Game Theory and Economics

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Module No. # 03
Illustrations of Nash Equilibrium
Lecture No. # 11
Further Aspects of Second Price Auction

Welcome to lecture 11, of module 3, of the course called game theory and economics. Let me just briefly recapitulate what we have been doing in this course so far.

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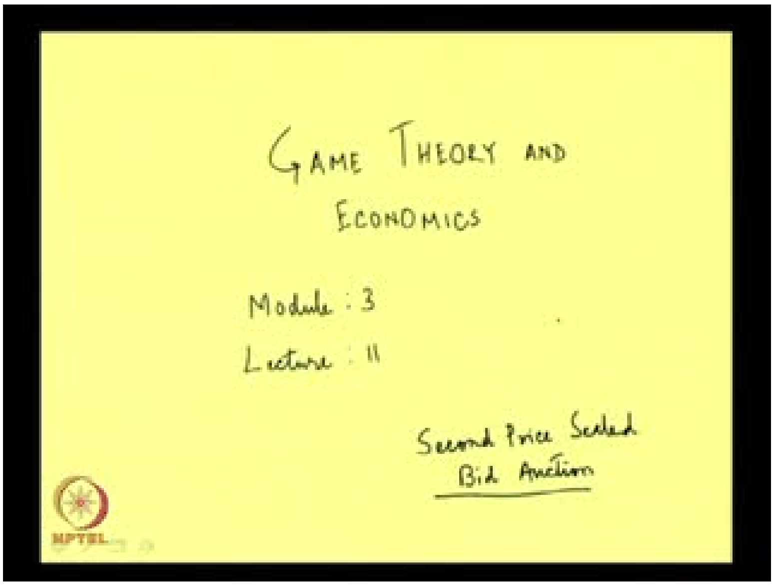

In the last lecture, we have started discussing the theory of auctions in game theory. The first auction that we are discussing here is the auction, which is known as the Second Price Sealed Bid Auction. This is just one of the different kinds of auctions that are used; in particular, auctions are used to allocate kind of precious goods or an article among different people.

So, there might be different contending parties, we will like to have that article, and they have different valuation of that article, what the auctioneer or the person would decides who will get the article does is to that he collects bids from different people and allocates that article to that person who submits the highest bid. In second price sealed bid auction, obviously, bids of other people are not known to any particular bidder. When a particular person is decided to be that person who has submitted the highest bid, he gets the object; but the price that he pays is not equal to the bid that he has submitted. In fact, the price that he pays, is the highest bid of the set of people which can be selected if he is taken out of the set of bidders; that is, the price that the person who gets the object pays, is the highest bid of the other people besides himself; so that is what known as second price sealed bid auction and so following will be the payoff here.

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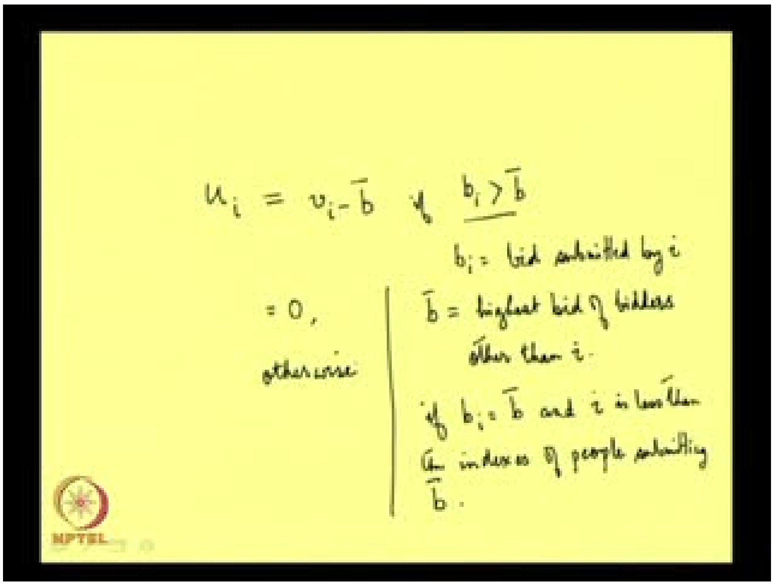

Here, if b i is greater than b bar, then this person is getting the object and the price that he pays is b bar, where b i is the bid submitted by i and b bar is the highest bid. So, this is the case, if b i is greater than b bar; however, person i will get the object not only in this case, but also in the case when b i is equal to b bar, and i is less than the indexes of people submitting b bar.

So, it is possible that the highest bidder that is, i has submitted a bid which is same as the bid of the highest bidder if he is taken out of the group. In that case, he may get the object if his index - that is i - is less than the indexes of the other people which means that he values the object more than the other people.

In these cases, i will get the object; if i does not get the object he gets 0. So, we write payoff is 0 otherwise means, if b i is less than b bar or if b i is equal to b bar, but i is greater than the indexes of the people submitting b bar, so this is the setting. Now, what will be the set of Nash equilibria in this particular second price auction.

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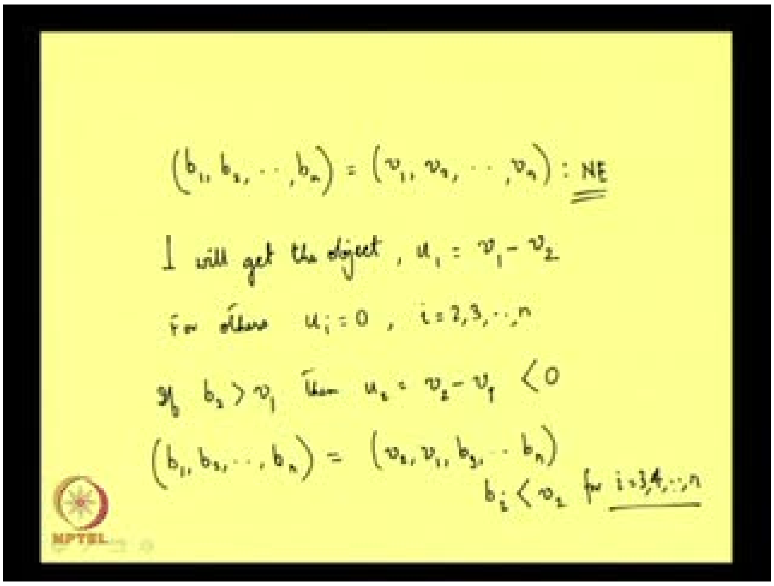

So, there can be several Nash equilibrium in this auction, particular Nash equilibrium that may seem very natural is this one. The specialty of this Nash equilibrium is very obvious; each person is submitting the bid which is equal to his or her valuation. In this case, the person 1 will get the object and the price that he will pay is equal to v 2 - v 2 is the second highest bid here.

So, this is his payoff v 1 minus v 2, for others payoff is 0, so I am claiming that this is a Nash equilibrium why? This is Nash equilibrium because, if player 1 deviates suppose, he deviates and bids something more than b 1 which is equal to v 1 here. So, if he bids more than v 1, then he is still getting the object and his payoff remains the same. So, by bidding more you are not improving your payoff, if he bids less then, as long as this less bid is more than v 2 again his payoff is remaining the same.

If he bids less than v 2 then he does not get the object, then his payoff becomes 0 which is strictly worse off. So, by changing the bid, player 1 is not better off, so that is why from player 1's point of view he is doing the optimal thing, what about the other players? If any other players suppose player 2 bids more than v 2 as long as he is bidding less than v 1 or equal to v 1, he is not getting the object, so his payoff remaining is 0. He can get the object if he bids more than v 1, but if he bids more than v 1; if b 2 is more than v 1 then what is the payoff? It is v 2 which is his valuation minus v 1, because in that case the second highest is v 1 and which we know is less than 0.

So, player 1 by deviating and bidding something more than v 2 is not better off, he can be strictly worse off and obviously by bidding less he does not make any difference to the outcome. So, his payoff remains at 0 and this kind of logic can be extended for other bidders also, as long as they are bidding less than are equal to v 1 their payoff is going to remain as 0.

