Game Theory and Economics
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Module No. # 03
Illustration of Nash Equilibrium
Lecture No. # 12
First Price Auction

Welcome to a lecture 25, of module 3, of the course called Game Theory and Economics. Before we start today's lecture, let me briefly recapitulate what we have been doing so far. What we have being doing is that we have been discussing the application of Nash equilibrium in auctions.

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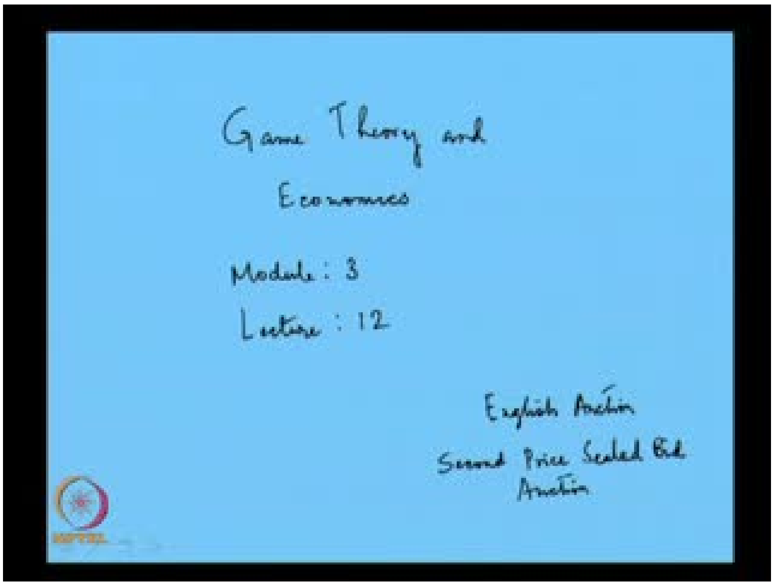

The first kind of auctions that have discussed so far is what is known as the English auction, also called the second price Sealed Bid auction, so that was what we have discussed before.

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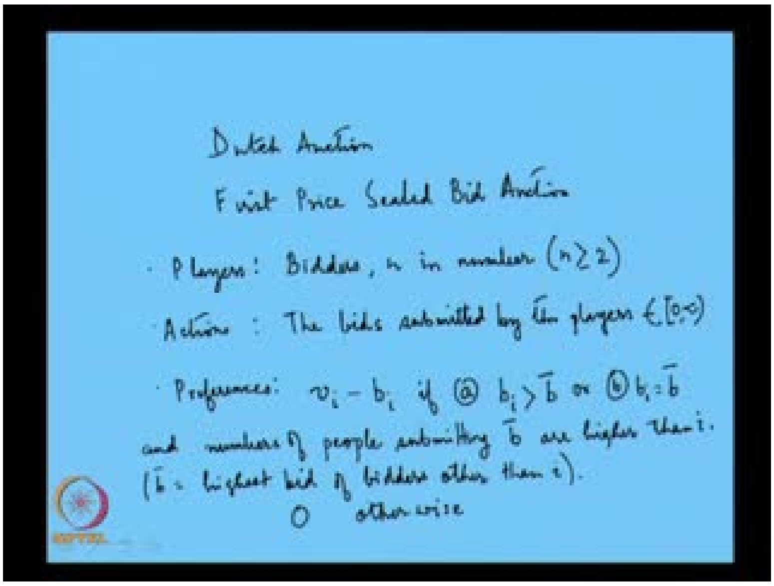

We have seen that there are infinite numbers of Nash equilibria in second price sealed bid auction. What we shall start discussing today is what is known as Dutch auction, so the difference of this auction with the hum second price sealed bid auction is that in this auction, the price that the winner gets of the auction is not the second price - second highest price, but the first price that is the price that he is quoting, it is going to be the price that is paying for the good.

Why is it called the Dutch auction? What is used to happen in Holland in many cases is that price has to come down from above. When it comes down from above, auction prices, whenever the auctioneer finds at least one person who is ready to pay that price, that person gets the goods by paying that price. Here, the second highest price does not matter for this auction. The person who is ready to get the good is going to pay the price, which is the highest price, not the second highest price and that is why it is called the first price sealed bid auction.

In terms of our language, in terms of game theory, it can be represented as the following. Finally, preferences of the players, here a difference will come compared to the second price auction. When a person gets the commodity, it is given by - its valuation for the commodity is v i minus b I, if a or b (Refer Slide Time: 05:06). So, if i is getting the commodity, then he is getting a valuation of v i, which is how he values the commodity.

From this, we have to subtract the price that he is paying which is b i, it is not the second highest price; it is the price that he is quoting. This will happen in two circumstances, one is if he is bidding that is b i is strictly greater than the highest bidding of the other players which is b bar, it can also happen if v i minus b i can also happen. If person i is bidding same as the highest bidding of other players, but person i's index that is, i is lower than those players who have submitted b bar and 0 otherwise.

So, if the bidding of this player is less than b bar or it is equal to b bar, but index of other players are less than i, then person i does not get the commodity and he gets a payoff of 0, so this is the setting.

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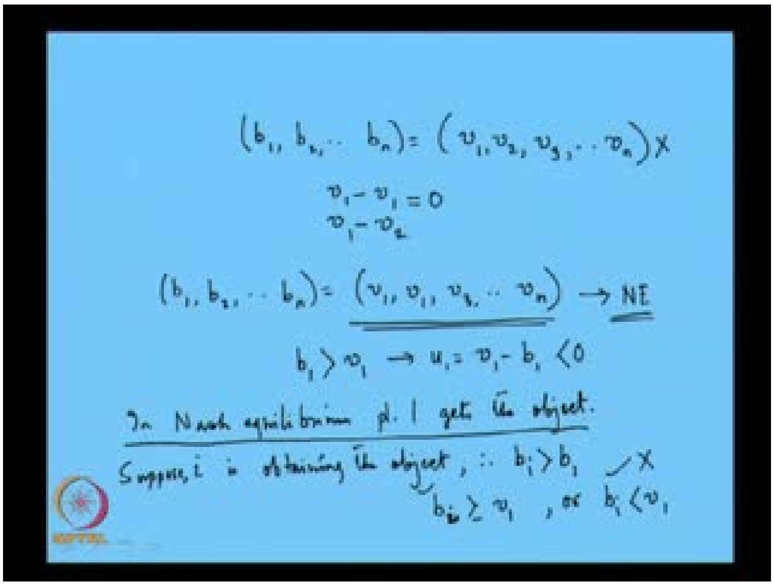

Now, let us look at what kind of Nash equilibrium that can be obtained from this setting; let us look at the following biddings. This was Nash equilibrium in the second price auction, but notice this cannot be Nash equilibrium here, because what will happen is that in this case, player 1 is getting the object by bidding v 1, his valuation. Every other player is bidding his own valuation, but player 1 can still get the object. If he reduces his bidding from v 1 to v 2, in that case, his payoff is going to go up. His payoff is going to go up to v 1 minus v 2, whereas presently he is getting v 1 minus v 1 is 0.

So, this is not Nash equilibrium in this setting, what can be Nash equilibrium? Let us take the following, this is a Nash equilibrium, what is happening is that player 1 and player 2 both are submitting bids which are equal to v 1, other bids are less than v 1, because we know that v 3, v 4, etcetera is less than v 1.

