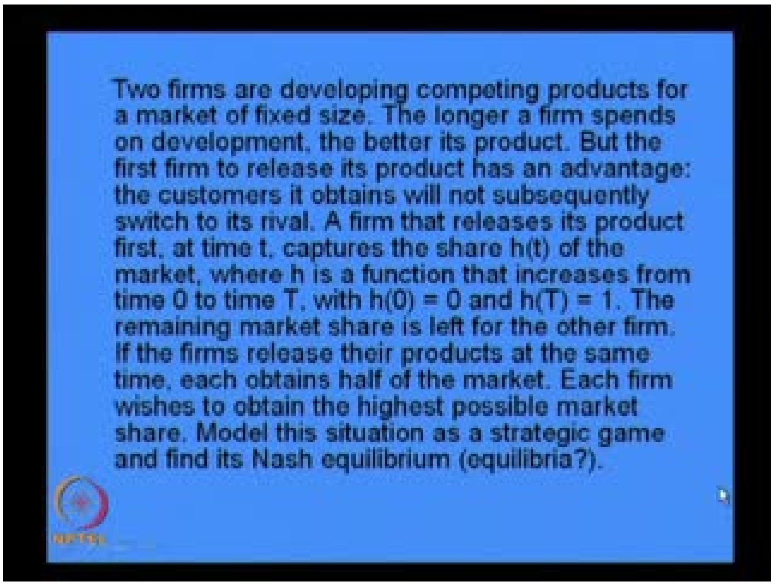

What is the range of t? It can take any value between 0 and not infinity, but capital T. So, that is a maximum deal, which one can spent in developing the product, which is given by capital C. As I have said, if I spent h t amount of time developing it, I get h t share of the market, provided my t is less than the t of the other firm. If I get h t fraction of the market, my rival gets 1 minus h t fraction of the market. Each firm tries to maximize the share of the market because if you remember, what is being said is that the each firm wishes to obtain the highest possible market share.

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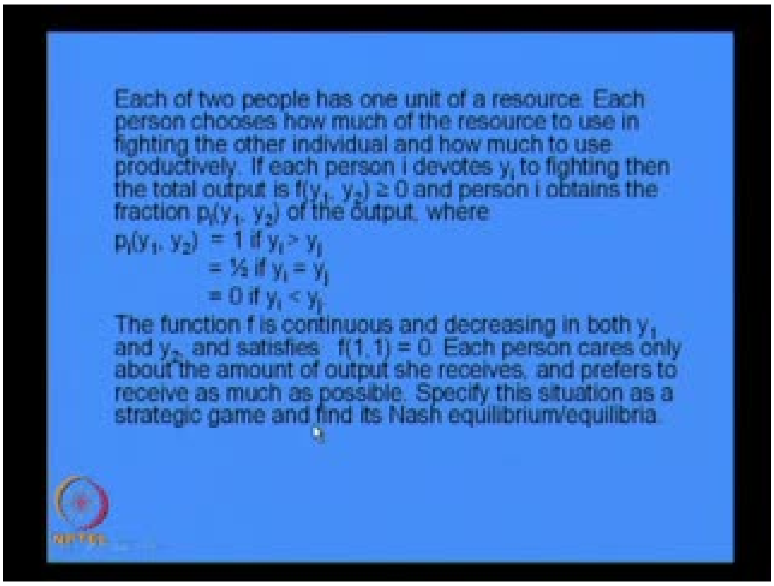

My aim will be to maximize h and that is the portion of the market that I get, so that is the game. Each of these people has one unit of resource. Each person chooses how much of the resource to use in fighting the other individual and how much to use productively. If each person devotes y i to fighting, then the total output is y i f of y 1 f y 2, which is greater than equal to 0. Person i obtains the fraction p i of y 1 y 2 of the output, where p i y 1 y 2 is equal to 1. If y i is greater than y j, which is equal to half, if y i is equal to y j is equal to 0, if y i is less than y j.

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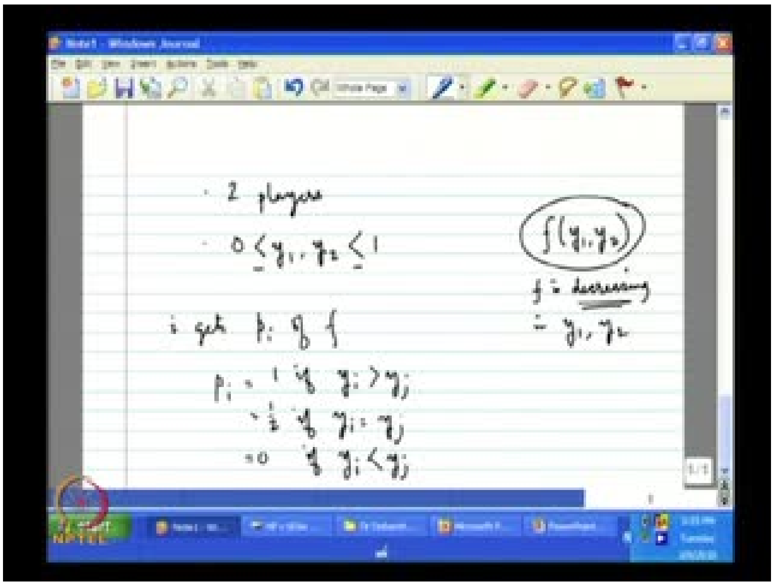

The function f is continuous and decreasing in both y 1 and y 2. It satisfies f of 1 and 1 is equal to 0. Each person cares only about the amount of output she receives and prefers to receive as much as possible. Specify this situation as strategic game and find its Nash equilibrium or equilibria. There are two players and they are fighting with each other. They are devoting y 1 and y 2 of resources to fight with each other. So, this is their actions y 1 y 2. They can take the value between 0 and 1 and highest. So, the highest value for y 1 is 1 and the lowest value is 0 for y 2.