If they bid more than v 1 then they can get the object but the payoff becomes negative, so that is why this is Nash equilibrium. So, everybody bidding his or her valuation is Nash equilibrium. In this particular Nash equilibrium, player 1 is getting the object, but as I said that the number of Nash equilibria in this game is infinite, there can be infinite number of Nash equilibrium.

One particular Nash equilibrium that is of interest is the following; here in short what is happening is, that player 1 is submitting a bid which is equal to v 2; player 2 is submitting a bid which is equal to v 1, and other players are submitting bids which are less than v 2 - strictly less than v 2. I am clamming that this is a Nash equilibrium, why this is so? First observe that in this particular configuration of bids, it is player 2's bid which is v 1 - the highest, so player 2 gets the object but what is his payoff? It is v 2 minus v 2, because player 1 has submitted the second highest bid which is in fact v 2 and so player 2's payoff from this equilibrium set of actions is 0.

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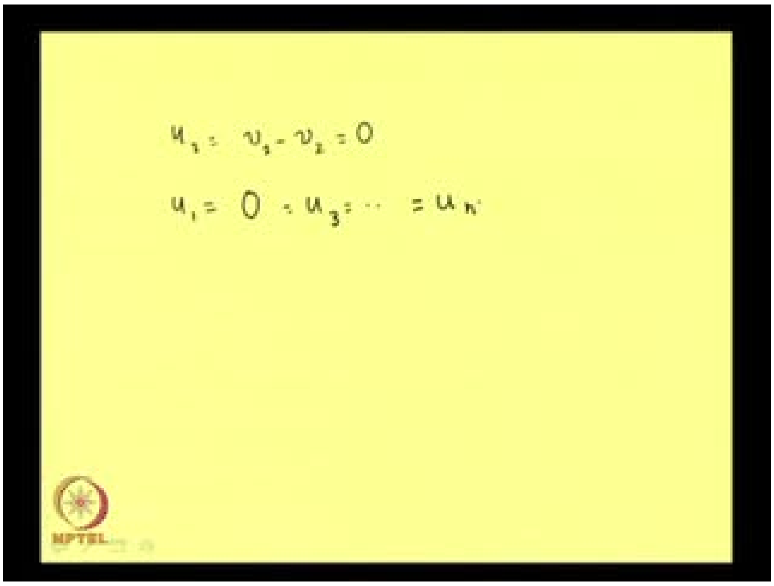

What is player 1's payoff? Player 1 is not getting the object, so it is anyway 0 and this is true for other players also.

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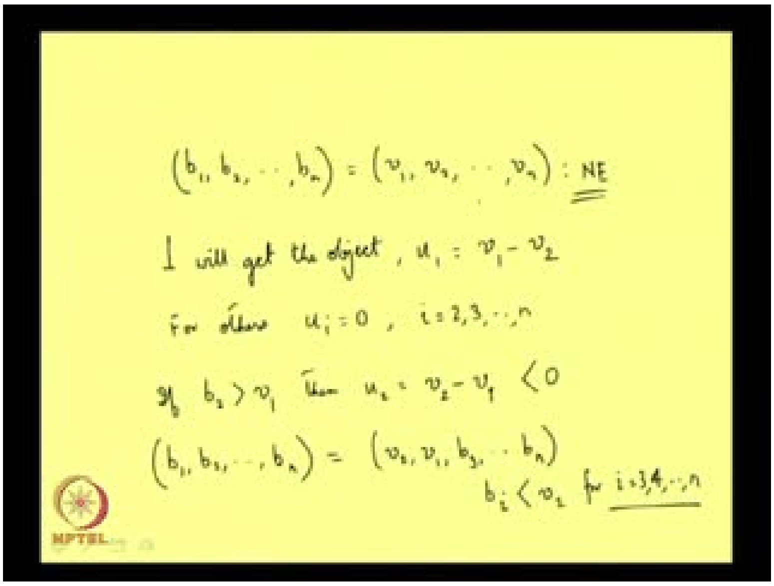

In this case, if player 2 can deviate from v 1 to something less, but if the deviation is greater than v 2, he is still getting the object. As long as he is getting the object his payoff is remaining same at v 2 minus v 2 that is 0. By bidding more, again he is still getting the object, so the payoff remains the same 0. If he bids equal to v 2 or less than v 2 then what happens? He is not getting the object, but even if he is not getting the object his payoff is remaining 0, so by deviating player 2 cannot improve the payoff. So, from player 2's point of view this is optimal; from player 1's point of view is this optimal? Player 1 is bidding v 2.

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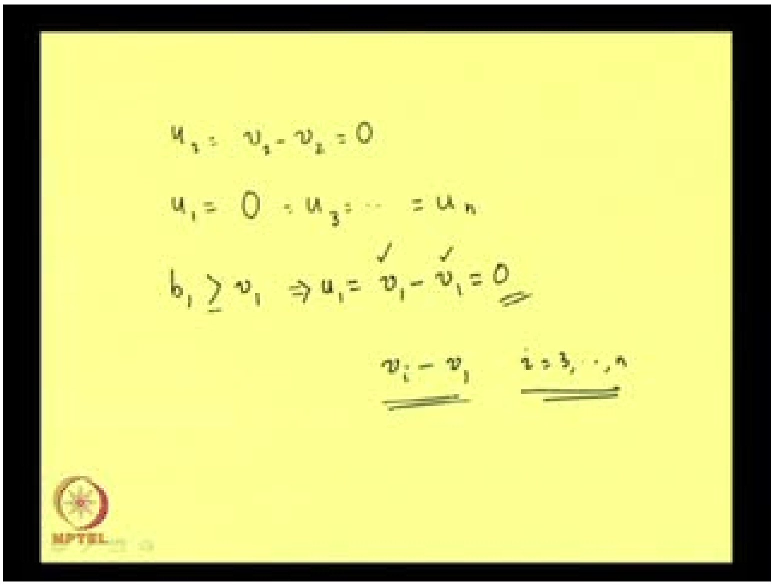

He can get the object if he bids more than v 1, but if he bids more than v 1 what happens? Suppose, b 1 is more than v 1 what happens to u 1? u 1 is equal to minus v 1, this is the valuation and this is the price that he pays because, player 2 is bidding v 1, so his payoff remains 0 even if he gets the object. Obviously, by doing something else by bidding less than v 1, he is not getting the object and therefore, - in fact this is true even, if it is equal to v 1 by bidding less than v 1, he is not getting the object - the payoff remains at 0.

Hence by deviating the player 1 is not being better off and this is true for other players also, that is, player 3, 4, all of them can get the object, but if they get the object their payoff is going to be negative, because v i minus v 1 will be the payoff and where i is we know this to be a negative thing.

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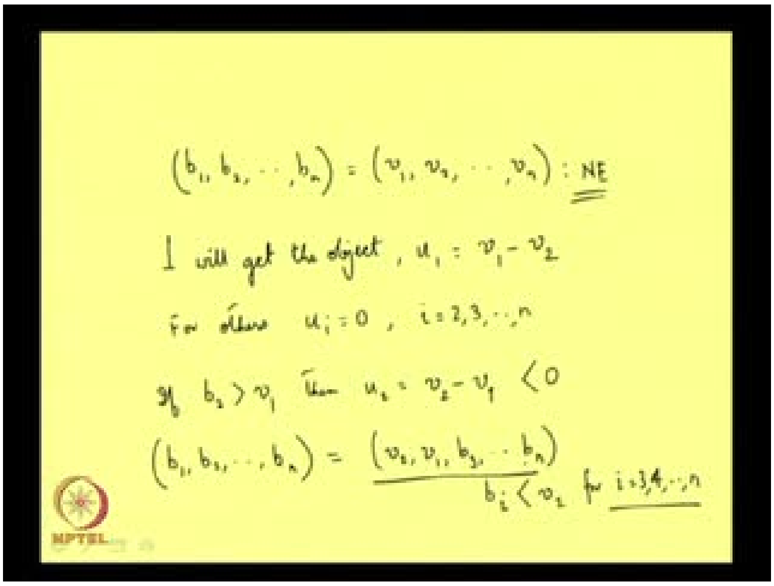