Player 1 here is also getting the object and he is getting a payoff of 0, if he deviates, if player 1 deviates and bids something more, suppose he bids b 1 which is greater than v 1, then payoff is going to be v 1 minus b 1 which is less than 0, whereas presently he is getting 0. By deviating upwards that is bidding something more, he is strictly worse off, if he bids something less, he will lose the object, he is not going to get the object. If he loses the object, his payoff becomes 0, so he is not better off. Therefore, this is Nash equilibrium, I have not specified the calculations of other players, but it will be Nash equilibrium for the following reason that right now other players are not getting any payoff, they were getting 0 payoffs.

Now, if they want to get the object, then they will have to outbid player 1, but if they outbid player 1, then the price that they will be coating will be more than v 1, which is more than their individual valuations. If this is more than your individual valuations, then you are making a negative payoff which means that by trying to get the object, they are going to get a negative payoff. Otherwise, if they do not get the object, their payoffs remains 0, whatever they bid, from other players point of view also, deviation is not profitable; hence, this is Nash equilibrium.

One interesting characteristic of this first price sealed bid auction is that in the Nash equilibrium player one will definitely get the object. Notice, this was not the case of second price sealed bid auction, there it could happen that other players are also getting the object. The reason is the following that player 1, if he does not get the object, it must be the case that someone else is getting the object, so let us call that someone to be i. If i is getting the object, then it must happen that b i is greater than b 1. Now, if b i is greater than b 1, then it can happen in two cases; b i is greater than equal to v 1, because I do not know exactly the value of v 1. So, there are two cases; one is b i is greater than equal to v 1, or b i is less than v 1, these are the only two possibilities.

Now, if b i is greater than equal to v 1, then the player i who is getting the object is making a negative payoff, because his valuation is v i which is less than v 1. So, he is getting the object, but his valuation is minus of b i minus v i which is a negative quantity.

He can do better by setting b i is equal to v i in which case he will get 0, so this is not Nash equilibrium, the other possibility is that b i is less than v 1.

Now, can this be Nash equilibrium? If b i is less than v 1, this is not Nash equilibrium for the following reasons; in this case, player 1 can always outbid player i, because his valuation v i is v i. So, he can always effort to bid something which is more than b i, but less than v 1. In fact, he can bid equal to b i and get the object, because the time breaking rule is that player 1 gets the object if he diverse with someone else.

So, again this cannot be a Nash equilibrium, because in this case, b 1 will be equal to v i, or more than v i which means that player 1 will get the object, so player i cannot get the object in this case also.

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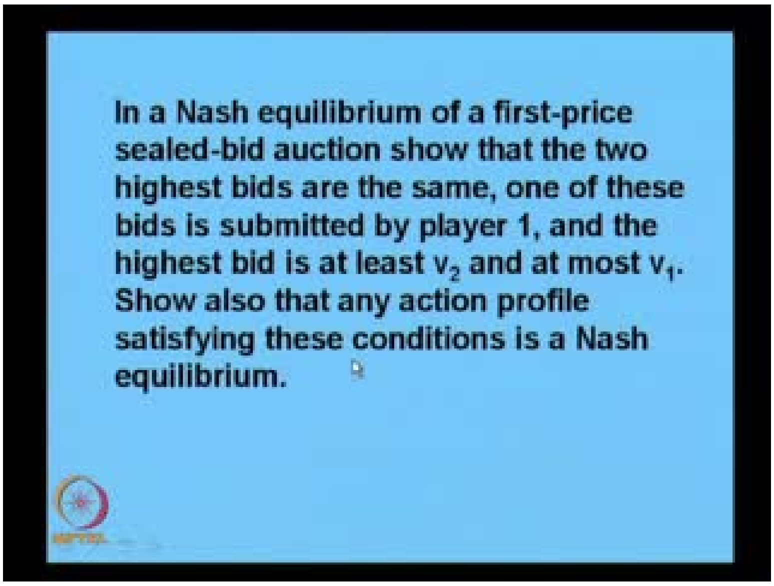

So, we are left with a possibility that in Nash equilibrium, player 1 always gets the object. There is another interesting property of Nash equilibrium in first price auction, which will be represented in terms of an exercise is the following. This is the exercise: In a Nash equilibrium of a first price sealed bid auction show that the two highest bids are the same, one of these bids is submitted by player 1, the highest bid is at least v 2 and at most v 1. Show also that any auction profile satisfying these conditions is Nash equilibrium.

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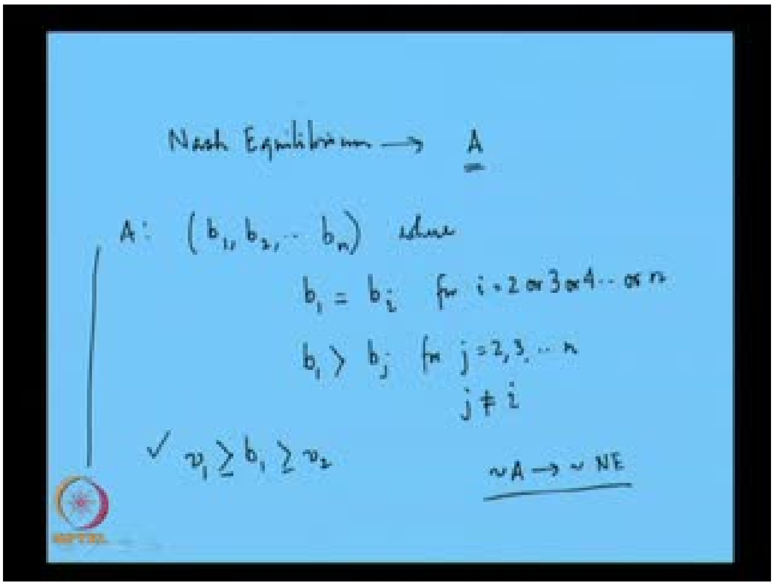

So, what we are required to show is the following. What is being said is that if you have Nash equilibrium, this Nash equilibrium must satisfy some characteristics. Let us call these characteristics as A; what are these characteristics? How is A described? A will be specifying an action profile, where the bids are suppose b 1, b 2, b n. Where b 1 is equal to b i for i is equal 2 or 3 or 4 etcetera or n. These two that is b 1 is greater than b j for j, 2 j is not equal to i. So, this is what is being said. Another part of this A is that b 1 is greater than equal to v 2, but less than equal to v 1, so this is how A is described, A is describing some property of a particular kind of profiles.

In these profiles, player 1 is making the highest bid and his bid is matching with the bid of another one person, just one person. We can in fact say that it can be possible more than one person is bidding same as player 1, so b 1 is equal to b i, where i can be 2 or 3 etcetera or n. Bids of other players, other players means player 1, besides player 1 and player i, the bids of other players we call it b j, those bids will be less than b 1. Finally, we have described important characteristic that the highest bid that b 1 must be lying between two bounds. Its lower bound is given by v 2 and the upper bound is given by v 1. So, we have to show that if I have Nash equilibrium, then that Nash equilibrium must satisfy this characteristic A.

Now, if I have to show that Nash equilibrium satisfies A, then I can show the same thing by showing that not A implies not Nash equilibrium. That is, if I do not have satisfaction of A, then I do not have Nash equilibrium that will guarantee me that Nash equilibrium is implying A. This comes from the fundamental mathematical logic, suppose A is sufficient for b, then b is necessary for A. So, we have to show that if A is not satisfying some of these three characteristics, are not satisfied and then we will not have Nash equilibrium.