When they fight with each other and devote this y 1 and y 2, obviously they will be left with less y’s or resources that they had to begin with. So, they will get the total output. If they devote y 1 and y 2, which is given by f of y 1 and y 2, where f is decreasing in y 1 and y 2. So, more he spent in fighting with your enemy, you are left with less output and the total output could be a result of the resources of both the people and it will go down.

The point is that if I win that is even and if I get the total output f, it may happen that I do not get the entire thing, but a part of it depending on y 1 and y 2. In particular, I gets p i of f, where p i is given by 1, if y i is greater than y j half. In short, if I have spent more resources than my rival, I get the full of this entire thing. If my spending of resource is equal to his spending of resource, I get half of this f. If I am spending less resource than my rival, then I am not getting any part of this f.

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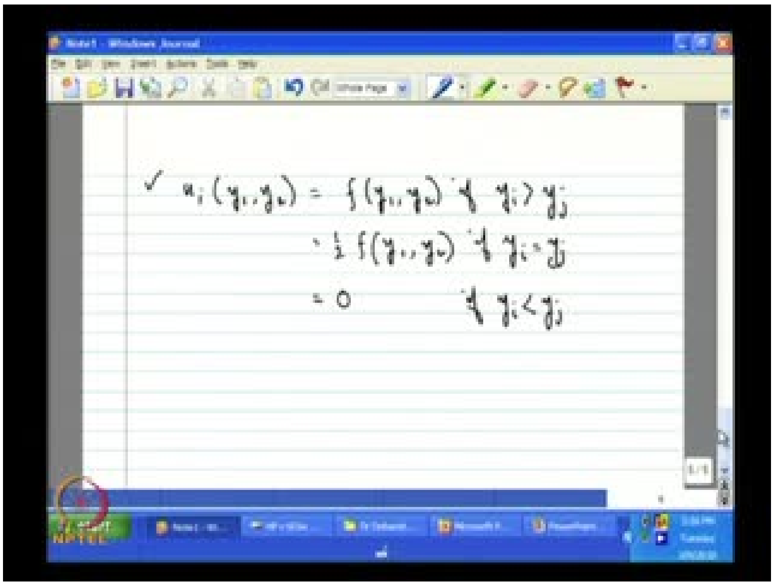

In other words, u i is equal to f of y 1 y 2. So, this is how it looks like. The question is which is the Nash equilibrium? This is what the players want to maximize; they want to maximize this u. If y 1 and y 2 go on rising, this f goes on declining. So, they would like to choose as little y as possible. Now, what are the Nash equilibira here?

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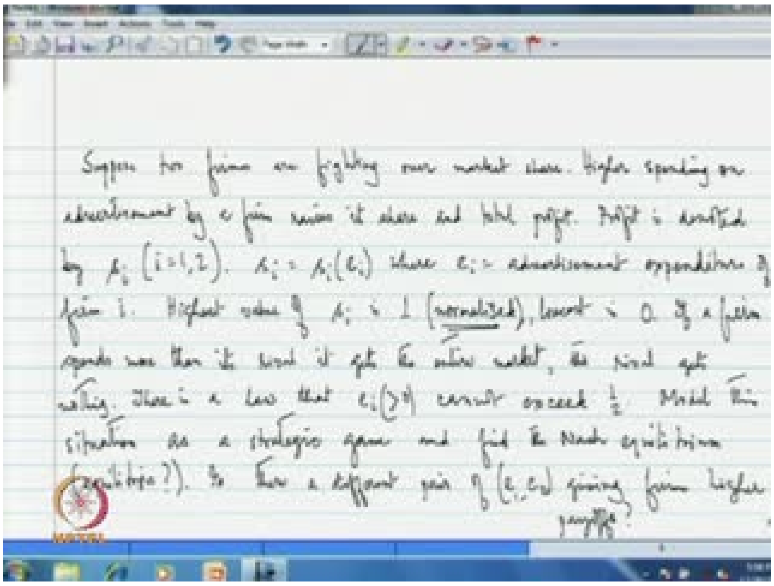

One exercise that we shall do about war of attrition is the following. So, this is the question – suppose, two firms are fighting over market share. Higher spending on advertisement by a firm raises its share and total profit. Total profit is detonated by s_i and s_i is a function of e_I, where e_i is the advertisement expenditure by firm i. Highest value of s_i is 1. So, this is a normalized value and lowest is 0. So, s varies between 0 and 1.

If a firm spends more than its rival, its gets the entire market and the rival gets nothing. There is a law that e_i cannot exceed half and it is greater than 0. Model this situation as a strategic game and find the Nash equilibrium. Is there a different pair of e_1 e_2 giving firms higher payoff?

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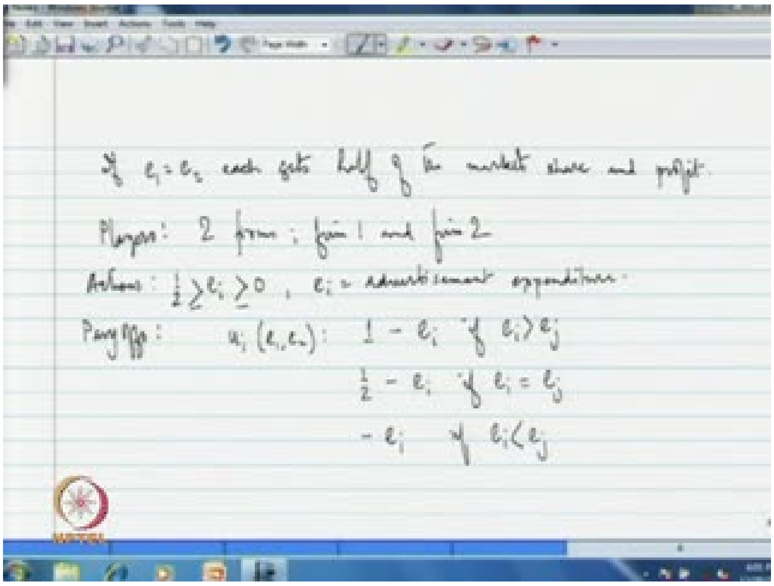

One more point, if market share and profit is equal. Each gets half of the market share and profit. So, we have to model this situation as a strategic game. So, who are players? Here, the players are the two firms - firm 1 and firm 2.

