Game Theory and Economics

Prof. Dr. Debarshi Das
Department of Humanities and Social Sciences
Indian Institute of Technology, Guwahati

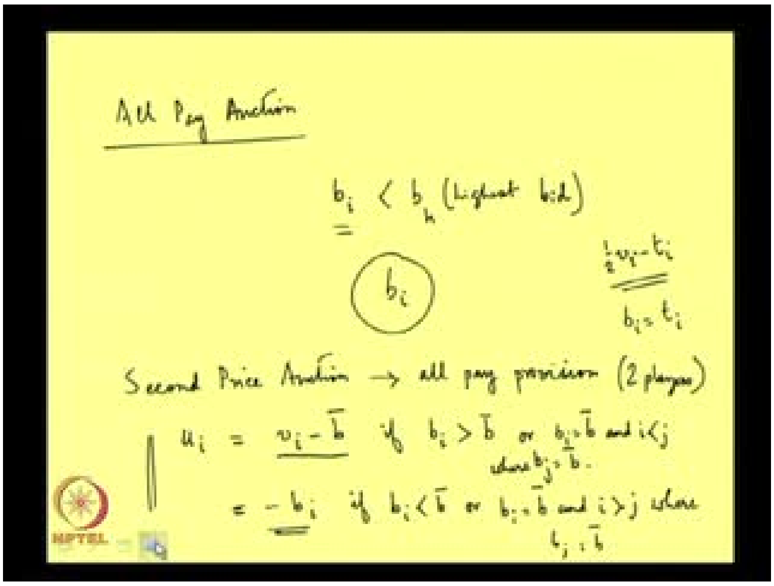
Module No. # 03
Illustrations of Nash Equilibrium
Lecture No. # 13
All Pay Auction, Multi Unit Auction

Welcome to lecture 13 of module 3 of course called game theory and economics. What we have been discussing so far is that, we have been discussing various aspects of auction through the model of game theory. The two kinds of auctions that we have discussed are second price sealed bid auction and first price sealed bid auction.

(Refer Slide Time: 00:46)

So, today, what I propose to do is, to look at other sorts of auctions which are prevalent, and then end this topic today itself. One kind of way auction which is neither second bid second price or first price, is what is known as all pay auctions.

(Refer Slide Time: 01:29)



In all pay auction, the person who is not getting the object also pays, which means that suppose, I have bid b i, but b i is less than the highest bid, suppose, this is b h, in that case, I do not get the object, but nevertheless I have to pay this b i. So, this is little bit tough on the people who are not getting the object, but this kind of auctions are there. For example, it may happen that if we can think of any real life example of this kind of auction - all pay auction, you can think of various political lobbies. That is trying to influence the functioning of the government, the various business lobbies.

For example, this business lobbies try to influence the functioning of the government, the decision of the government - may be they pay bribes to the ministers or bureaucrats. Now, it may happen that I pay bribes, but the decision of the government does not go in my favor. In that case, I pay something which is b i, which is like a bid - bid for the decision to go in my favor, but at the end of the day, if the decision of the government does not go in my favor, it is not that I get back this money which I have already paid to the various functionaries of the government.

So, there are some real life examples of this all pay auction. Now, interestingly, if we apply this all pay auction, in case of the setting of suppose second price auction, then what happens? Suppose, this also has all pay provision and then what happens is that this second price auction becomes similar to the war of attrition game that we have seen before, except for some few cases. Why I am saying this is that, in second price auction, if we have this all pay provision, then what is the payoff of any individual? It is v i minus b i. If b i is greater than b bar, remember b bar is the highest bid of the other players or b i is equal to b bar and i is less than j, where j is equal to b bar.

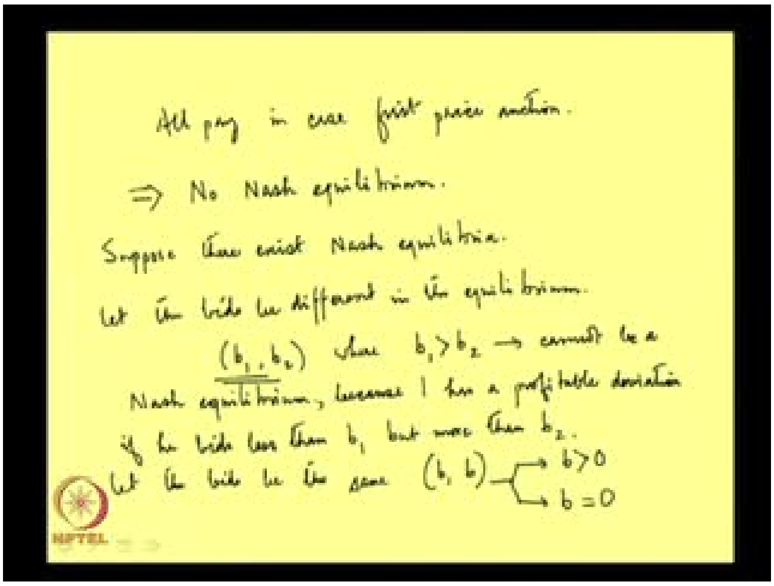
So, it may also happen that I bid the highest bid of the other players, but my index which is i, is less, therefore, I get the object. In that case, my payoff is this much, v i minus b i. It may happen that I do not get the object; in that case I get 0. This happens if b i is less than b bar that is my bid is less than the highest bid of other players or my bid is same as the highest bid of the other players, but my index is more than the indexes of these players who are bidding the highest, in that case also my payoff is 0.