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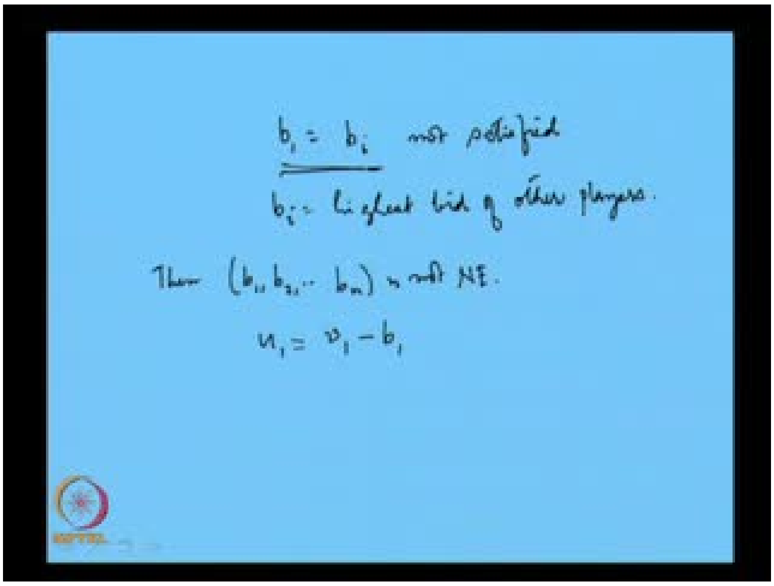

Suppose, this is not satisfied, the bidding of player 1 is not equal to the highest bid of other players, if this is not satisfied, if this is not satisfied, then obviously this is not Nash equilibrium. This is for the reason that in that case, player 1 will lower his bid and bid equal to b 1, because his payoff is given by v 1 minus b 1; as b 1 declines, his payoff rises.

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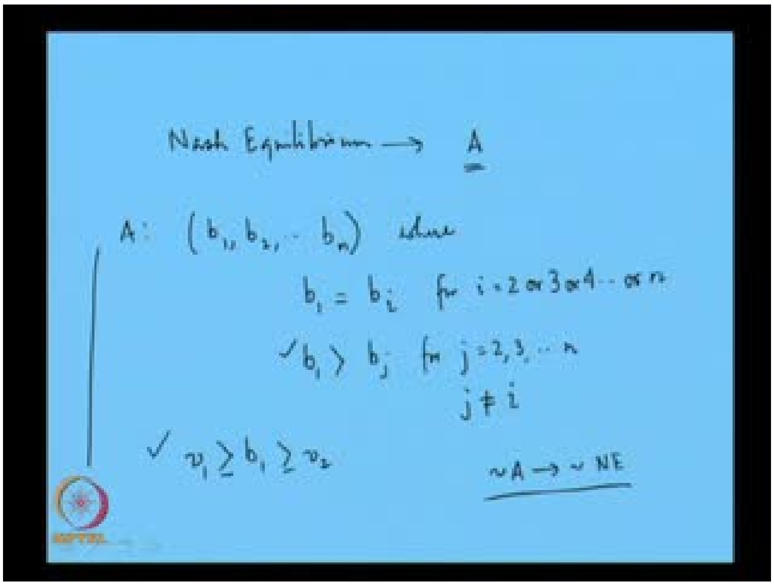

So, he will always like to bid as little as possible, so this is u 1, hence if this is not satisfied, we do not have Nash equilibrium. Player 1 will have profitable deviation, what is the other characteristic that must be satisfied? This highest payoff must be greater than the other bids.

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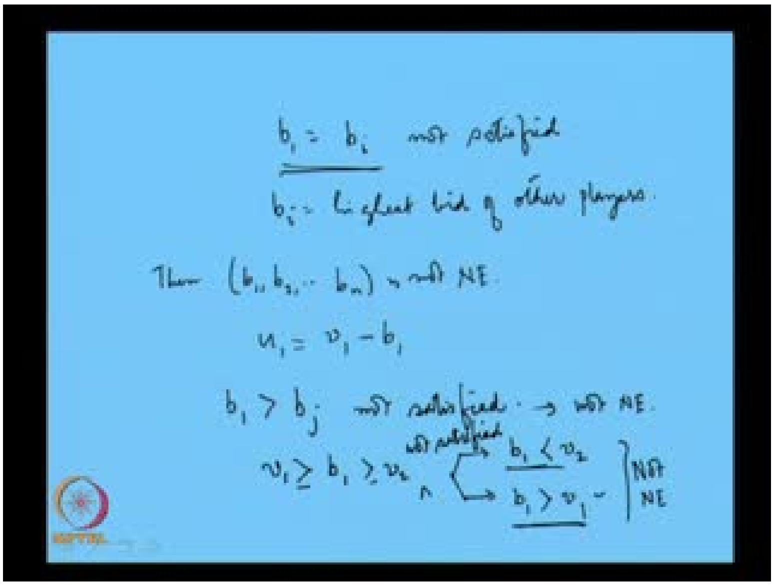

This is very easy, we have seen this before and we have proved this before. If some other player is bidding more than b 1, then that player will get the object, which we have just proved that cannot happen. In Nash equilibrium, player 1 must get the object, so no further elaboration is required for this point.

The third point is this, if this is not satisfied, then what happens? Which means that there are two possibilities, b 1 is strictly less than v 2. If b 1 less than v 2, then that cannot be Nash equilibrium, because other players, for example player 2 - player 2 in that case outbid player 1 and get the object. As long as his bid is less than v 2, he will make a positive payoff which he is not getting here. Because, I know that in Nash equilibrium, so far the Nash equilibrium that we have considered, player 1 is getting the object, so player 2 payoffs is 0.

So, he can better his payoff by getting more than player 1's bidding which is b 1. He gets the object and makes a positive payoff, so that player 2 will have a profitable deviation in this case. If b 1 is greater than v 1, is that Nash equilibrium? Again the answer is no, because here player 1 is getting the object, but he is making a negative payoff, so that is not good for him. He can at least bid to b 1 is equal to v 1 and that will give him 0 payoff, again from here also there is profitable deviation for player 1.

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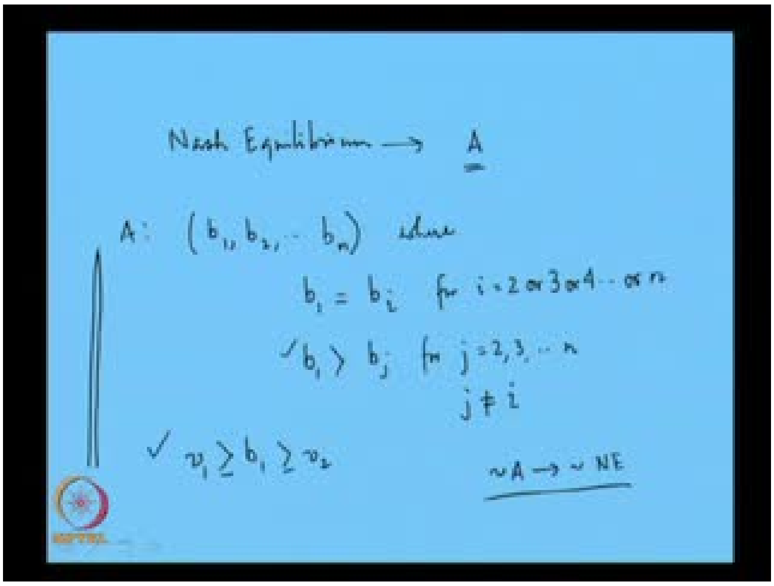

So, we have basically shown that if these conditions of A are not satisfied, then we do not get Nash equilibrium which means that Nash equilibrium implies A, the other part is this. It is written in the question itself that any profile satisfying A is Nash equilibrium.