Actions - what are the actions that they choose for e i? e i is greater than 0, it can be equal to 0 also and e i is less than equal to half. Here, e i is the advertisement expenditure less that this. It is greater than equal to 0. So, u i is the payoff of firm i. It is a function of two variables - e 1 and e 2. Here, i can be 1 or 2 and it is equal to s of e i. If it is spends more than the other firm, it gets the entire market. It means it gets 1 minus e i, if e i is greater than e j. It is half minus e I, if e i is equal to e j. It is equal to minus e I, if e i is less than e j. Remember, the total market is the market share and the highest market share that is possible is 1. If my advertisement expenditure is more than the other firm’s advertisement expenditure, I get the entire market. The highest value of s in that case is 1 and if our spending are same, I get half of that entire market. So that basically is normalized to half and so this becomes the payoff function of each player.

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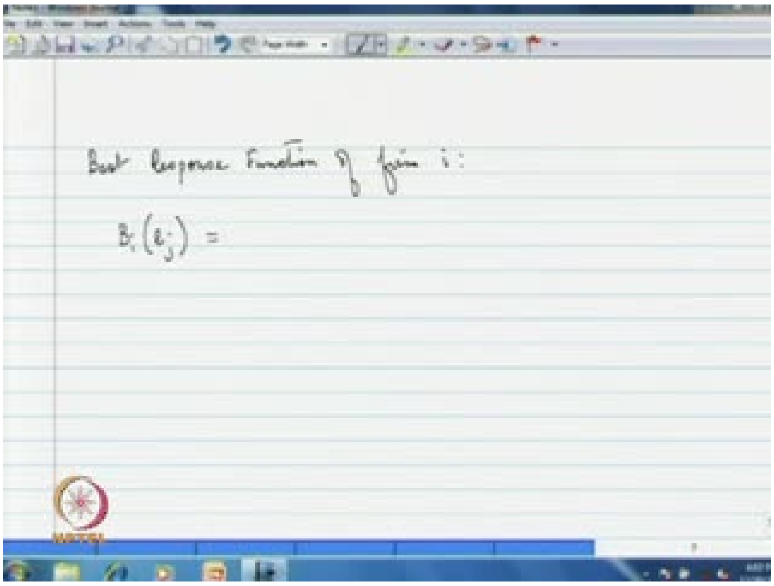

We have to find out what is the Nash equilibrium. For that I have to find out what is the best response function of firm i set, so it is a function of e j and it is given by what? Remember, the highest value that e j can take is half and the lowest value that it can take is 0.

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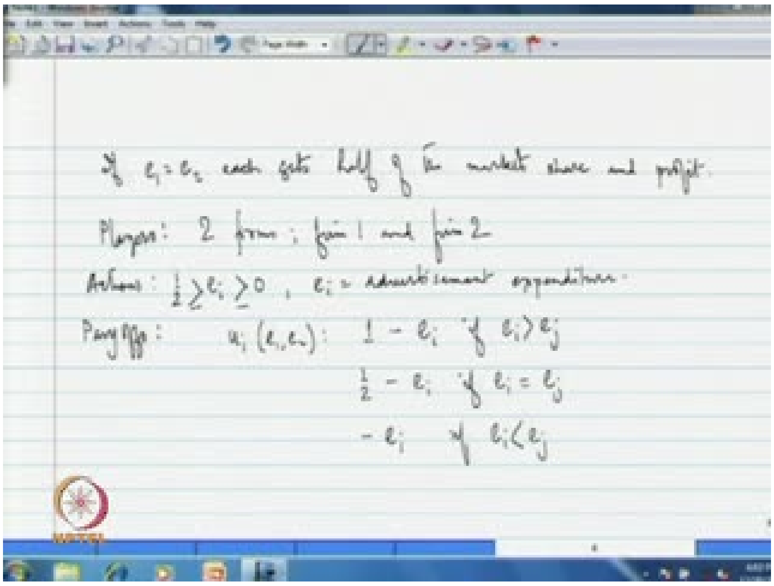

Always the firm i will like to have his expenditure slightly higher than the other firm's expenditure. In doing so, it will have an expenditure greater than e j, but by what and by how much? There is no any unique e i, which is greater than e j, but lowest among all the possible values that it can take.