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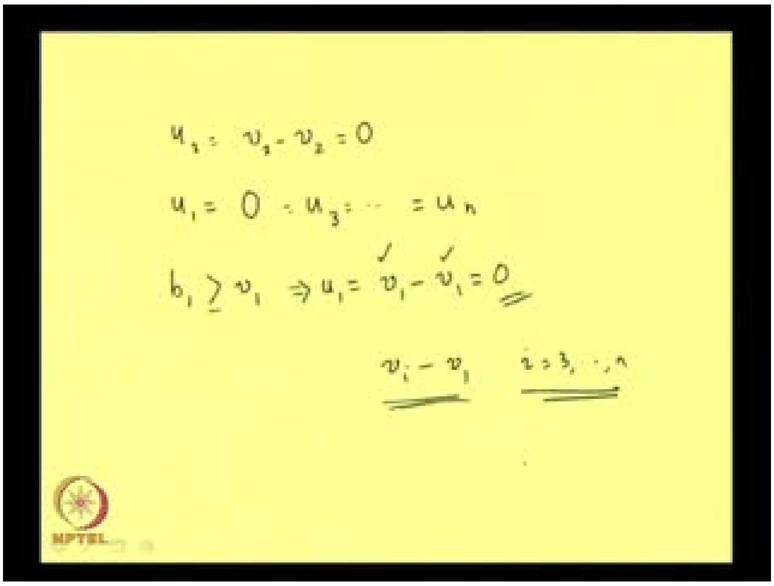

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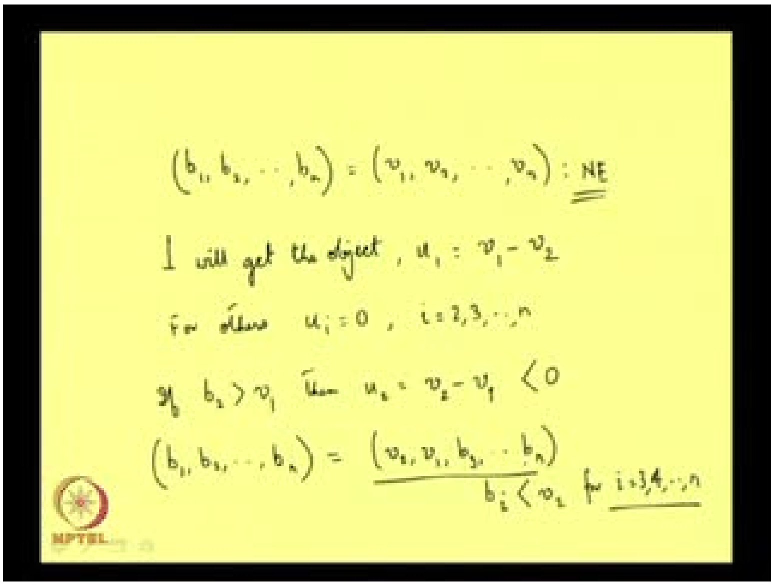

So, this set of actions is indeed Nash equilibrium. Now, this is something which is little bit curious because, in this Nash equilibrium what player 2 doing is bidding more than his valuation, that is, he is bidding v 1 which is greater than his valuation which is v 2. It may seem little unnatural because a player is risking the case - is risking a lot - in fact if he is bidding more than the valuation because, if the second highest is also more than the valuation, then the first player is going to make a loss.

In this equilibrium, he is not making a loss because the second highest bid is equal to v 2, so player 2 is breaking even, but never the less its seems a little unlikely that any player will bid more than the valuation, because it becomes little risky. This idea is sort to be captured by the following proposition that in second price auction for any player i, v i weakly dominates all other actions.

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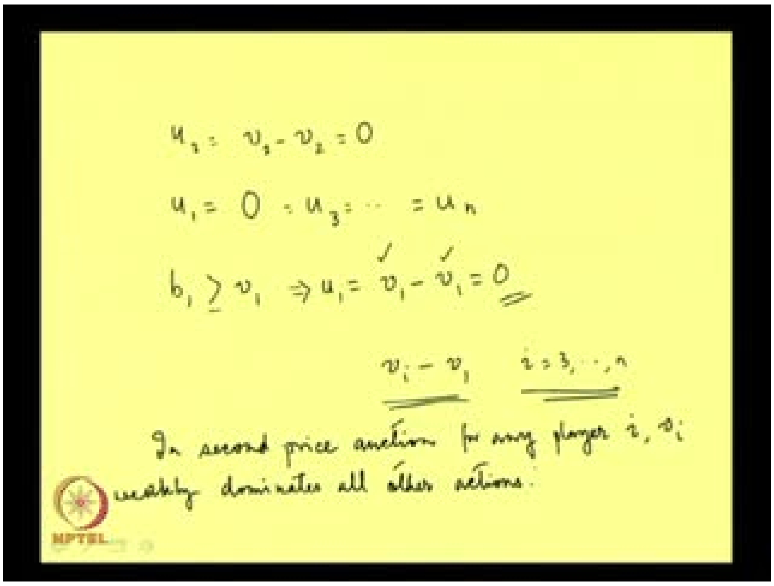

So, for any player i if that player bids v i, that is, weakly better than bidding anything else, which means that in some cases this person i will be strictly better off by not bidding anything else. In some cases he will be indifferent if he bids v 1 compared to any other bid, to exactly show how we can make this proposition is the following.

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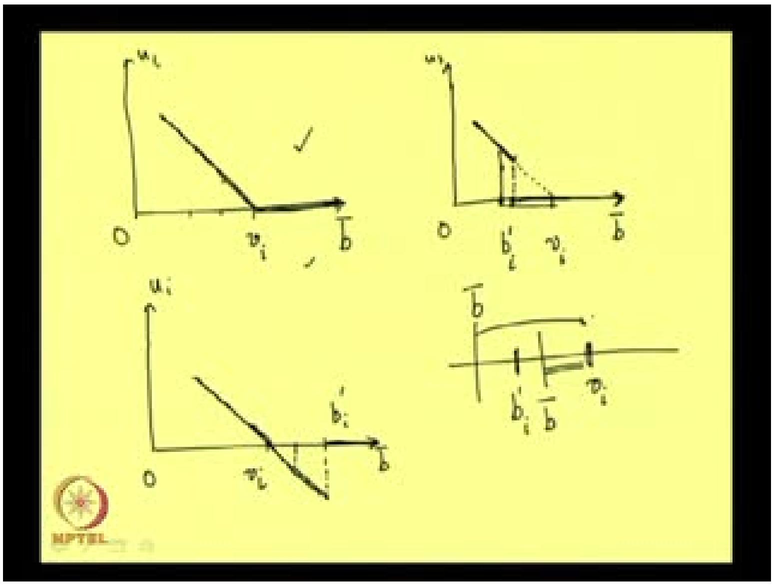

Suppose, in this line I am representing b bar - b bar is the highest bid of other players - if I take out the player i and here is v i. Now, if b bar is less than v i then, player 1 player i will get the object - suppose, b bar is here - he will get the object and the payoff that he gets is this much, where this value is same as this value (Refer Slide Time: 19:15).

Similarly, if b bar is here this value is same as this value and therefore, by connecting all these points I get the line which represents the payoff of player i. If b bar is equal to v i, it does not matter whether, i gets the object or does not get the object, the payoff remains 0. Because, even if he gets the object, the price that he pays is equal to v i, so the net payoff is 0.

If b bar is greater than v i obviously, he is not getting the objects, so payoff is 0; so it remains coincided with the axis. So, this is the case of v i is equal to the person suppose bidding equal to v i, then what happens? Suppose, the person is not bidding v i but less than v i; he is bidding b i dashed.

Now, if he is bidding b i dashed then, if the other players highest bid is less than this b i dashed then he is getting the object obviously. The price that he is paying is the difference between v i and b dashed, so this part is same as this part, so this is 45 degrees then once again, but if b bar is greater than b dashed then this person is not going to get the object. So, at this point it stops here and after this b bar is greater than b i dashed, the person is not getting the object therefore, the payoff is going to coincide with the horizontal axis.

Remember, if he had bid equal to v in this case, he would have got more than what he is getting right now. So, this dotted line is what he could get if he had bid equal to v, what happens if he bids more than v? Suppose, b i dashed is here obviously, for all these values of b bar - this is b bar - which are less than v i, this 45 degree line will denote his payoff.