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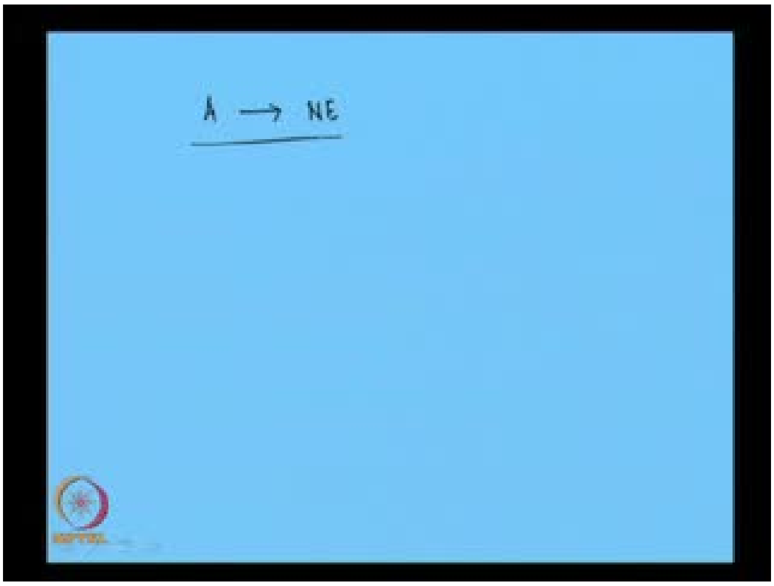

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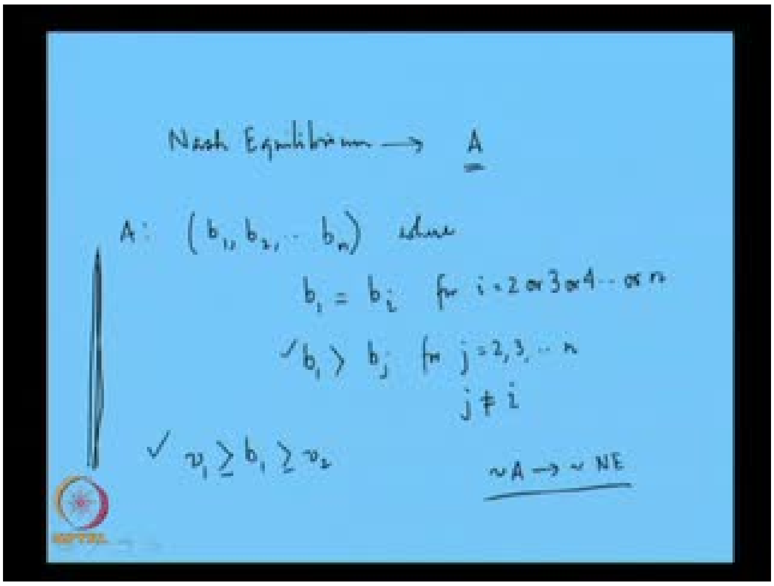

We have to show that A implies Nash equilibrium; this is not very difficult to show. So, any profile satisfying these characteristics is Nash equilibrium. This is true because, look at that possible deviation of any player, in this case, player 1 is getting the object, either he is getting a 0 payoff or he is getting a positive payoff. If he is getting a 0 payoff which means that b 1 is equal to v 1. From this, if he bids more, then he is going to get the object, but his payoff is going to turn into negative. So that is not good for him, because right now he is getting 0 payoff from 0 to negative, is bad.

If he bids less than, he is not going to get the object, which means that his payoff remains 0. So, deviation for player 1 is not profitable, this is the case where b 1 is equal to v 1. If b 1 is less than v 1, the player 1 is making positive payoff. If he raises his bid, he is still getting the object, but his payoff is going to go down, because the payoff is given by v 1 minus b 1.

If he reduces the bid, he is going to lose the object, so payoff becomes 0 which is worse than a positive payoff. So, deviation by player 1 is not profitable, for the other players, any player, it may be 2 or 3 any player, if they want to change the outcome by bidding more than player 1, they are going get the object. But, as we have seen before, the payoff is going to turn to negative, right now they are getting 0 payoff, so that is not good.

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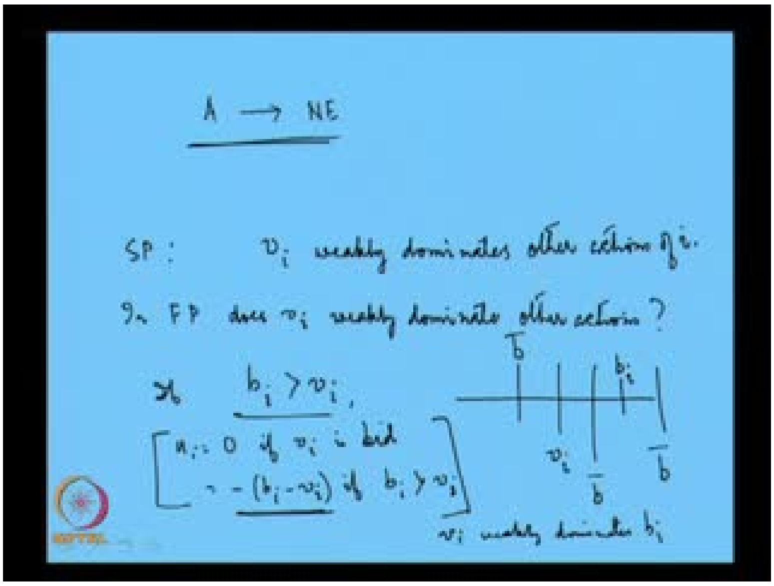

If they bid something less, then they are still not getting the object, so outcome does not change, so payoff remains 0, hence deviation by other players is not profitable. Any profile which is satisfying this set of characteristics is going to be Nash equilibrium. Therefore, we have proved this that A implies NE, so unlike what is the moral of the story? Moral of the story is that unlike second price auction, first price auction has - the equilibrium in first price auction has some definite characteristics. These characteristics basically bound the equilibria into some limits. They are not as wide as in the case of second price auction, they cannot be anything, they have to satisfy some characteristics. Now, in second price auction there are some other characteristics, for example, v i weakly dominate other actions of i, so this was second price. Now, one may wonder if these properties hold from first price auction also.

So, is it the case in first price auction? Does v i weakly dominate other actions? It turns out that some of the actions are weakly dominated by v i. So, suppose this is v i and I am considering b i which is greater than v i, then one can see that v i weakly dominates any action which is greater than v i. This is because it depends on where b bar is; if b bar is here, then bidding v i gives you 0 payoffs, and bidding b i gives you negative payoff.

So, u i is equal to 0, if v i is bid is equal to negative of (Refer Slide Time: 31:01). So, in this case, if b bar is less than v i, then v i is the better action than b i. If b bar lies here, then by bidding b i you are getting the object, but your payoff remains at this value which is negative, whereas by bidding b i, you are getting the object, so your payoff is 0.

If b bar is here, greater than b i, then both the bids that v i and b i give you 0 payoff, because you are not getting the object, so the player is indifferent. So, we see that considering all possible cases, we see that bidding v i is either better or it gives you the same payoff as bidding b i.