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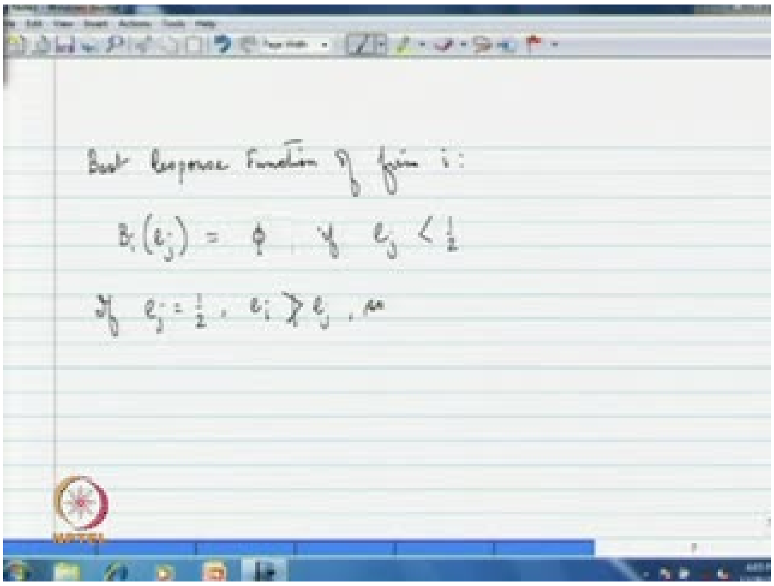

Therefore, it is given by this null set, if e j is less than half. This is one case - e j is less than half, if e j is equal to half. What happens, if e j is equal to half? Remember, in that case, e i cannot be more than e j because the highest value that e i can take is half. So, what player i can do in this case is make his a expenditure equal to e j.

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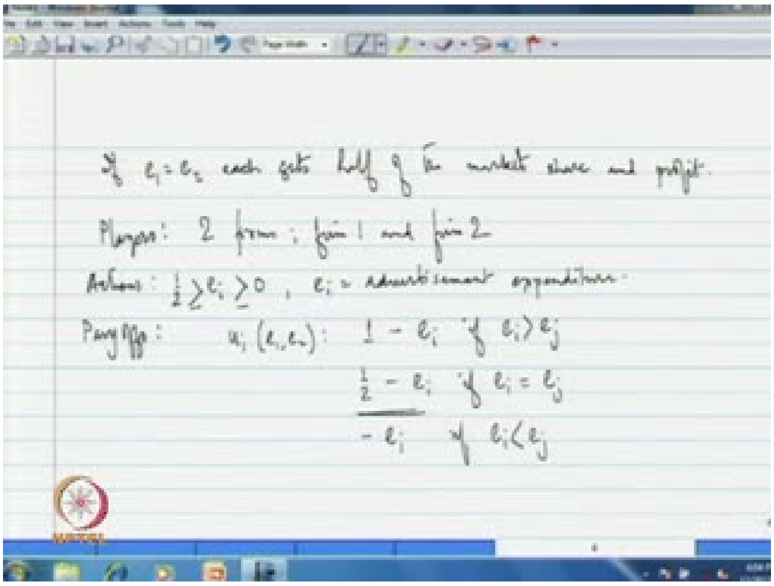

In that case, e is getting e i is equal to half and then e is getting half minus. Here, half is equal to 0 or he can spent 0 in which case, his payoff remains 0. These are the two possibilities that he can do.

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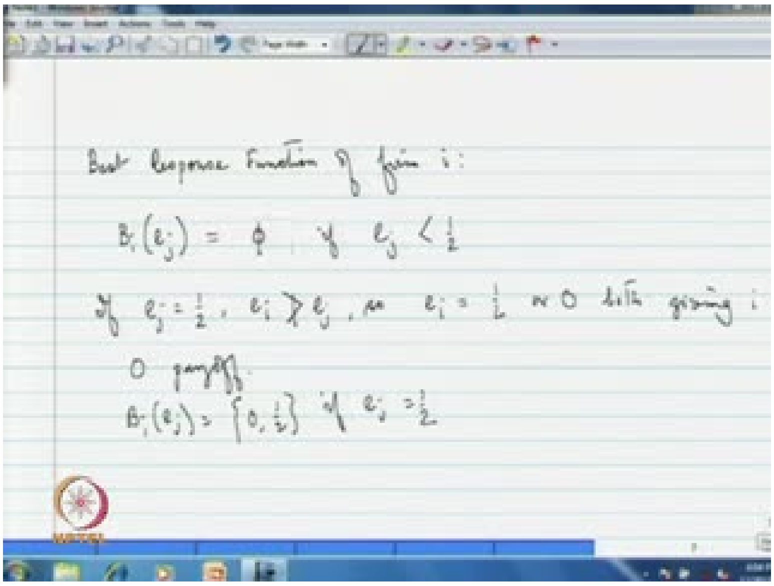

In this case, e i can be equal to half or 0 or both giving 0 payoff. So, in this case, what we are having is this; this is the case and these are the best response functions. Let us not plot the two best response functions and try to find out what is the Nash equilibrium.

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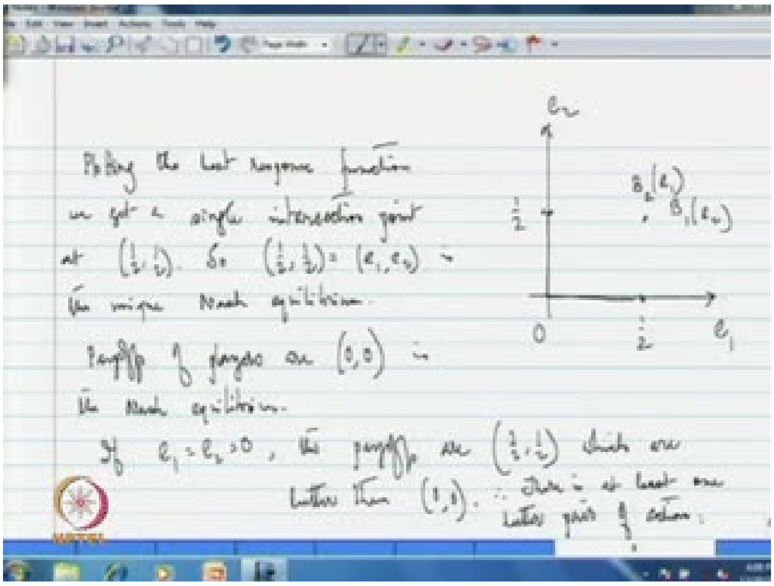

Here, you have e 1 and e 2 and let us say these are the values half and half. We have seen that if e 2 is less than half, then there is no best response for firm 1. For this range, there is no best response. If e 2 is equal to half, then the best e 1 could be either 0 or half. So, this is one possibility and this is another possibility, both are equally good. So, this is B 1. What about B 2? We have seen that if e 1 is less than half, then there is no best response for firm 2. If e 1 is equal to half, this is one possibility that e 2 a becomes 0 or e 2 becomes half.