Now, look this payoff structure is same as the war of attrition payoff structure, except for b i is equal to t i, because what was happening in that case is that, this is equal to minus b i, because when I am not getting the object I am still paying, so this was the difference which was not there in case of second price auction, but right now we have this case of this thing. This should be b bar, so this is the case where I have only two players, if I have only two players, payoff function of each player is equal to v i. The valuation of his valuation minus b bar b bar is the second highest if I get the object. Even if I do not get the object by, but my index is lower, then my payoff is this much.

If I do not get the object, then whatever time I have spent in waiting for the object, whatever time I spent in case of war of attrition, but here in case of second price auction, whatever bid I have made which is given by b i that is my loss for from this whole exercise, because I am not getting the object, so this is the case. What is important is that this is exactly like the war of attrition game, where if I get the object, then my payoff becomes v i minus t j. Here, t j is equal to b bar; b bar is the bid of the other player. If I do not get the object that I gave up earlier, then my rival in case of war of attrition, I lose the entire time that I spent the value of that time.

Here, it is minus b i, there it was minus t i, except the difference is that in case of tie, in case of war of attrition, people who are getting half of v i minus t i, so that was the case in case of tie, but here we have the object going to the person who values the object more. Tie breaking rule is different here, but beside that this game is same as the game of war of attrition.

(Refer Slide Time: 09:38)

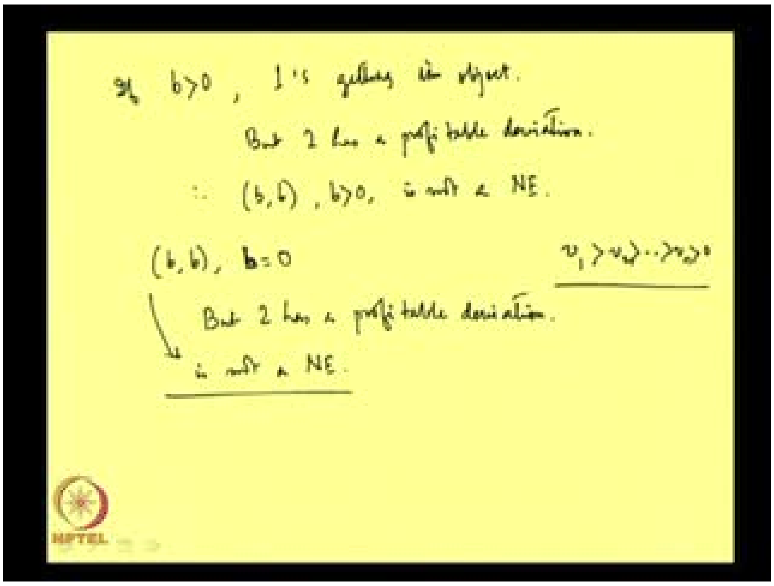


Now, this is one interesting thing, the second interesting thing is that if I apply this idea of all pay, in case of first price auction, then what is the equilibrium that emerges from here? The answer is that if I have a first price auction, where people have to pay irrespective of the fact that they get the object or not, then there is no Nash equilibrium. What is the proof? Proof is the following. Suppose, there is Nash equilibrium, now let the bids be different in the equilibrium. If there are equilibrium, then they can be of two types, either in the equilibrium the bids are different or in the equilibrium the bids are the same.

Let us take the first case that the bids are different. I have b 1, b 2, where suppose without loss of generality b 1 is greater than b 2, now this is Nash equilibrium. This cannot be Nash equilibrium, because player one who is biding more will deviate and bid something less. If he bids something less, he can improve his payoff, because one has a profitable deviation, if he bids less than b 1, but more than b 2.

This cannot be Nash equilibrium, because at least one person is having a profitable deviation. What can be the other possibility? The other possibility is the bids are the same. Suppose, the bids are b b, can this be Nash equilibrium? Here, I have two possible cases, one is that b is greater than 0 or b is equal to 0.

(Refer Slide Time: 13:30)

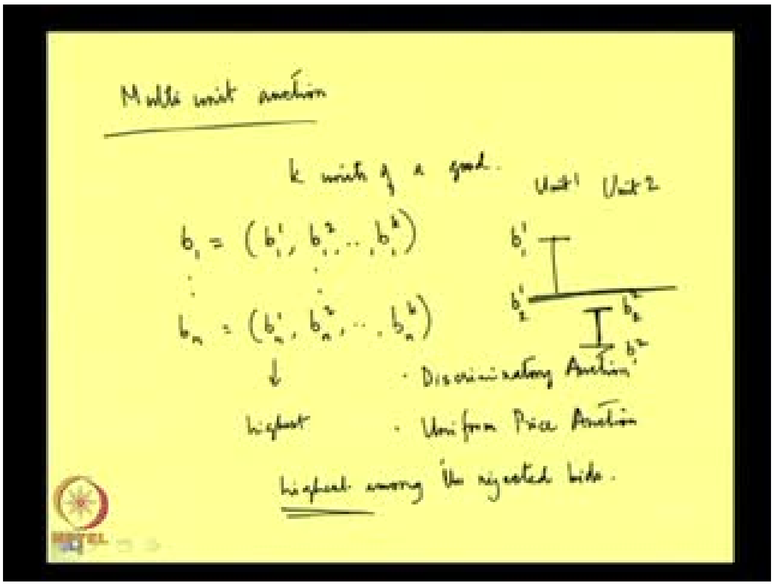


Now, if b is greater than 0, then who is the person who is getting the object? One is getting the object, but is this Nash equilibrium? This is not equilibrium, because two has a profitable deviation. The reason is being, here at b b this player two is paying b, which means his payoff is minus b. This is upset for him to go on paying this b while not getting the object at all, so he can at least bid less than b and that is a profitable deviation for him, this is not Nash equilibrium.