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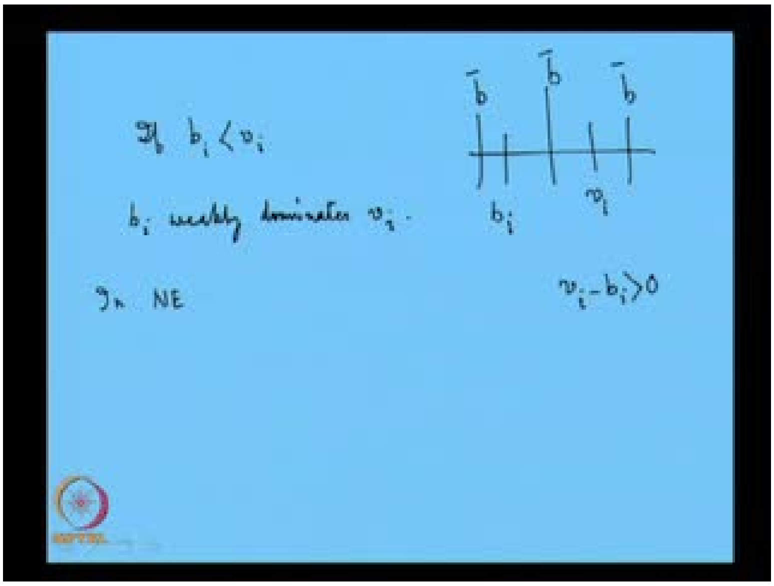

Therefore, v i weakly dominates b i, but is that true for any action which is less than v i? The answer is no. So, if b i is less than v i, in this case, what is important to see is that by bidding v i, you will always get 0 payoff in case of first price auction. Because, if you get the object, your payoff is v i minus b i and b i is equal to v i, so your payoff is 0. If you do not get the object, obviously your payoff is 0, so by bidding v I, you are always going to get 0, but bid by bidding b i can you do better? The answer is yes. Suppose, b bar is here, less than b i, then by bidding b i you are going to get v i minus b i, which is positive. So, there are cases where b i is better than v i. If b bar is here, then by bidding b i you are not getting the object, so your payoff is 0, but we have seen already that by bidding v i also the payoff is always 0, same logic holds if b bar is greater than v i; in this case, in both b i and v i the payoff is 0.

In some cases, bidding b i is better than bidding v i, in the other cases, they are same as far as payoff is concerned. So, if b i is less than v i, b i weakly dominates v i. So, any action which is less than the valuation of the person is a weekly dominating action, compared to the valuation of that person.

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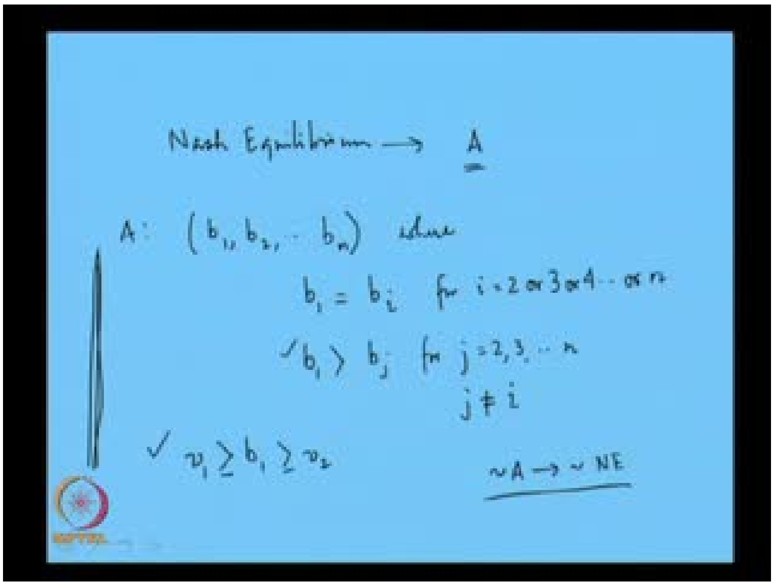

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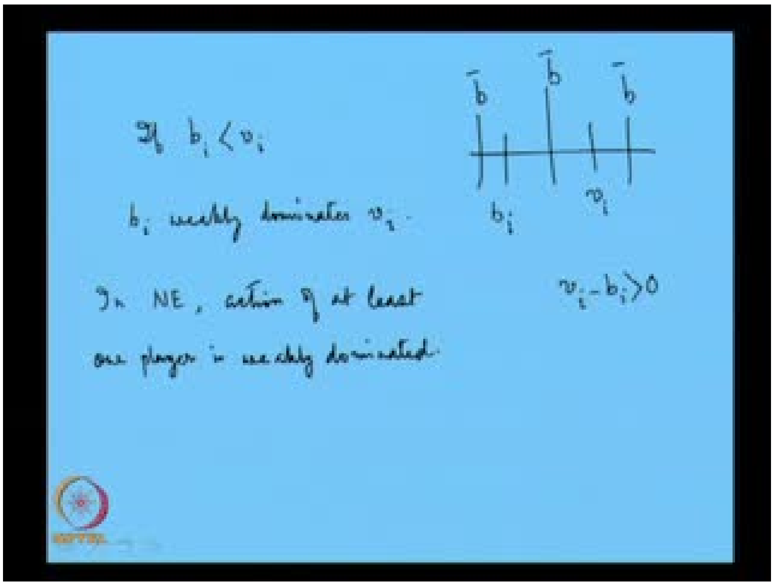

This implies, if I put these two things together that any action which is greater than the valuation is weekly dominated, any action which is equal to the valuation is also weakly dominated. We have seen this here; it means that in Nash equilibrium which satisfies these characteristics; remember this characteristic, which means in Nash equilibrium, action of at least one player is weekly dominated.

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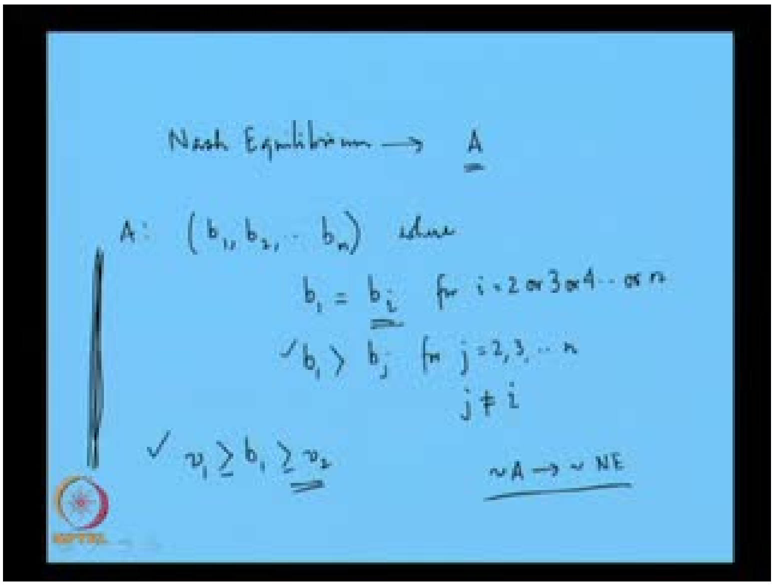

One is basically talking about this action, I know that there is at least one player who is taking the action b i, I know that b i is going to be greater than or equal to v 2. Even if it is equal to v 2, if i is equal to 2 that is player 2 is taking this action, even then we have seen that v 2 for player 2 is a weekly dominated action, which means in that case also one action is there in the equilibrium profile which is weekly dominated.