Now, if his pay is v i dashed and b bar is here then, he is getting the object but he is getting the object and his net payoff in that case is negative, it is v i minus v bar. So, I have this extension of this 45 degree line here and this goes all the way until I reach b i dashed. So, I am basically venturing in to the negative territory which I was not doing here; that means, by comparing all these three possible cases, the person is at his best that is he is doing the optimal thing by bidding only v i by not bidding something more. If he bids something more than in some cases he is worse off; if he bids less, again in some cases he can be forgoing or payoff which he could have earned if he had bid v 1 v i, so that is the idea.

So, in terms of this - this is v i - and suppose I am considering this b i dashed which is less than v i; if b dashed is here I could have earned this value by bidding v i. By bidding b i dashed I am not getting that value because this good is going to someone else that is the idea. So, bidding one’s valuation v i is weakly better than bidding some others valuation. Obviously, if b bar is here it does not matter whether a person is bidding b i dashed or v i, in both cases the person is getting the object and the payoff is this much.

So, in some cases, the payoff from both the actions, from these two actions are the same, but in some cases v i is giving the person a better payoff, that is why this is a weakly dominating action, and this is true for any bid which is greater than v i also that is why we are saying that v i weakly dominates all other actions.

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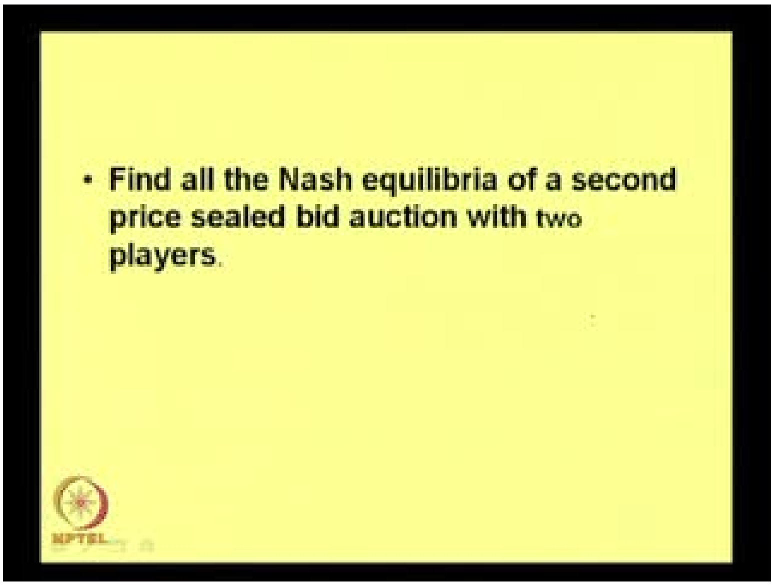

Let us do one exercise which is a generalization of this frame work. Find all the Nash equilibria of a second price sealed bid auction with two players. We have seen that there are plenty of possibilities of Nash equilibria here, in particular it is not necessary that person 1 will get the object, it may happen that person 2 gets the object. So, if I take a simpler case of two people not n people, just 2 bidders then, how will the complete set of Nash equilibria look like.

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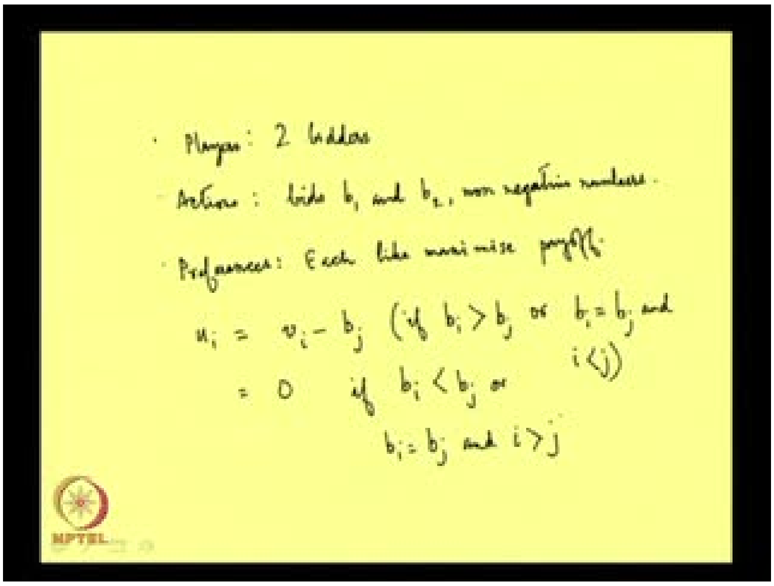

So, let us try to solve this problem, so players: 2 bidders, actions they are bids b 1 and b 2, non-negative numbers; preferences: each likes to maximize payoff and u i that is the payoff of player i is v i minus b j. If b i is greater than b j or b i is equal to b j and i is less than j is equal to 0, if b i is less than b j. If your bid is less or b i is equal to b j and i is greater than j, so this is the payoff function of player i, i can be 1, i can be 2, we need to find out what is the Nash equilibrium.

So, we shall go along the familiar way we shall try to find out the best response functions and try to see at what areas or what points the best response function of these two players are intersecting with each other, so for player i, b i b j this is what we need to find.

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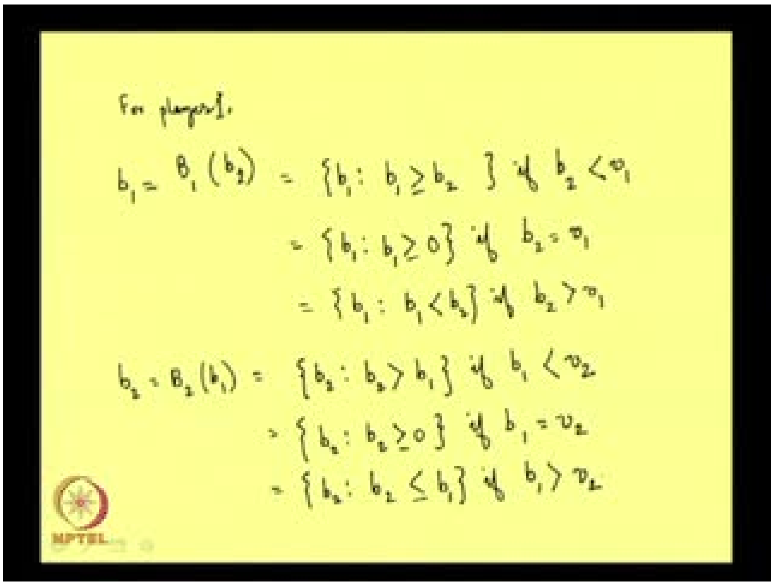

Remember, what is important here is how different this small b j is compared to the valuation of player i. For player 1, suppose, if b j that is a bid of player 2 is less than v 1 in that case it will be worthwhile for player 1 to outbid his rival and how he does out bid the rival? He bids either equal to his rival or he bids more than the rival.

In both the cases he is indifferent because, he is getting the object and paying the price which is same as b 2. So, let us write down the best response function instead of the general case write down the best response function of player 1. So, this is b 1 because things are not symmetric that is why we are writing these things, this two best response functions separately is equal to b 1 such that b 1 is greater than equal to b 2 if b 2 is less than v 1; if b 2 is equal to v 1 then by bidding less, player 1 is getting 0 by bidding equal or less greater than b 2 also b 1: b 1 is greater than equal to 0 if b 2 is equal to v 1.

In this case by bidding anything player 1 is getting no payoff - he is getting 0 payoffs. So, he can bid anything, there is no maximum to his payoff function. What happens if b 2 is strictly greater than v 1? In that case, player 1 will like to lose the object, because if he gets the object then, the price that he will be paying will be more than his valuation which is not good for him. So, in this case player 1 will be bidding something strictly less than b 2, if b 2 is greater than v 1.

So, this is the best response function of player 1, what about the best response function of player 2? Likewise, if b 1 is less than v 2, then it is worthwhile for player 2 to win the object. So, he will win the object, if he bids strictly more than the other peoples bid, if b 1 is strictly less than v 2. If b 1 is equal to v 2 again, player 2 gets nothing, even if he gets the object he is getting nothing; if he does not get the object obviously, he is not getting anything, so in this case b 2 is this.