So, this is B 2, B 1 as a function of e 2 and B 2 as a function of e 1. Therefore, plotting the best responses, this is the Nash equilibrium; this is the unique Nash equilibrium. Interestingly, the payoffs of the two players are 0 because both are choosing their highest possible advertisement expenditures, the market is getting share and the payoff is becoming half minus half and therefore 0. Is there any other pair, which gives the players better payoff than the Nash equilibrium payoff? If different pairs e 1 and e 2 giving the firms higher payoff, it appears that there is at least one pair. If e 1 is equal to e 2 and it is equal to 0, again the market is getting divided here as half and half, which is better than 0 0. So, there is at least one better pair of actions and that is it.

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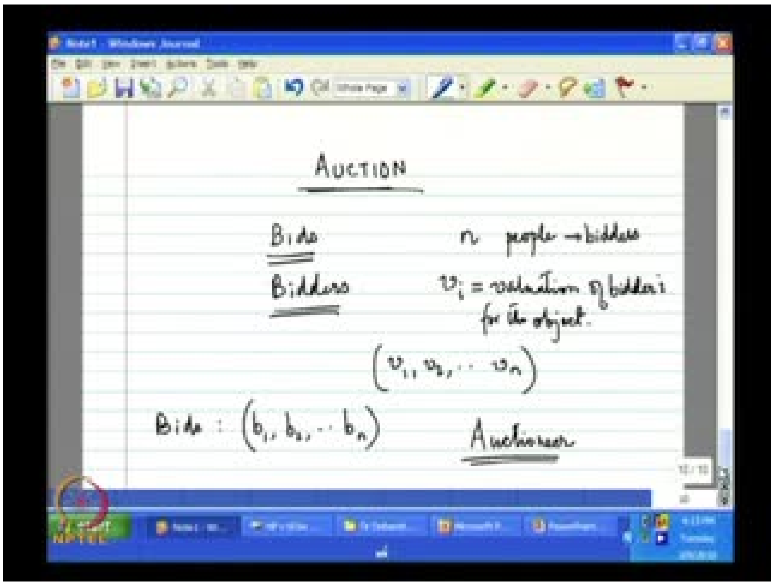

What we are going to take up now is another application of Nash equilibrium, which is known as the auction. So, what does auction mean? Auction is a problem of allocating of some precious resource among contenting parties. This precious resource can be anything, it can be an object of art, it can be oil filled, it can be mobile frequency, it can be radio frequency. So, all these things are sold by the government or by the person who is owning it. It is not necessarily the owner always and two people, who want to have that object. The people who want to have the object will compete with each other and try to out to do each other by announcing high and high prices. These quotations of the prices are known as bids. So, we say bidder people submitting bids are called bidders and they are quoting prices.

As we see, the price that I quote or that object is not necessarily the price that I pay. There are different kinds of auctions. In some auctions, you pay the price that you quote. In some case, you pay the price, which has been quoted by someone else, but nevertheless, the person who is quoting the highest price will generally get the object, so that is the idea. Suppose, there are n people or n bidders and there are two sets of things to remember here. Each bidder will have some valuation for the object and it is not necessary that the valuation of any particular bidder will be same as the valuation of any other bidder. These valuations can differ across the bidders, so we shall call these valuations as v. In particular, v i is the valuation of bidder i for the object.

Now, this is one set of variables that we shall remember that is v 1, v 2, v n. There is another set of values that one needs to remember. These are the bids and bids are b 1, b 2, b n. This is the symbol that we are going to use for the prices that are quoted by each of the bidders. So, bidder 1 is having a valuation 1 and bidding b 1, bidder 2 has a valuation of v 2 and bidding b 2 and like that.

Our task is study of auction theory. Incidentally, auction theory is a separate branch within economics. It is to see different kinds of auctions. It is the allocation of the object efficiently and the case that the person, who values the object most is getting the object. Does it or can it go to someone else? We also want to look at whether the person, who is conducting the entire thing, who is known as the auctioneer is getting the maximum possible revenue from the entire bidding process. So, does it maximize the revenue? These are some of the questions that we shall be interested in and we shall like to answer those questions.

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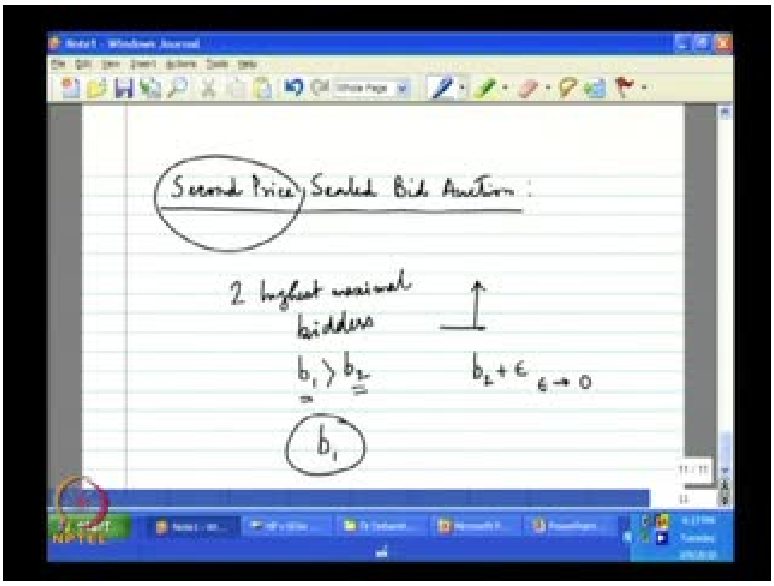

Now, as I just said that there are different kinds of auctions. The first kind of auction that we shall deal is known as second price sealed bid auction. So, think of the general idea of auction that we have many people, who have valuations for this object as v 1, v 2 etc. We are going to assume that these v 1 and v 2 are known to everyone. So, it is a common knowledge, how I value the object and I also know what are the valuations that any other players posses for that object.