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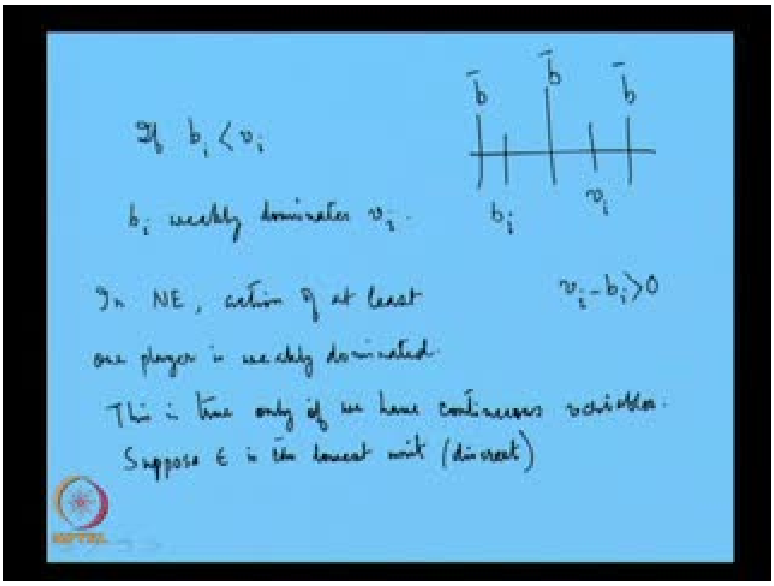

Obviously, if b i is greater than v 2, then that is going to be weakly dominated. Now, this is true, but this is true if we consider only the cases where we have continuous variables. If we do not have continuous variable, if that variables are discrete which means there are some units of the variables below which we cannot divide the variables. In that case, however we can have equilibrium in which nobody’s action is weakly dominated.

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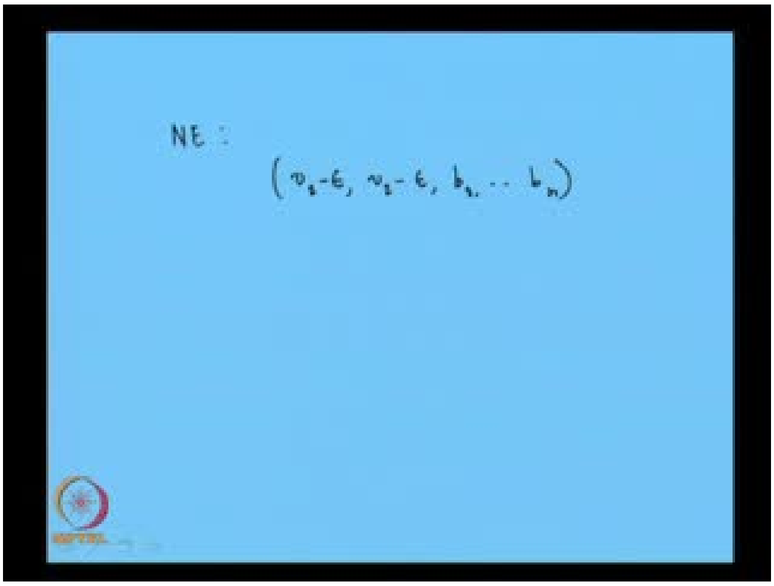

I can give you the example in the following sense, suppose, epsilon is the lowest unit - discrete unit, in that case, what is Nash equilibrium? What is one example of Nash equilibrium? I can take the following Nash equilibrium.

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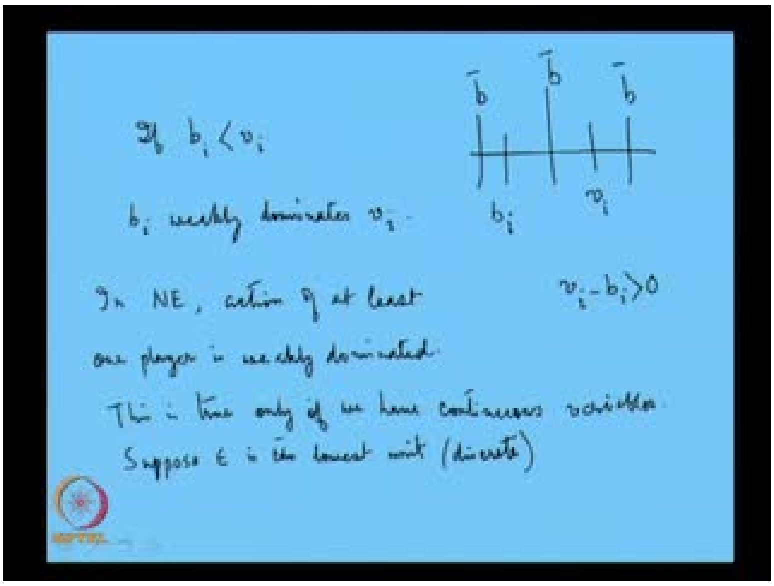

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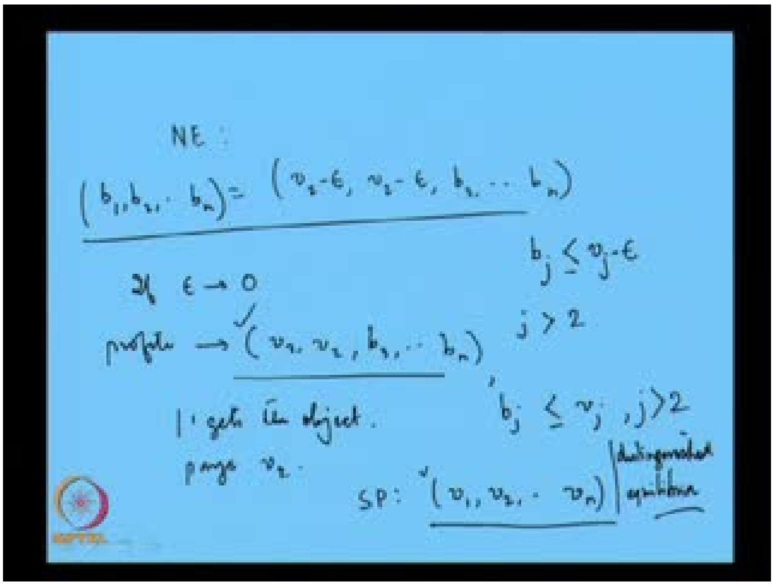

Here, epsilon is the smallest unit in which these valuations can be divided into - by the way this is wrong spelling. Here, player 1 is bidding v 2 minus epsilon, player 2 is also bidding v 2 minus epsilon and other players are bidding b 3, b 4, and etcetera, where b j supposes is less than equal to v j minus epsilon for j greater than 2.

For other players, they are bidding something either equal to v j minus epsilon or less than that. Now, I am claiming that this is a Nash equilibrium, why this is a Nash equilibrium? Because, here player 1 is getting the object as we already know that must happen in Nash equilibrium, but player 2 is bidding same as player 1, if player 2 bids a little bit more, then he will get the object; if he bids v 2, he will get the object.

But, if he bids v 2 his payoff remains at v 2 minus v 2, which is equal to 0. Right now, also he is getting a 0 payoff by not getting the object. So, by deviating player 2 is not better off. If he deviates to some less value, then obviously he is not getting the object still and so his payoff remains at 0. A player 1 will obviously not deviate, he is getting positive payoff. If she bids more, he still gets the object and the payoff goes down. If he bids less his payoff becomes 0.

Other players if they change the outcome, then they get negative payoff. If they do not change the outcome, their payoff remains 0. So, this is Nash equilibrium, but look what is an interesting in this Nash equilibrium is that everyone is bidding less than his valuation for the object. If everyone is bidding less than the valuation, then the actions are not weakly dominated. Because, we have seen that those actions which are equal to or more than the valuations are weakly dominated not actions are less than the valuation.