The last case that is left, is this a Nash equilibrium? That b b, and the value of b is equal to 0. Here, notice though player one is getting the object, the difference between the previous cases is that player two cannot bid less than this. The previous argument cannot be applied in this case, but in this case, player two can bid more than b and get the object that is going to be profitable, because if you remember one assumption that we had taken before, at the start all the valuations are positive.

So, whatever be the valuation of player two, it might be very little, but as long as this player two bids more than 0, he can make a positive payoff; right now he is making a 0 payoff. From here also player two has a profitable deviation, so this is not Nash equilibrium. Basically, we have exhausted all possibilities, we have not found any Nash equilibrium in case of first price auction where everybody is paying his or her bid; this was the case of all pay auction.

(Refer Slide Time: 16:31)



Now, we also have another kind of auction which is known as multi-unit auction. The auction here is auctioning not a single good but different unit of good. It may happen that the auctioneer has k units of a good in which these k units, the auction likes to auction off and there are bidders.

Since there are k units of a good to be auctioned off, a bidder when he submits his bid is not going to quote one price, but vector of prices having k elements. So, may be b 1 will look like this, b 1 1, b 1 2, b 1 k, similarly the last person. In this way, everybody is going to submit bids for any unit; the highest bid will be picked up that person who is submitting the highest bid is going to get that unit that is very clear. However, it may happen that the price this person is paying to the auctioneer varies according to differ rules of the game.

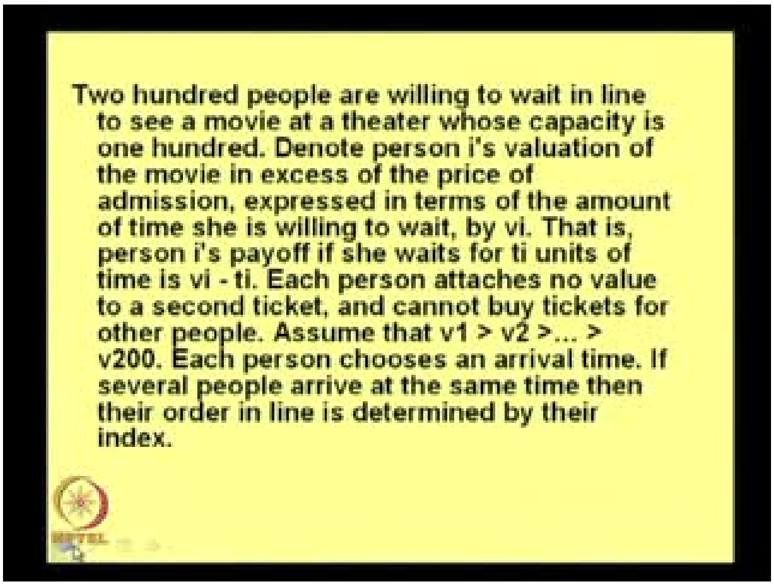
There we are going to discuss only two rules of the game, one is what is known as discriminatory auction, and the other is what is known as uniform price auction. Basically, in discriminatory auction, what price the players is paying for the units that they are getting; this model is trying to imitate the model of first price auction.

So, for any unit, the person who is getting the unit is going to pay the price which he himself is quoting, so it is like the first price auction, we are taking many units. In case of uniform price auction, what we are trying to imitate is the second price auction. It is the price that this people are paying to the auctioneer, are the price that is being paid is the highest among the rejected bids. To elaborate what the idea is for each unit, there is one price, which is winning the auction and there are many prices which did not win that unit. There is a highest among those prices; this is the highest rejected bid for that unit.

Now, if we compare all the highest rejected bids of all the units, there will be one price which will be the highest among these highest, so that is what we mean by the highest among the rejected bids. This highest rejected bid is the price that all the people who are getting the object are going to pay. The price that the winners are paying in this case is the same; it is uniform that is why it is called uniform price auction. You can see that it is like the second price auction, in the sense that the price is not the bid that I submitted, but it is a price which has been rejected. But, there is a complication here; it may happen that the price that I am paying is more than what I bid for.

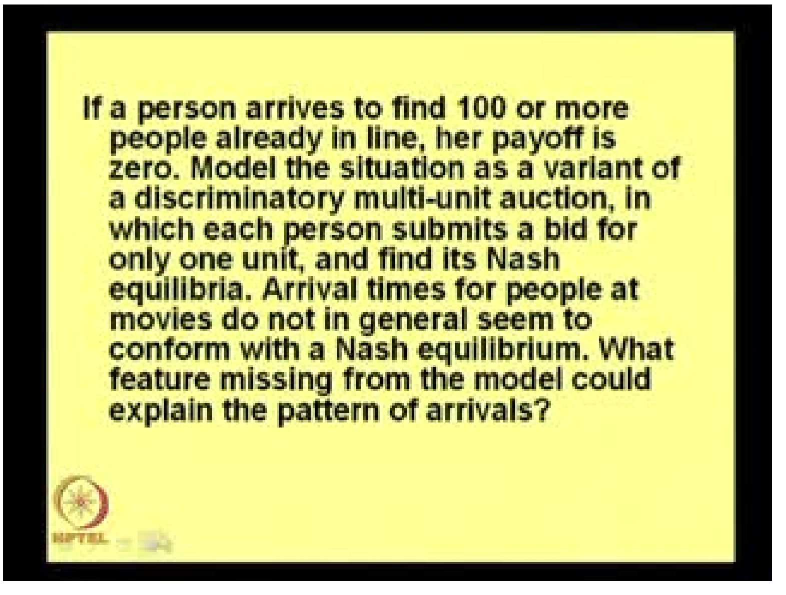
For example, take this case, this is unit 1 and this is unit 2; in unit 1, player one has submitted b 1 1, player two has submitted b 2 1. In unit 2, player two has submitted higher bid bid 2, it is b 2 2 and player one has submitted b 1 2. Now, here, the highest among the rejected bid is this much, but this bid is higher than the bid that player two submitted for this second unit.