Now, in general, the auctions that we are familiar with price starts from a very low value and it goes up. Now, if it goes up, then I can imagine that any person or any bidder will have a maximal amount of biding, beyond which he or she will not go. If that is the case, then ultimately the auction is a fight between two people. If I think that they are maximal's are different; maximal bidding that each person will submit is different from others, then it is a file between two people or two highest maximal bidders.

If that is the case and suppose, b 1 is greater than b 2 that is player 1 is ready to pay the maximum amount as b 1 and the maximum bidding that player 2 will do is b 2. What is the optimal bidding for player 1? In this case, observe that player 1 will not optimally bid b 1 and he will bid something as b 2 plus epsilon. If he is bidding more than player 2, he is getting the object. So, there is no point why he should bid b 1. Epsilon can be taken to 0 and it can approach 0, which means that player 1's optimal bidding can be thought to be equal to b 2 itself.

He will not bid his own maximal team, so this is the general idea of auction that we have. This should be replicated in a kind of sealed bid auction, where people do not observe other bidding. I do not see what the other people is doing. I do whatever I submit and whatever my maximal bid is. If my bid is the maximal and my b 1 is the highest, then I will get the object, but the price that I will pay is not this b 1, but this b 2. So, by this technique of sealed bid, what we want to do is to replicate the original auction that we are talking about, where people can see each other's bids. Here, people cannot see other bids; nevertheless the auctioneer receives all these bids from people from the players. He picks up the bid, which is the highest and decides the player, who has submitted the highest bid.

He will get the object, but he will not pay the price that he has bid. He will pay the price of the highest of the other players other than him and that is why it is called second price auction and not the first price, but the second price. It is sealed bid that I do not know, what is the price submitted by other players and that is why it is sealed bid. It is important that it is sealed bid because if you remember, in this strategic game, I do not know the action taken by any other players. So, it has to be a sealed bid auction and this is the rule of the game.

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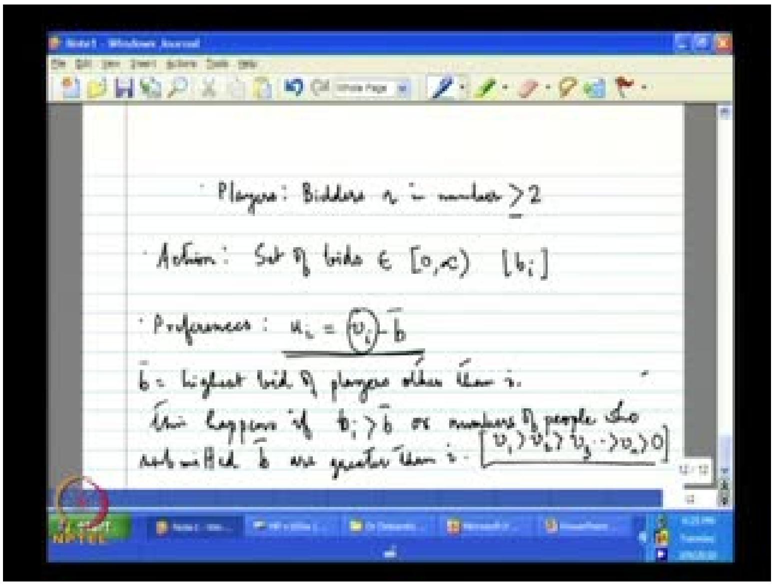

If I want to write this in terms of the language of game theory, the language that we have introduced so far, I have to specify three elements. So, the players and who are the players? The players are the bidders and they are any number n. It can be any value greater than or equal to 2. There cannot be any auction, if there is just one player or one bidder to get the object.

What does action decides on? So, it is set of bids, which is any non-negative number. So, b i is being determined by player i. In general, let us write what is the payoff of player i. Player i will get the object under two circumstances - one is that his bid that is b i is higher than any other bid, all the other players are bidding something, which is less than his bid. That is one possibility, but what happens if there are two people, who have bid the highest bids. For example, it may happen that b i, which is the bid of player i is equal to b 1 or suppose b 7, then who gets the object. So, this is case of tie and we will follow the rule of tie breaking that if b 7 is equal to b 10 that is the bid of player 7 is equal to bid of player 10 and this is the highest, then 7 will get it. Why player 7 will get it? Because player 7 is less than player 10.

What is the rational for that? Why are we assuming that the player with the lower number is getting the object? The reason is that or the kind of rational is that we are going to assume that v 1 is strictly greater than v 2 to v n strictly is greater than 0, which means that player 1 values the object most and player 2's valuation for the object is less. The least valuation for the object is for player n, but which is still greater than 0. These are different numbers that also has to be remembered. So, in a case, where bid of two players are equal, then the object is being allocated to person with the lower number because that person values the object more than the other person. So that is the rational for tie breaking.

Now, player i will get the object under two circumstances - one is b I, it is greater than every other bid, then player i will get it. Player i can get the object, if suppose b i is equal to some other b, but i is less than the number or the index of other b. In these two cases, i will get the object. If i get the object, then what is his payoff? Let us try to write that. So, it is written as v i minus b bar, where b bar is the bid of the highest bid.