This is a profile where nobody's action is weakly dominated, if epsilon goes to 0, then this profile approaches v 2, it approaches this. In this equilibrium, player 1 still gets the object pays v 2; this equilibrium is similar to other equilibrium in the second price auction. This in the sense that in this profile also, in second price auction, player 1 is getting the object, he is paying a price is equal to v 2. As far as the amount of money which is collected from the auction is concerned, these two profiles, this is in first price and this is in the second price are known as revenue equivalent. They both of them are giving the same revenue to the auctioneer.

Other, similarity between these two profiles, Nash equilibrium profiles is that in both these profiles, action of no 1 is weakly dominated. So, these two profiles are called distinguished equilibrium.

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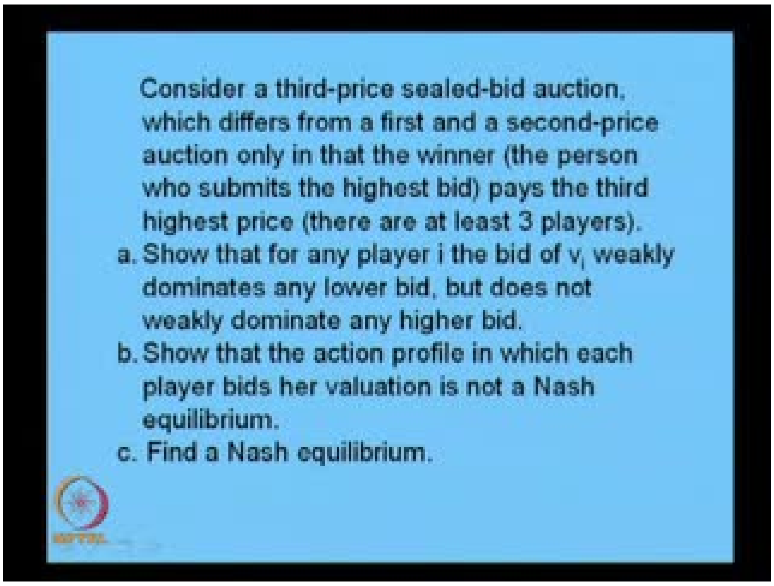

Let us do one exercise which extends our first price and second price into what is known as third price auction, consider a third-price sealed bid auction which differs from a first and second price auctions. Only in that the winner that is the person, who submits the highest bid pays the third highest price, there are at least three players.

Now, the thing is that we have to answer the following. One is, show that for any player i, bid v i weakly dominates any lower bid, but does not weakly dominate any higher bid. Secondly, show that the action profile in which each player bids a valuation, is not Nash equilibrium, thirdly find Nash equilibrium.

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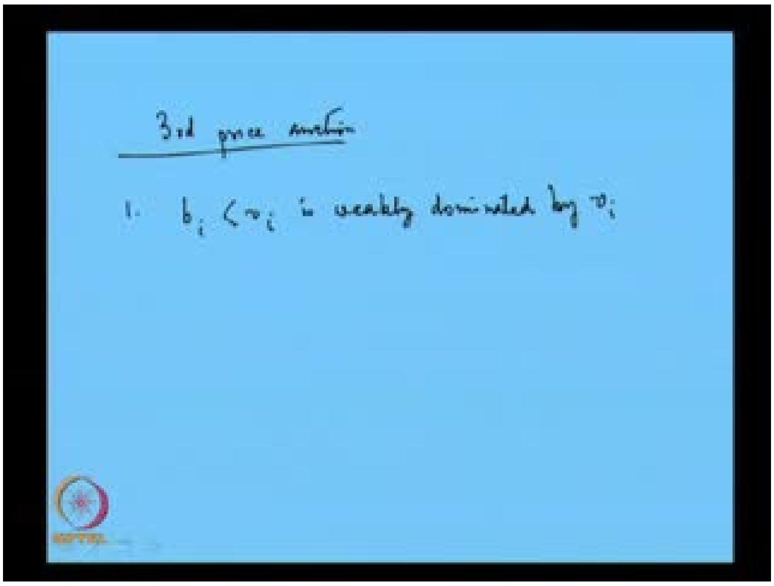

So, the questions that we have to answer; firstly, b i less than v i, is weakly dominated by v i, this we have to prove. So, this is more or less - this is the end of the lecture for today. Before we wrap up, what are the things that we have discussed?

We have looked to the various aspects of the first price sealed bid auction. We have seen that first price sealed bid auction satisfies many interesting properties; in equilibrium, player 1 always gets the object, in equilibrium 2, bids must be same, one of them is player 1's. Also we have seen that this Nash equilibrium, if it a continuous variables, will have at least one action which is weakly dominated. If we do not take continuous variables, if it is discrete variables, it is possible to find Nash equilibrium, in which the actions are not weekly dominated; thank you.

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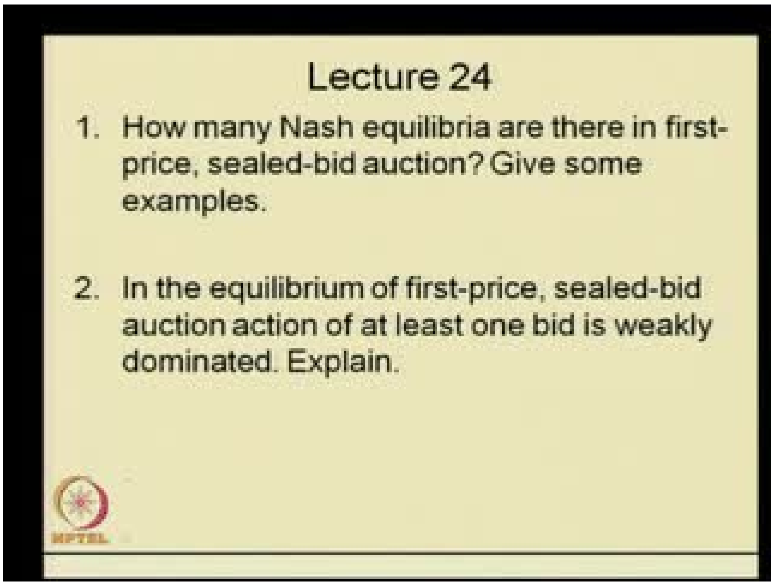

How many Nash equilibria are there in first price sealed bid auction, give some examples?

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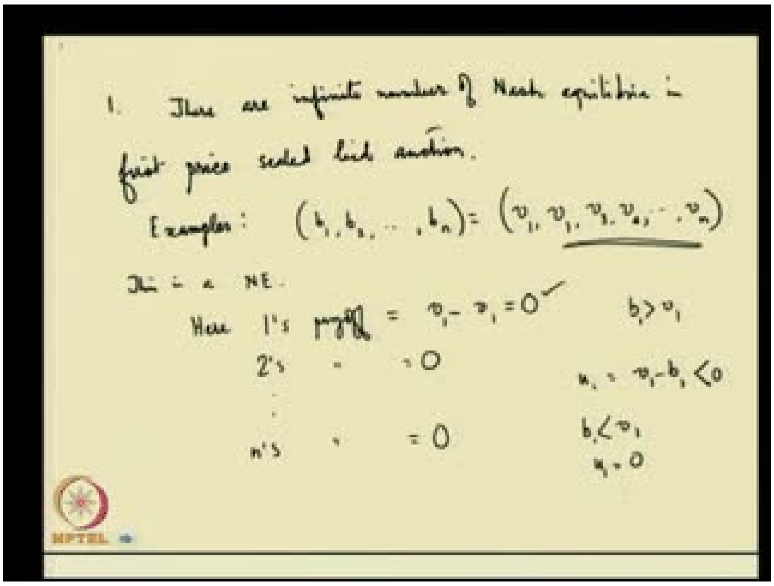

So, the answer to the question is that there are infinite numbers of Nash equilibria in first price sealed bid auction. We were supposed to give some examples - examples of such Nash equilibrium. For example, suppose b 1, b 2 and b n, these are the bids, suppose player 1 is bidding same as his valuation v 1, player 2 is bidding the same thing and the other players are bidding their own valuation.