(Refer Slide Time: 22:38)



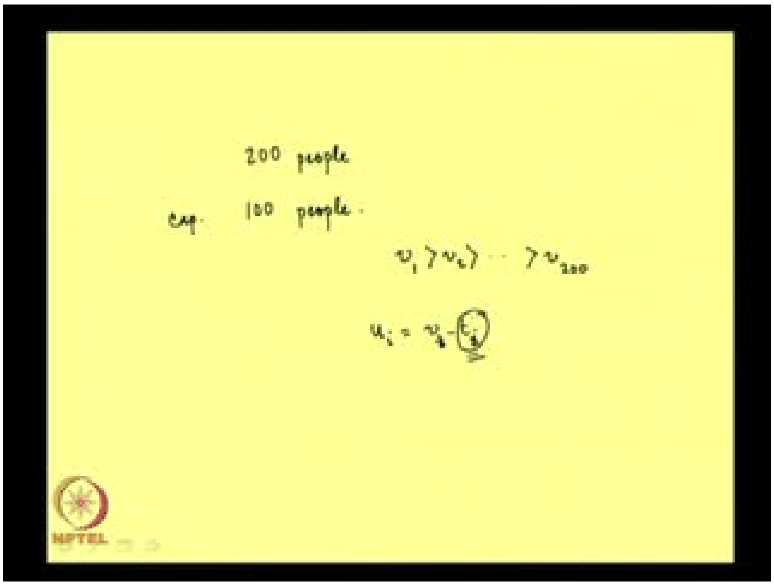
There is this edit complication, what we are going to do is to look at one exercise from multi-unit auction, here is the exercise. Two hundred people are willing to wait in line to see a movie at a theater whose capacity is one hundred. Denote person i’s valuation of the movie in excess of the price of admission, expressed in terms of the amount of time she is willing to wait, by v i. That is person i's payoff if she waits for t i units of time is v i minus t i. Each person attaches no value to a second ticket and cannot buy tickets for other people. Assume that v 1 is greater than v 2 etcetera, greater than v200. Each person chooses an arrival time. If several people arrive at the same time then, their order in the line is determined by their index.

(Refer Slide Time: 23:41)



If a person arrives to find 100 or more people already in line, her payoff is 0. Model the situation as a variant of discriminatory multi-unit auction, in which each person submits a bid for only one unit and find its Nash equilibria. Arrival times for people at movies do not in general seem to conform with a Nash equilibria. What feature missing from this model could explain the pattern of arrivals?

(Refer Slide Time: 24:26)



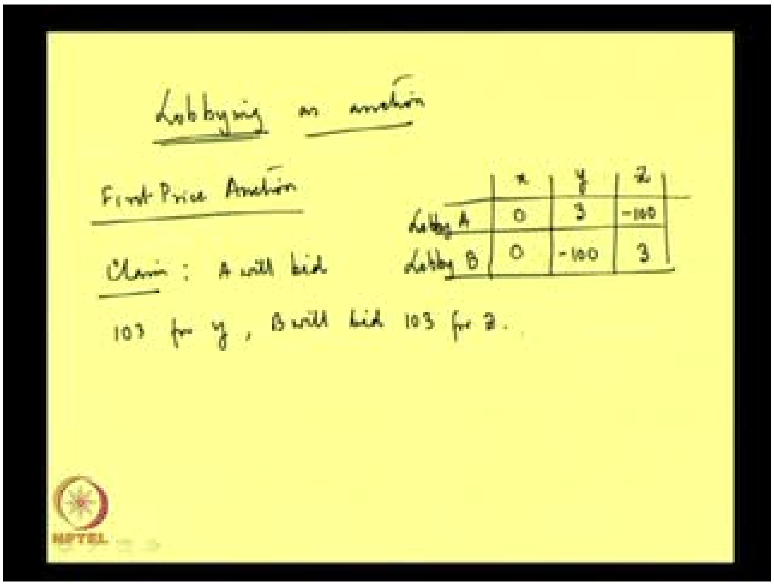
To translate this story in to simple English, there are 200 people, they want to watch a movie, but in that movie theatre the capacity is for 100 people. So, the tickets are going to be distributed at a particular point of time. Now, if I come early, then I have a better chance of getting a ticket. It may happen that I am the first person to arrive at the ticket counter, so I will be the first in the line, so I will definitely get the ticket. But, if there are more than 100 people already in the line, when I arrived there, then I will not get the ticket, I will not be able to watch that movie and in that case my payoff will be 0. How much people value watching the movie? That varies from person to person, so this is what is paying assumed.

The indexes is such that the first person values the movie the most, the second person the second highest, etcetera. How to write the payoff then? Payoff is written as v i minus t i. There is this valuation from the movie which I get v i, but I had to stand in the line for some time to get the ticket, so that is denoted by t i. Both this are expressed in the unit of time, so I do not have any problem of unit, I can subtract t i from v i, because they both of them are in terms of time. So, in this setting I have to find out what is the Nash equilibrium action.