Finally, if b 1 is greater than v 2 player 2 will likely lose the object, so in this case he loses the object by bidding less than b 1 or equal to b 1, so this is the complete best response functions of two players.

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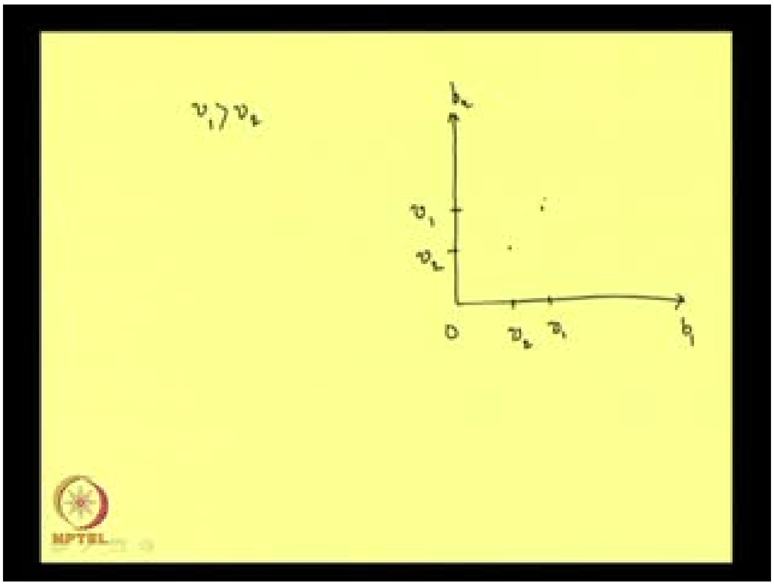

We shall now plot these two best response functions and try to see what are the overlaps, so this is b 1, this is b 2, remember, v 1 is greater than v 2; this is suppose v 2, this is suppose v 1.

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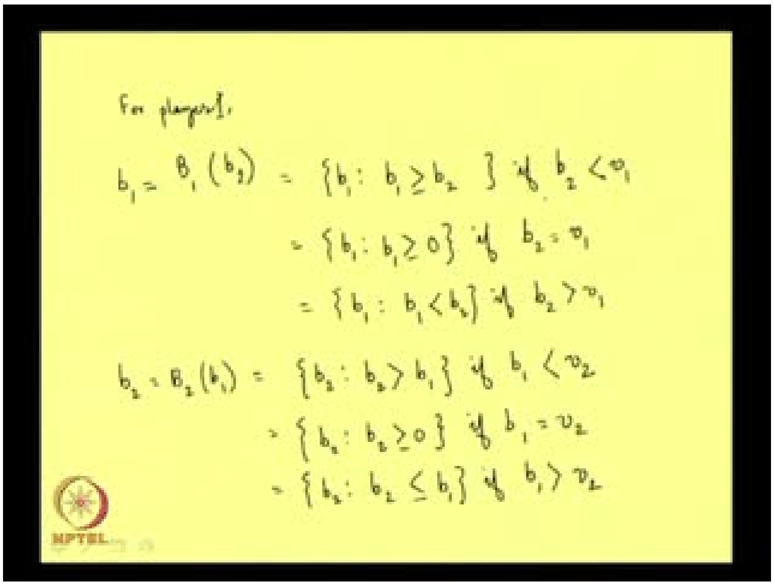

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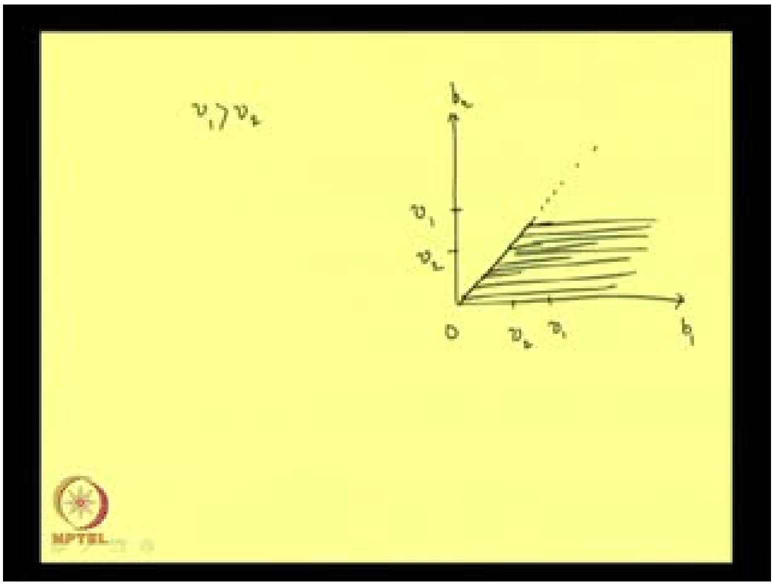

If b 2 is less than v 1 then player 1 will like to bid more than b 2, so I have this 45 degree line dotted, if b 2 is less than v 1 player 1 is bidding more than player 2 - more than and equal to also.

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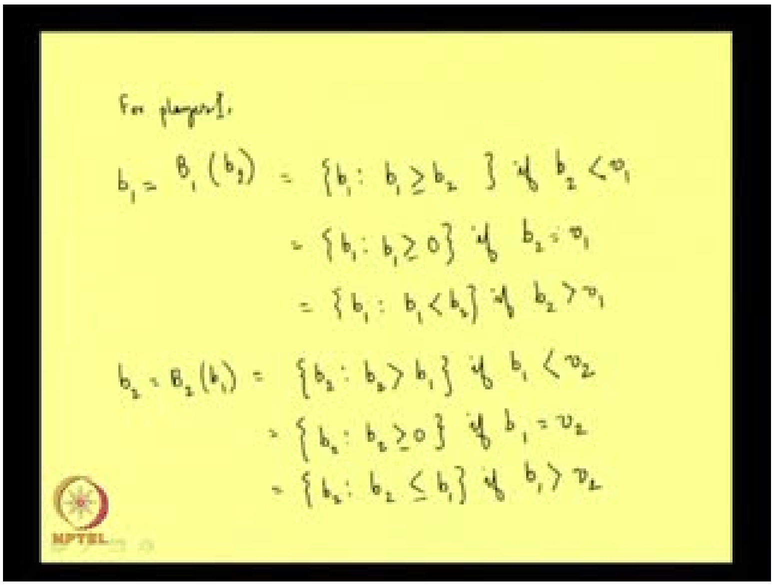

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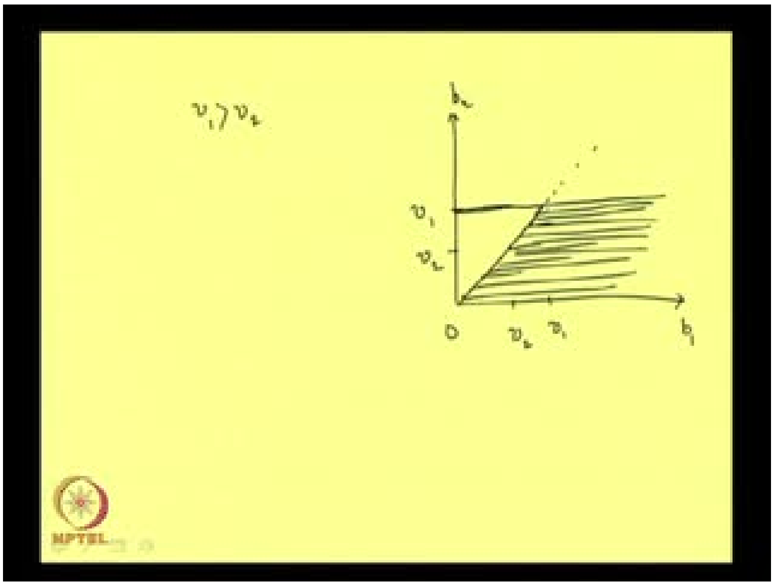