This will happen, if b i is greater than b bar or number of people, who submitted b bar number. Let us say because there might be more than one person, who has submitted b bar. Numbers of people who submitted b bar are greater than i. So, under these two circumstances, player y will get v i minus b bar, when his bid is the highest - single highest bid, then he will get the object. The price that he pays is the bid of this person and the person, who has bid as highest bid among the set of other players.

So, his payoff in that case is the valuation of his for the object minus the price that he is paying that is b bar. He can get the object also, if his bid is same as some other people's bid. Both these bids are highest and his number is less than the number of the other person.

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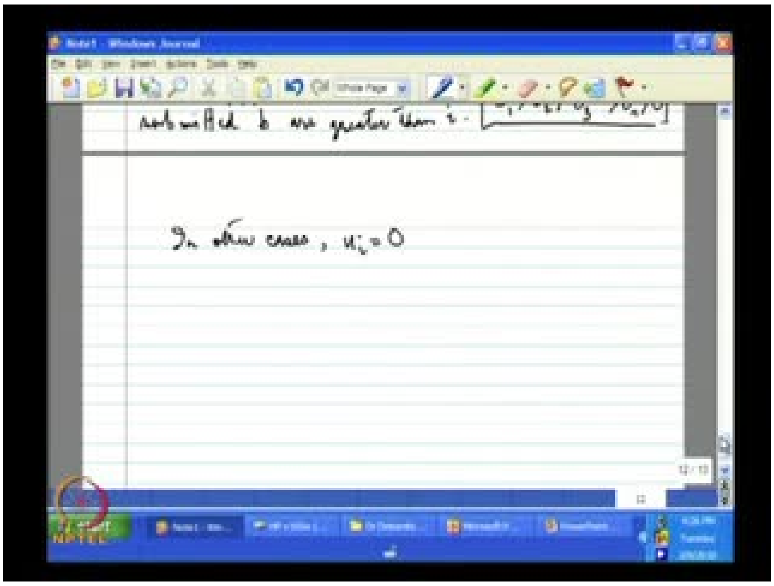

In other cases, u i is simply equal to 0. What can be the other cases that b i is less than b bar that is one case or b i is equal to b bar, but i is greater than the number of the people, who have submitted b bar. In those cases, we can say that u i is equal to 0. Notice, it is not necessarily that this value v i minus b bar is positive. It may happen that this value is 0 and this may turn out to be negative also because it may happen, a person has a low v but has a submitted a very high b i. So, bid can be in theory and it can be higher than your valuation. In that case, it may happen that you win the object, but the payoff that you get is negative or it can be 0. This is the general setting and we shall stop here.

We shall try to see what are the Nash equilibria, which can be derived in the case of second price sealed bid auction. So, before we finish this lecture, let me take you through what we have discussed in this lecture. We have ended the discussion of war of attrition and we have looked at the various aspects of war of attrition, but the simple thing that has to be highlighted is that in equilibrium, in the original model of war of attrition, people submit defer. People take different kinds of action; in particular one person just withdraws from the fight and immediately there is no fight in equilibrium, but we have seen that there can be variations, whereas in the equilibrium, there are fights.

For example, in the last case, we have seen people fighting for a long time. In fact, we have started the discussion of auctions and we have started the discussion of second price sealed bid auction. See you in the next class. Thank you.

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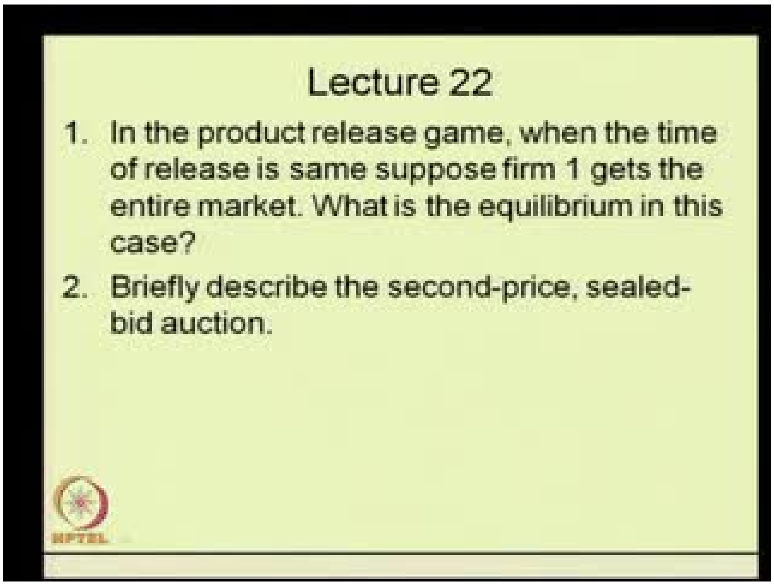

In the product release game, when the time of release is same and suppose, firm one gets the entire market, then what is the equilibrium in this case?