Remember, what is happening is that every person is choosing at t, t is the waiting time, larger is t, the less well of you are, because this t i is going up, so u is going to go down. At the same time, if I choose a small t, I will not be able to watch the movie, because there will be more people already in the line, so that is the thing why I am calling it a multi-unit auction?

This is a multi-unit auction, because there are 100 units, in the sense that 100 is the capacity. These 100 units are sold off and each person is submitting the bid for only one unit. So, he or she is bidding for only 1 unit, the price that they are paying is in terms of t, greater is the t, less well of they are, so this is how it can be seen as a kind of auction.

(Refer Slide Time: 28:07)



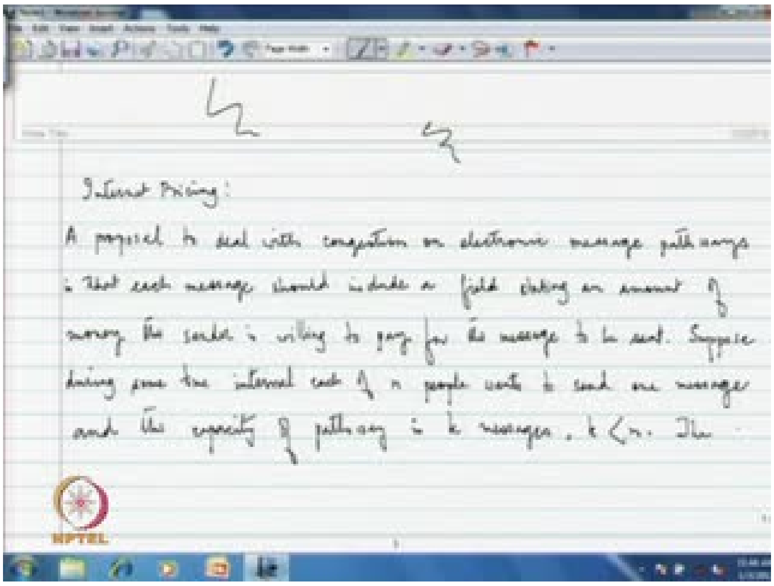
Now, we have to find the equilibrium of this game, so that is a more or less - to close this chapter on auction, just one more thing that I wanted to discuss is the following, is the case of lobbying; lobbying can also be seen as auction. What I mean by lobbying is the following, suppose, there are different social groups, may be business groups, they want the government to follow a particular policy, the policy which is most beneficial for them. There might be divergences, I might like policy x to be followed by, my rival group might be wanting the government to follow some other policy may be y or z.

In that case, we shall try to impress the government, it may happen when the government is in a way selling these policies to people, whoever is bidding the highest, which a party bidding the highest that party is getting his or her policy of choice to be implemented by the government. In that case, can we apply the models of Nash equilibrium or model of auctions that we have discuss so far.

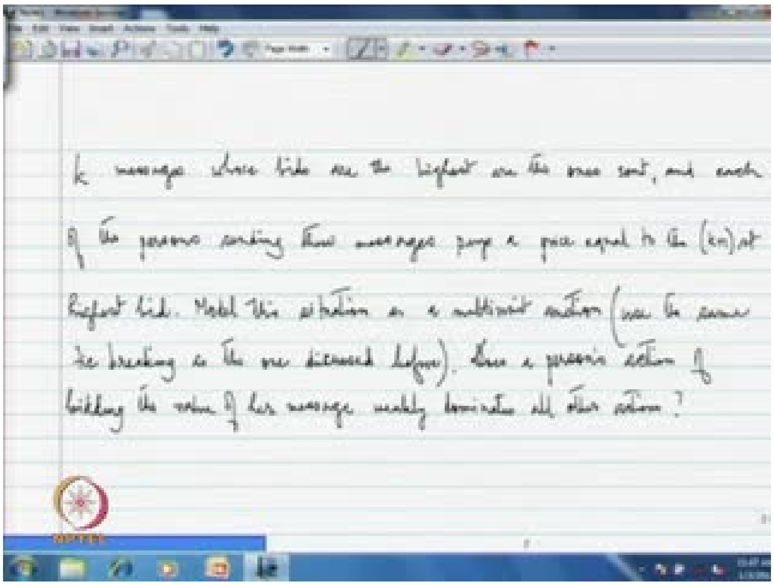
Let us take the following game. Suppose there are two lobbies lobby A and lobby B, and there are three policies that can be taken by the government. The numbers here indicate the benefits that the lobbies are going to get from the policies. In short, if x is the government's policy, neither A nor B will get anything from this policy. If y is implemented, then lobby A gets 3 payoff from the policy, lobby B gets minus 100 from the policy, so it is very bad for lobby B, but little good for lobby A. The opposite thing happens if z is implemented, B is getting 3 and A is getting minus 100.

Now, what is the rule of the game? What rule government is telling to these lobbies to follow? Suppose, the rule is the first price auction which means that each lobby will tell the government how much money it is ready to pay to the government, to gets its favorite policy implemented. So, if lobby A quotes a higher bid, then lobby A is going to implement his policy in that case lobby B is not going to pay anything to the government, because its policy is not going to get implemented, so that is the setting. Now, the claim is the following; claim is that A will bid 103 for y; B will bid 103 for z that is Nash equilibrium, so let us do one exercise on this idea of multi-unit auction.