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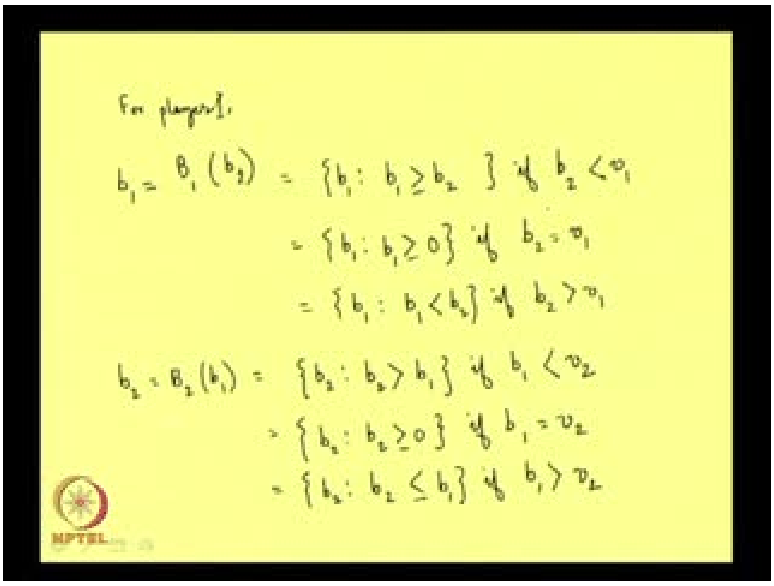

So, this line is included here, if b 2 is equal to v 1 any value; this is 45 degree line point, this horizontal line is included, if b 2 is greater than v 1 b 1 should be strictly less than v 2.

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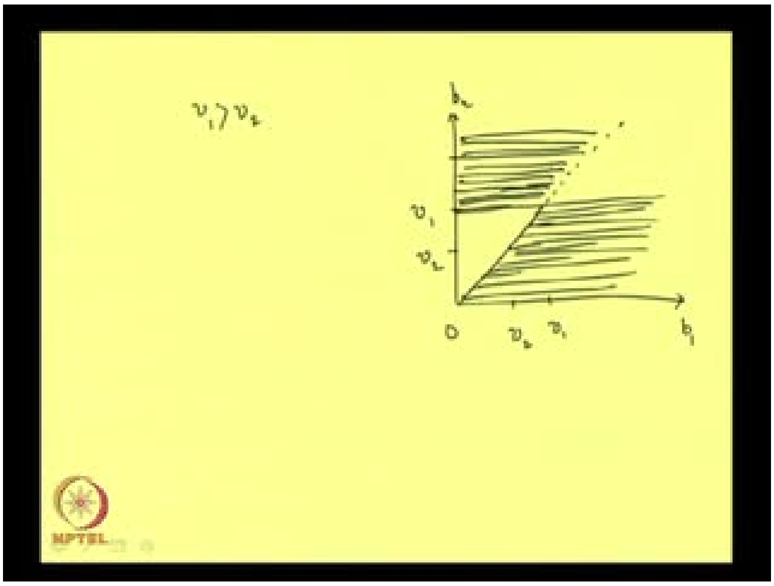

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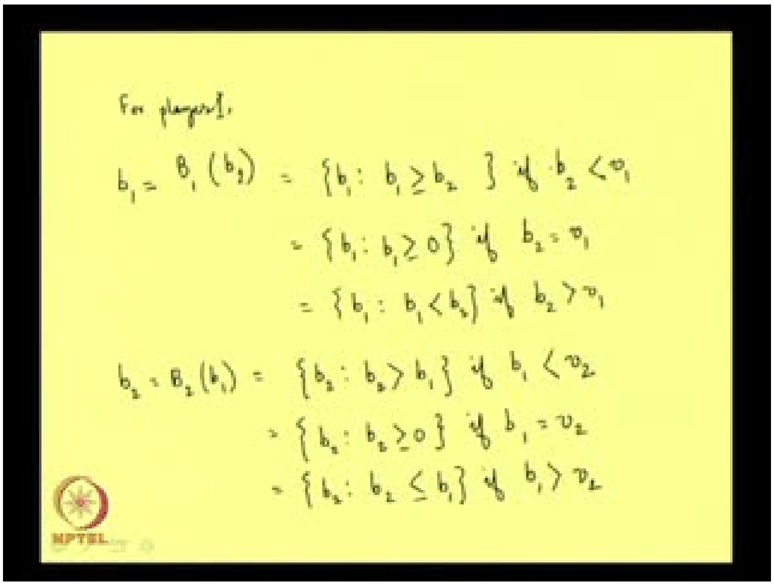

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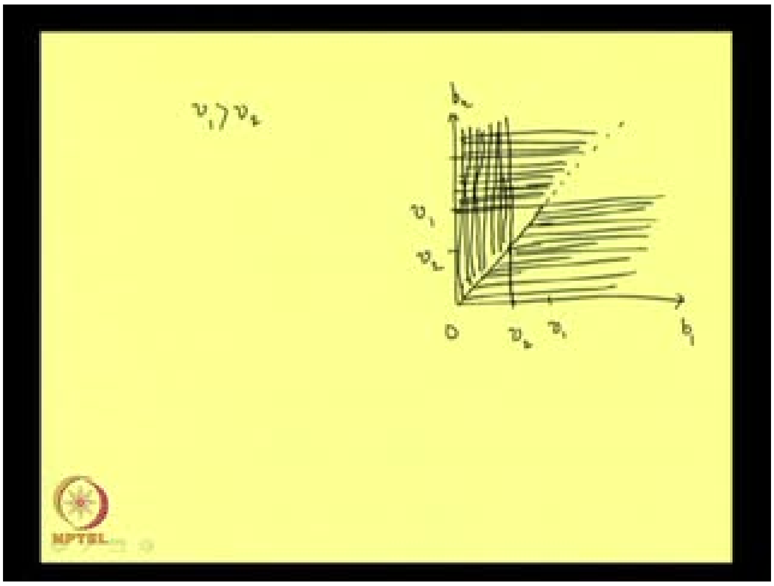

So, this 45 degree line is no longer included; this is the best response function of player 1; player 2's best response function is, if b 1 is less than v 2 then, b 2 is strictly greater than v 1. So, this is v 2 point I have this vertical at line here and all these points will be included, but the points on the 45 degree line will not be include.

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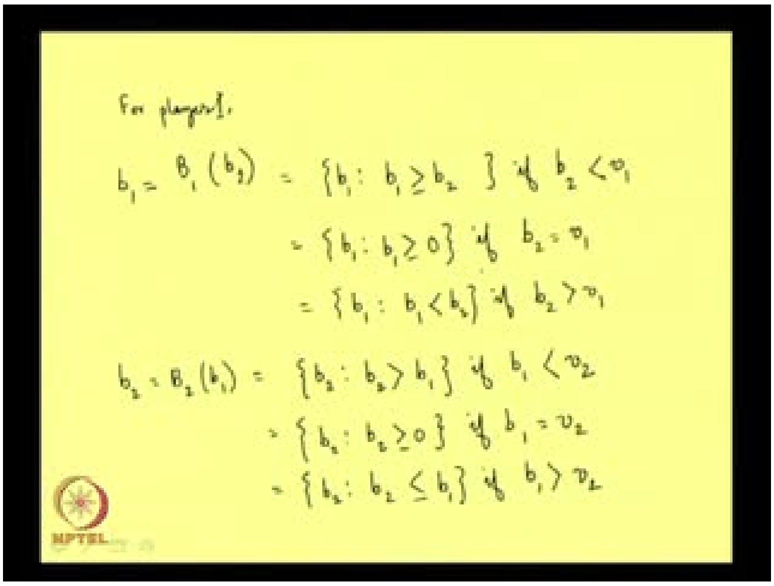

Secondly, if v 1 is equal to v 2; b 2 can be of any value - this is the vertical line - and if b 1 is greater than v 2; b 2 is less than equal to b 1, so this line is included.