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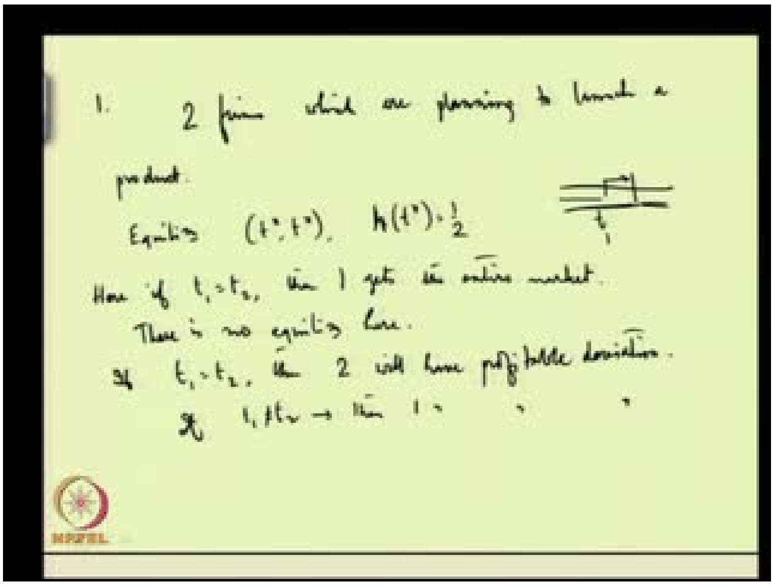

Let us try to remember what the product release game was and what the equilibrium was. So, there are two firms, which are planning to launch a product. If I launch earlier, if I launch at this time t 1, I get this much share of the market, but longer I wait for the product to develop in my research, I will get a bigger share of the market. So, more higher t, I choose, greater share of the market I get. There is a problem that if I wait for a long time, the other firm can launch his product earlier than mine. It can get a part of the market and I will be left with the rest of the market.

The equilibrium we saw was the following. It was given by t star, t star, where h t star equal to half. It means that they will they release their goods at the same time, t star. What is the property of this t star? At t star, if a firm announces its launches of its goods at t star, it gets half of the market. This half represents that share of the market.

Now, this was the case, where if the time is the same, then market is shared in equal parts. If t 1 is equal to t 2, then one gets the entire market question. It is what the equilibrium. Our claim is that there is no equilibrium here and what is the reason? Reason is that if t 1 is equal to t 2, then t 2 will have profitable deviation. Remember, t is equal to t 2, the market entirely goes to firm 1. So, the firm 2 will choose a time, which is less than t 1. We will get some portion of the market and it does not matter. It will get some positive portion of the market, so deviation by 2 is profitable.

Now, this cannot happen, if t 1 plus t 2 is equal to 0, then firm 2 cannot charge. It cannot fix a time less than t 1, which is equal to 0. In that case, firm 2 will choose a time little bit above t 1. So that if t 1 choose the 0, it get zero portion of the market. So that the rest of the market goes to firm 2. If t 1 is equal to t 2, there cannot be an equilibrium. If t 1 is not equal to t 2, it cannot be an equilibrium and then, it will have a profitable deviation.

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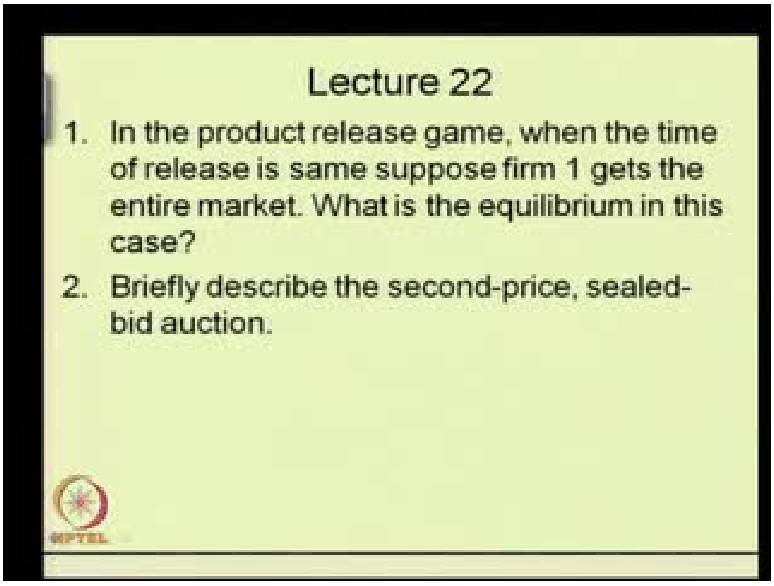

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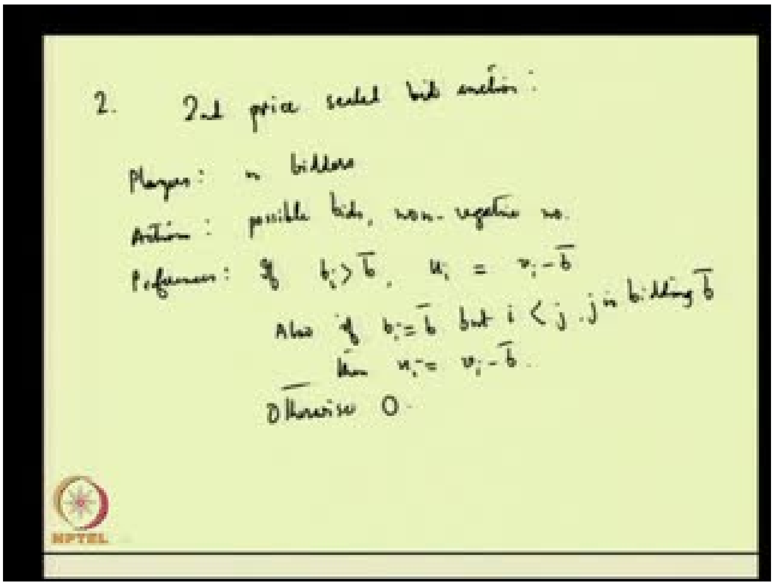

If t 1 is equal to t 2, firm 1 will choose a time just equal to t 2 and get the entire market. So that is the idea and there cannot be an equilibrium in this case. Briefly, describe the second price sealed bid auction players n bidders actions possible bids non-negative numbers and preferences if b i is greater than b bar where b bar is the highest bid of other players then payoff of player i is equal to v i minus b bar also if b i is equal to b bar but i is less than j, where j is bidding b bar then u i is same that is v i minus b bar otherwise 0. Thank you.