(Refer Slide Time: 32:14)

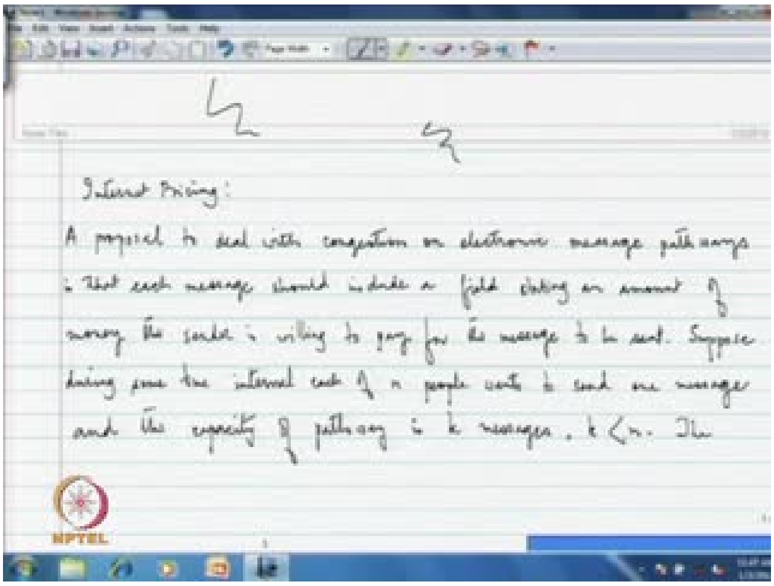


(Refer Slide Time: 36:03)



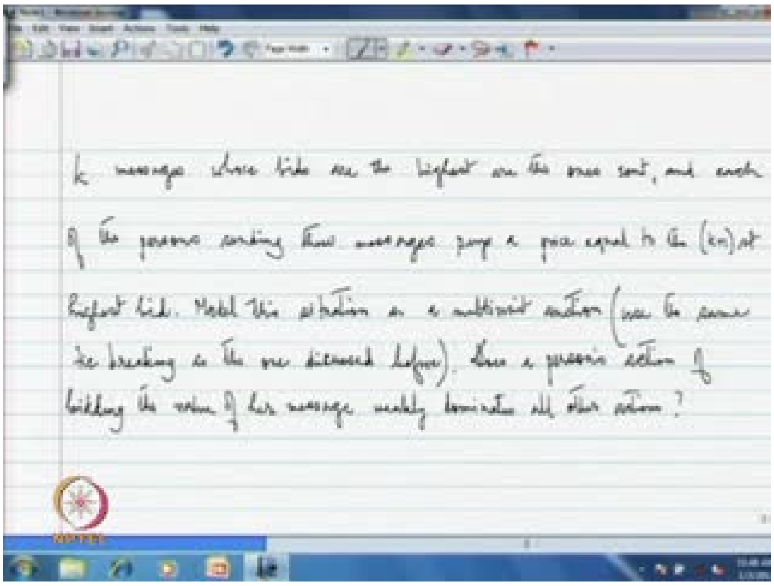
This exercise is called internet pricing, I shall write down the question first and then we shall solve this problem ((no audio 33:00 to 39:00)), so this is the question.

(Refer Slide Time: 39:14)



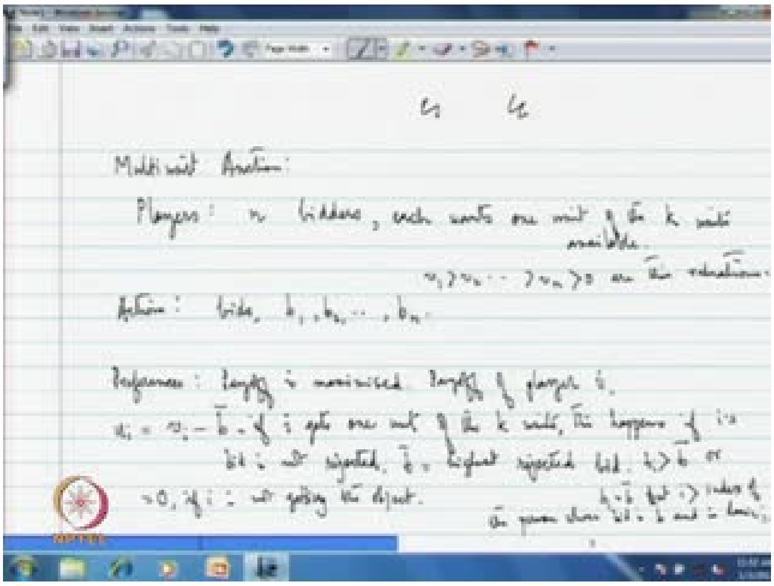
Let me read the question, a proposal to deal with congestion on electronic message path ways is that each message should include a field, stating an amount of money the sender is willing to pay for the message to be sent. Suppose, during some time interval, each of n people wants to send one message and the capacity of pathway is k messages, where k is strictly less than n.

(Refer Slide Time: 39:45)



The k messages, whose bids are the highest, are one sent and each of the people sending his messages pays a price equal to the k plus 1, it is highest bid. Model this situation as a multi-unit auction, use the same tie breaking as one discussed before.