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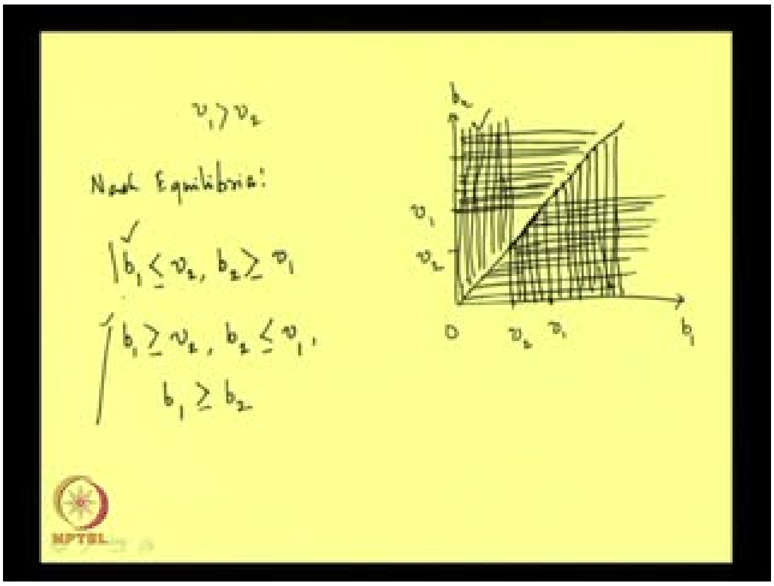

We have basically two sets of Nash equilibria here; first set, let us first describe this set which is on the north west, b 1 is less than equal to v 2 and b 2 is greater than equal to v 1. If I describe this set, b 1 is greater than equal to v 2, b 2 is less than equal to v 1 and b 1 is greater than equal to b 2. This line is basically important, I have to consider points only to the right of this line.

So, this is a set of Nash equilibrium and this is the set of Nash equilibrium. It is obvious why these are Nash equilibria a little bit of discussion is required here. Let us consider this set of Nash equilibria, here player 1 is bidding something less than or equal to v 2, player 2 is bidding either equal to v 1 or greater than v 1. So, player 2 will get the object in this case and player 2 will pay a price which is equal to b 1, so if b 1 is less than v 2 then player 2 will make some positive payoff. Can he deviate and be better off, if he deviates to something less than v 1 as long as this less than v 1 is greater than v 2, he is still getting the object and his payoff is remaining the same.

If he bids less than b 1 then he loses the object, so his payoff becomes 0. So, by deviating he is not being better off whatever player 1, player 1 is not getting the object. If he wants to get the object, he will have to bid more than b 2, but if he bids more than b 2, either he gets 0 payoffs when b 2 is equal to v 1 or he gets a negative payoff when b 2 is greater than v 1 and he outbids player 2.

So, by deviating he is not getting any positive payoff and therefore, this is Nash equilibrium. By the same logic other thing is also a Nash equilibrium, here player 1 is getting the object by bidding more than v 2 and player 2 is bidding something less than v 1, but at the same time b 1 is greater than b 2, so player 1 is getting the object. He is paying a price which is equal to b 2 and b 2 is less than equal to v 1, so it is either positive or 0 player 1's payoff, by deviating he can lose to player 2 that is the only thing he can do but if he loses to player 2 his payoff becomes 0, so which is not in any case better than what he is getting by the same kind of interior logic. We can show that player 2 also by deviating cannot be better off.

So, these are the complete set of Nash equilibria in case of two person, second price auction, let us do one more exercise and then, we shall talk about some other auction.

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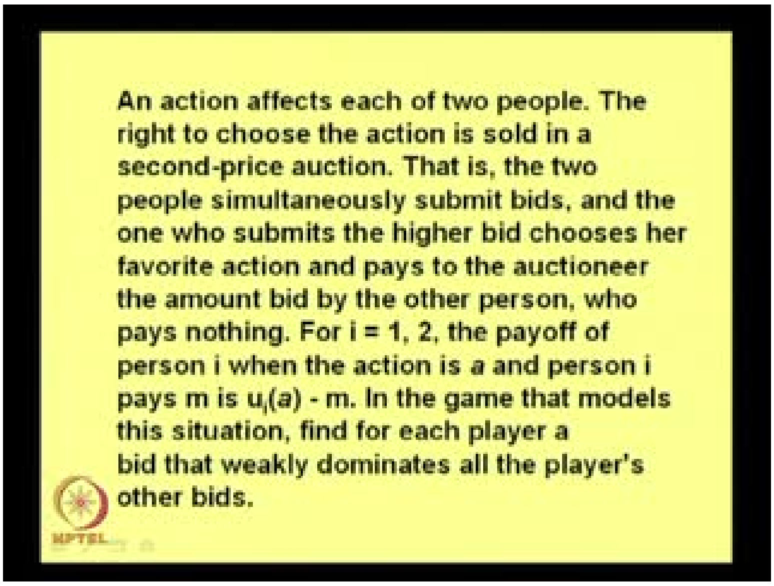

This is the exercise: an action affects each of two people - so it is a two person game. The right to choose the action is sold in a second-price auction, that is, two people simultaneously submit bids and the one who submits the higher bid chooses her favorite action and pays to the auctioneer the amount bid by the other person, who pays nothing. For i is equal to 1 or 2 the payoff of person i when the action is a and the person i pays m is u i a minus m. In the game that models this situation, find for each player the a bid that weakly dominates all the player's other bids.

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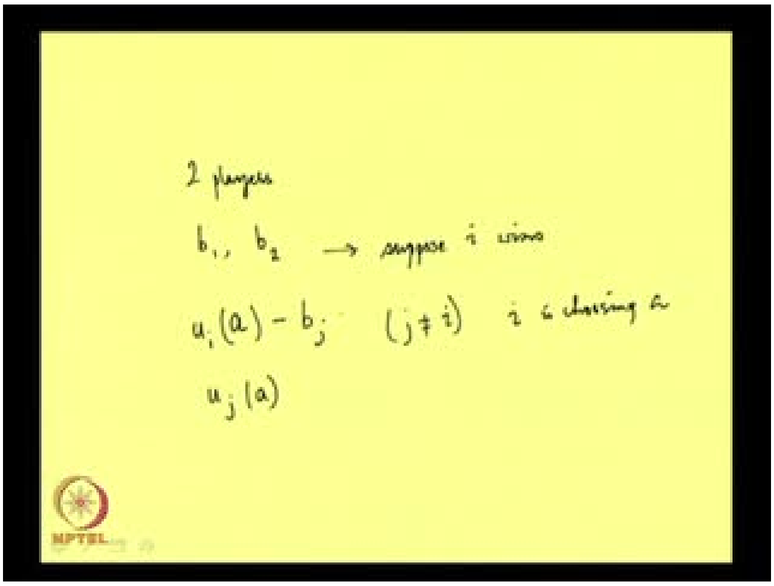

So, in short there are two players and they are bidding with each other, so let us call these bids as b 1 and b 2. Now, whoever is winning the second price auction he is getting to choose an action a and if he chooses the action a, suppose, i wins then the payoff for i is u i a minus b j - j is not equal to i. When i is choosing a, it effects the payoff of the other player also, the other player gets u j a, because this a is entering his utility, his payoff function also, though he is not paying anything in this case because he is losing the auction.

So, this is the setting and we have to find out an action for each player a bid that weakly dominates the players other bids. So, I do not have much time today, before we wrap up this lecture, let me introduce another kind of auctions that we shall be discussing in detail in the next lecture, which is called the First Price Sealed Bid Auction.

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Now, unlike the second price sealed bid auction, here the story is the following, the people are submitting their bids and the person who is submitting the highest bid is winning the good - obtaining the good - the price that he is paying here is not the second price, but the price that he himself is quoting. So, the price is equal to the bid that he himself has submitted. So, this is the case and this is also known as the Dutch Auction the other one that we have discussed before the second price is known as the English Auction.

This is called the Dutch auction because the sequential counter part of this auction; if this auction is conducted sequentially, it has a real life example and in Holland this use to be a common practice that the price that is announced by the auctioneer, it does not start from below, but it starts from above and then it comes down.

So, the auctioneer announces a very high price and asks the bidders whether is there anyone who is ready to pay that price, if he finds no one then he reduces the price a little bit, as soon as he gets any particular person who is ready to pay that price, that person gets the good and pays that price which he said that he will be ready to pay.