Now, my claim is that this is a Nash equilibrium, the reason is that we have to check whether players can deviate and be better off. Here, one's payoff in equilibrium is given by v 1 minus v 1, which is equal to 0, so he is not getting anything, though he is getting the object.

Similarly, two's payoff is not getting the object; other players, none of them are getting the object, so their payoff is 0. The question is can someone deviate and earn a positive payoff, the answer is no. For example, player 1 can deviate and bid something more than v 1, but remember if I bid more than v 1, I will definitely get the object, but my payoff will it turn out to be base which is negative.

So, it is not good in my interest to bid more than v 1. If I bid less than v 1, in that case I am not getting the object, player 2 will get the object. If player 2 gets the object, she will get something, I do not know but I am not getting the object, but I am not better off either.

So, here this is 0 which was the same thing that I was getting earlier in that equilibrium. So, this is equilibrium indeed for player 1, at least player 1 cannot deviate and be better off. Similarly, for other players also, if they have to get the object, they have to bid more than v 1 and if they bid more than v 1, then their payoff become negative. So, from Nernst point of view, this with any profitable deviation is possible.

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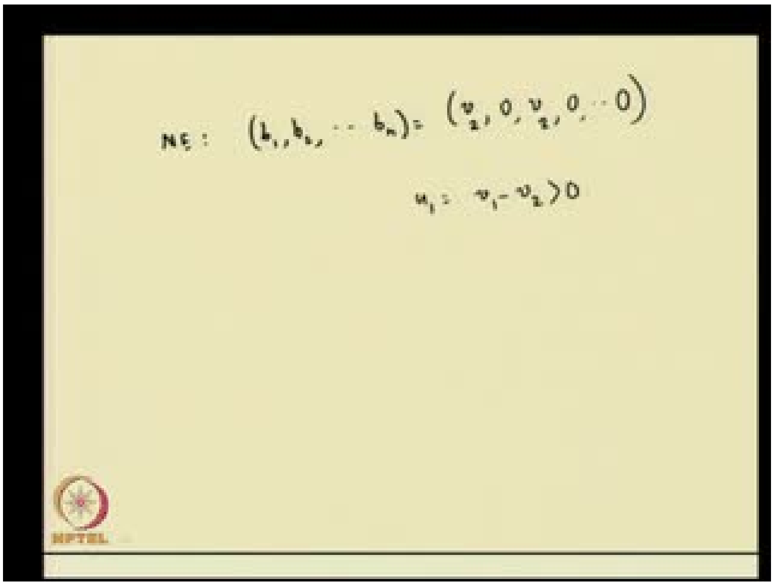

Another example we can give is the following, take this, let us say this is changed, we get more interested, because otherwise the payoff of the first player will be same as the payoff of first player within the previous equilibrium that we sighted. Here, what is happening is that player 1 will again get the object and his payoff, will be v 1 minus v 2, which is positive. So, he is now getting a positive payoff, can he be better off, he can bid more than v 1, but then his payoff falls.

If he bids something less than v 2, he loses the object which is strictly worse, so deviation by player 1 is not profitable. For player 2, if he has to get the object, he has to bid more than v 2, which means that his payoff will turn out to be negative which is worse than getting 0. For the other case also same logic applies, so this is the equilibrium.

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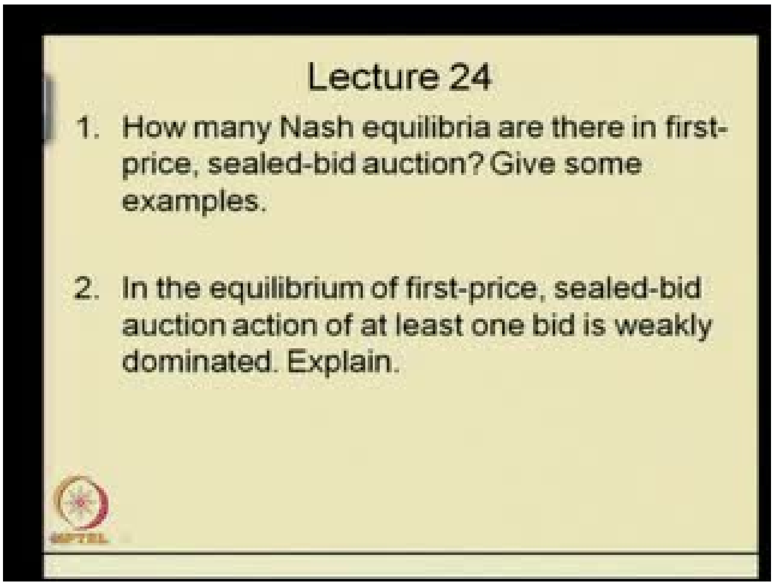

Second question, in the equilibrium of the first price sealed bid auction, auction of at least one bid is weakly dominated, explain?

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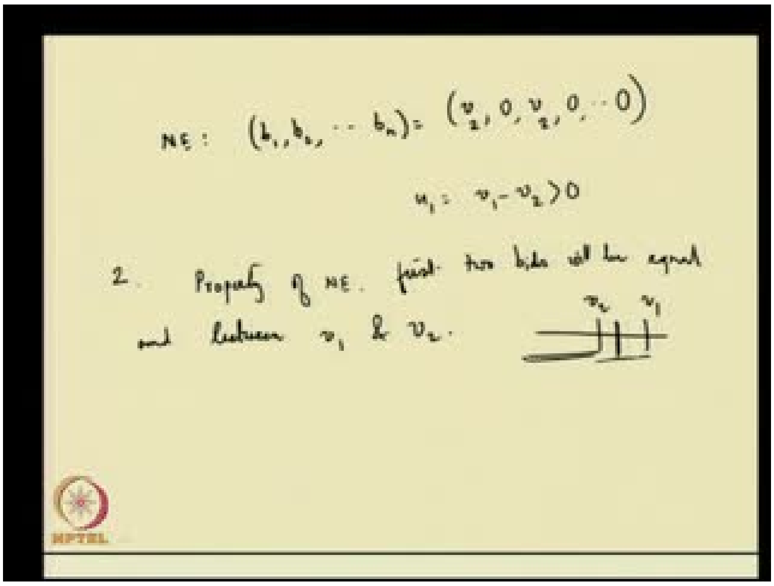

Now, interesting property of Nash equilibrium in the first price auction is that the first two highest bids will be equal between v 1 and v 2. So, we have here v 1 and here v 2, so the first two highest bids must be somewhere here. Now, we have another property of this first price auction that is, bidding more than 1’s valuation of bidding equal to 1’s valuation is weakly dominated.

So, if we have such equilibrium here, then at least one player who is coming from this side that is, player 2, 3, etcetera, n, at least one of them is bidding more than his or her valuation in this range. That action we know is weakly dominated, therefore, in each equilibrium, in first price auction, at least one player’s action is weakly dominated, so that is the meaning of this statement; thank you.