(Refer Slide Time: 40:23)

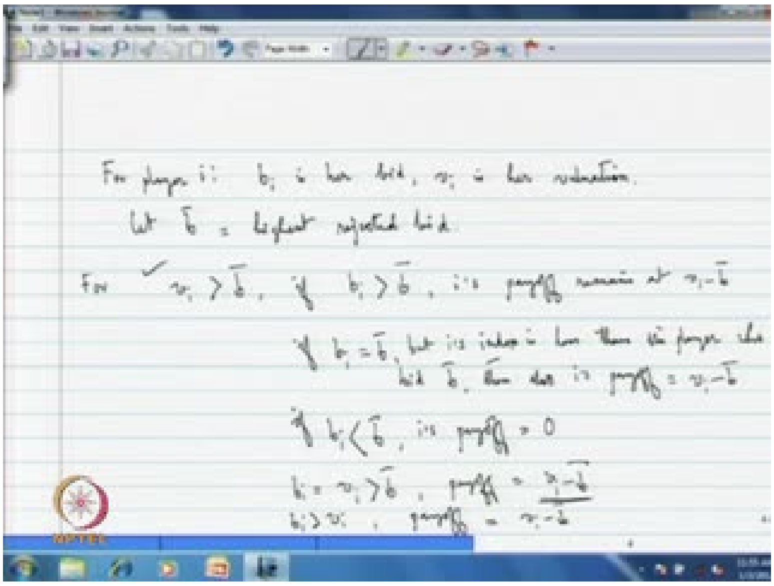


Does a person's action of bidding the value of her message weakly dominate all other actions? So, this is like the following multi-unit auction, so players, who are the player's n bidders, each wants 1 unit of the k units available and they have their valuations also.

So, v 1, v 2, etcetera v n, are greater than 0, are their valuations. Actions are nothing but bids b 1, b 2, b n; finally, preferences payoff is maximized, payoff of player i, let us call it u i, is equal to v i minus b bar. If i get 1 unit of the k units, this will happen if i's bid is not rejected, b bar is the highest rejected bid.

So, in this case, b i is greater than b bar, it can be equal to b bar also or b i is equal to b bar, but i is greater than index of the person whose bid is b bar and is denied. So, this is the case when i is getting the object or 0. If i is not getting the object, so in this case, his bid is being denied. So, this is how it can be seen as an auction game. Now, second part of the question is the following. Does a person's action of bidding the value of her message quickly dominate all other actions?

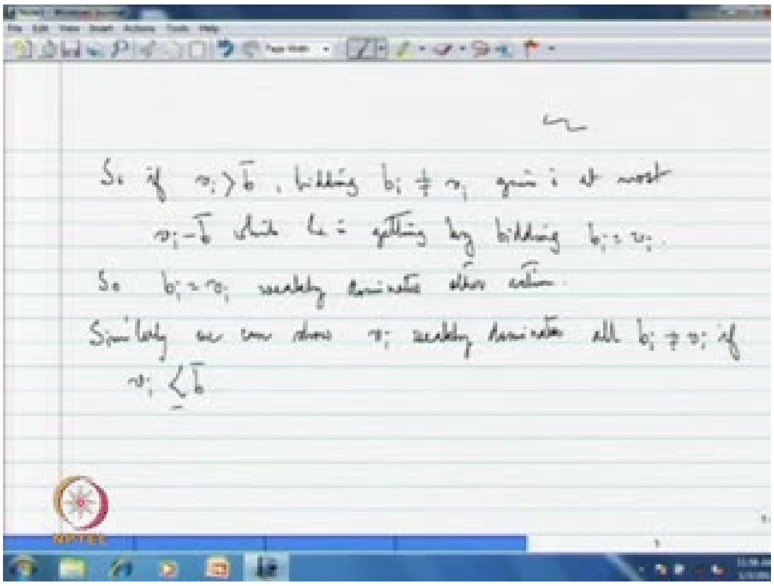
(Refer Slide Time: 44:34)



So, take the player i, for player i, b i is her bid and v i is her valuation. Now, let b bar is the highest rejected bid. For let us say, v i greater than b bar, if b i is greater than b bar, i's payoff remains at b i minus b bar, this is what he gets. If b i is equal to b bar, but i's index is less than the player who bid b bar, then also i's payoff is equal to minus b bar.

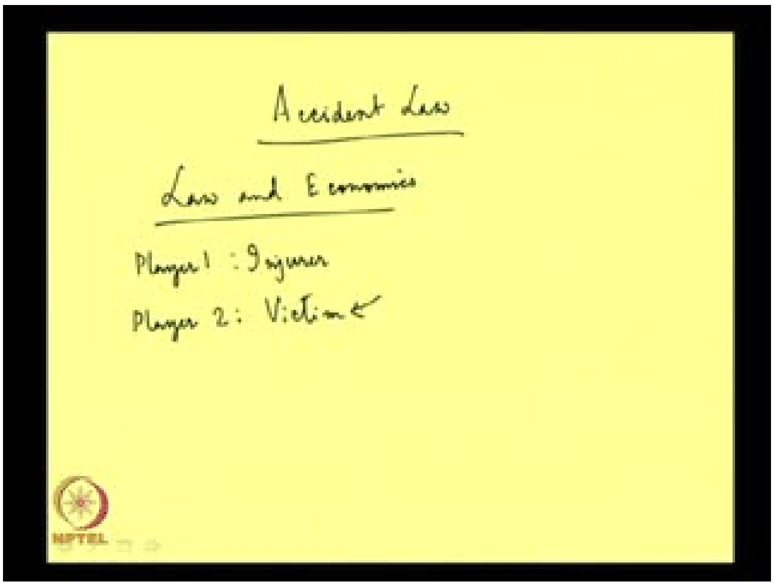
If b i is less than b bar, i's is payoff is equal to 0, he is not getting the object. What could be the other possibility that b i is equal to v i and this is greater than obviously b bar. In this case, payoff is equal to v i minus b bar. If b i is greater than v i, payoff is equal to same as v i minus b bar, by looking at all this different possibilities of b i, we are taking different values of b i. We are seeing that whatever be the value of b i, if it is not equal to v i, then payoff of i cannot be more than what he is getting by bidding just equal to v I, which means in this case, if v i is greater than b bar, bidding b i not equal to v i, gives i at most - v i minus b bar which he is getting by bidding b i is equal to v i.