So, in this case you see the story that we had it is a competition between the 2 highest bidders and the highest bidder is going to pay a price which is very close to the price offered by the second highest bidder does not hold, here the highest bidder is himself paying the price which he is quoting. Second highest bidder is does not maatterial it does not matter in this particular outcome, so that is why it is called a Dutch auction. We are going to look into the other aspects of this Dutch auction the next lecture, thank you.

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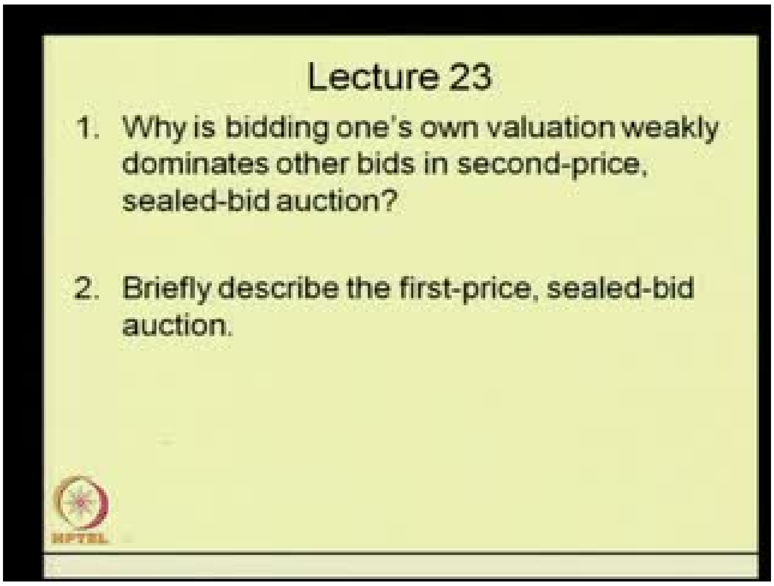

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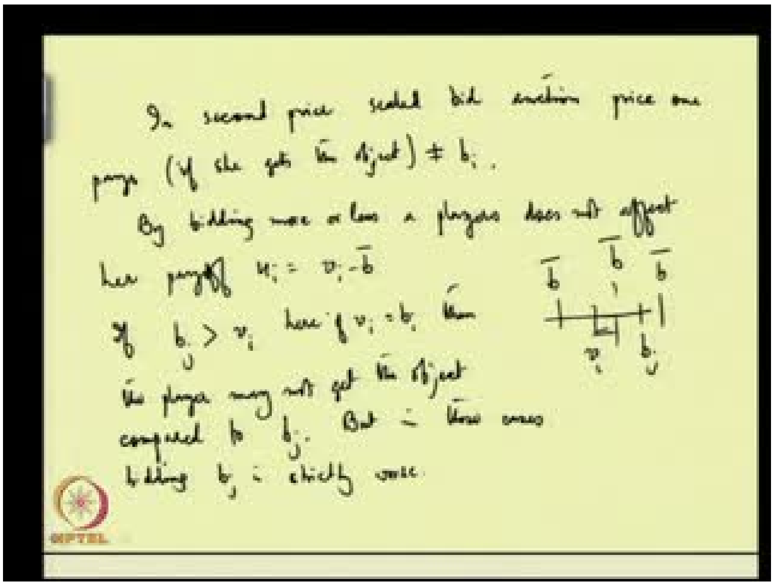

Why is bidding one‘s own valuation weakly dominates other bids in the second price sealed bid auction? In second price sealed bid auction price one pays if she gets the object is not equal to b i. So, because this is the price that I pay if I get the object is not my bidding, but the second highest price of the other players, so that is why this is called second price sealed bid auction. It might happen that the other price is less than what I have bid.

Now, if the price that I am paying is not my bid then bidding more or less I am not affecting my payoff, which is v i minus b bar, b bar is decided by someone else v i is given from outside, so u i is independent of the bid that I submit. However, by changing b i - changing my bid - I can change the chance of my getting the object. So, that is the only difference it makes to the payoff by bidding more or less. Now, there might be some cases, if i consider b j which is greater than v i; here if v i is equal to b i then, the player may not get the object compared to the case, compared to b j - so here is b j.

So, in some cases by bidding b j, I will get the object but by bidding v i I will not get the object, but in those cases bidding b i is strictly worse because these are the cases where this will happen. If b bar is between v i and b j in this cases, by bidding b j I will get the object, by bidding v i I will not get the object, but if I get the object I get a negative payoff of this much, so bidding v i is always better.

In all other cases suppose, b bar is here or b bar is here, it does not make a difference bidding v i, is same as bidding b j. Therefore, v i is weakly dominating b j, if b j is greater than v i; by a similar logic I can show that v i weakly dominates b j if b j is less than v i.

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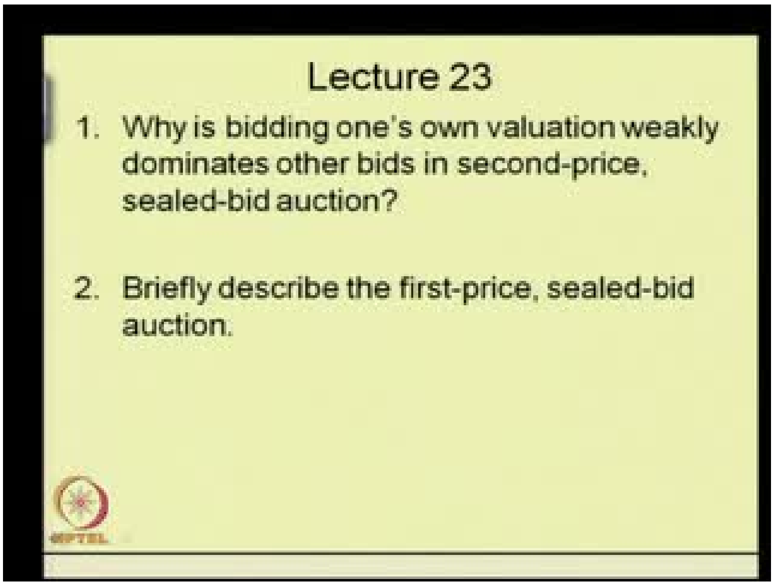

Briefly describe the first price sealed bid auction.

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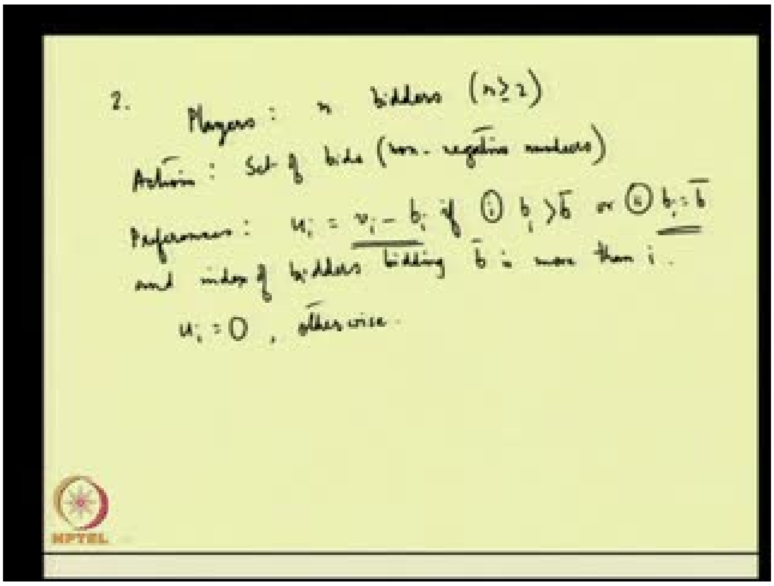

So, here player's n bidder's actions set of bids which are non-negative numbers and preferences is given by this.

So, in these two cases where if I bid more than the other players bid then, I get the object and the price that I pay is same as my bid which is b i and my payoff is therefore, v i minus b i or it might happen that I am bidding same as the highest bid of other players, but in that case also I can get the object if my index is less than the players who have bid b bar - in that case also I will get the object and get a payoff of this.

If these two conditions are not satisfied then I get 0, that is two conditions means, if my bid is not the highest bid or my bid is the highest bid, but there is another player whose bid is equal to my bid and whose index is less than my index in that case also I will lose the object, so I will get 0.