(Refer Slide Time: 47:47)



So, b i is equal to v i weakly dominates other actions. Similarly, we can show v i weakly dominate if v i is less than b bar or equal to b bar. So, this is the proof that if in this case of multi-unit auction, where person who is getting the object pays the k plus one th highest bid, once valuation of the object weakly dominates over all her other actions.

(Refer Slide Time: 50:08)



This was more or less our discussion of auction theory. What we are going to do in the next lecture, I am going to introduce the topic is known as accident law. Law and economics is basically a sub discipline within economics. Where what is tried to be analyzed is the affectivity of laws, whether these laws are in line with the economic principle such as efficiency, productivity and etcetera. Can we have certain laws - can we have certain laws, therefore which are efficient in the economic sense or can the laws being proved?

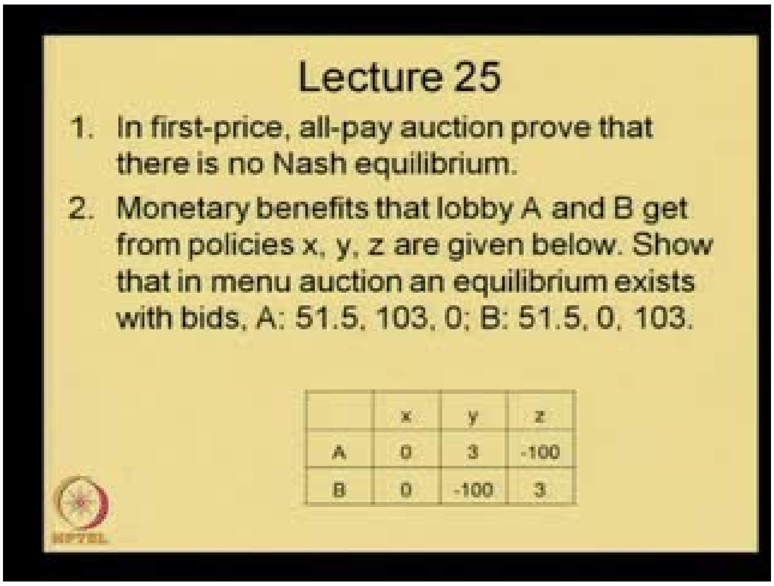
Here also we are going to look into a particular sort of law which is known as accident law. In an accident law, what is the general setting? There are two parties of this accident law, one is supposing the injurer, so this is player one, who will be called the injurer and there is player two who is the injured person, let us call him the victim.

So, to visualize this thing, there can be several examples, but one example could be that a person - a pedestrian is crossing the road, suddenly car comes and hits him. The law that occurs from this accident is occurring to the victim that is why he is called a victim. The car owner or the driver is not in at a loss, now he is not getting harmed in anyway. But, now the question is when this accident is taking place, there are two sides to the story.

It might happen that the injurer was carefully driving, whereas the victim was careless. He went into the street without looking at other different directions and that is why this loss - this accident happened or it could be the other way that the victim was very careful in crossing the street, but the injurer the car driver was rash and that is why this accident happened. Can there be a law which apportions, which assigns a kind of care that these two parties should take, such that this kind of negligence from each side is minimized and we get outcome which is efficient for the two parties taken together?

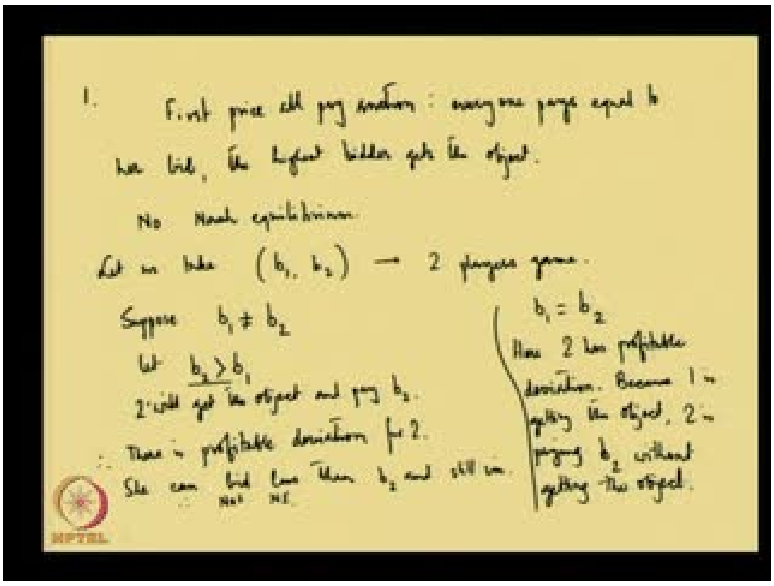
So, this is something which we are going to look it - look into in the next lecture. So, before we finish, what we have done in this lecture is that we have discussed the various aspects of auction theory. We have gone beyond first price auction and second price auction, we have started discussing what is known as accident laws; thank you.

(Refer Slide Time: 53:42)



In first price, all pay auctions prove that there is no Nash equilibrium.

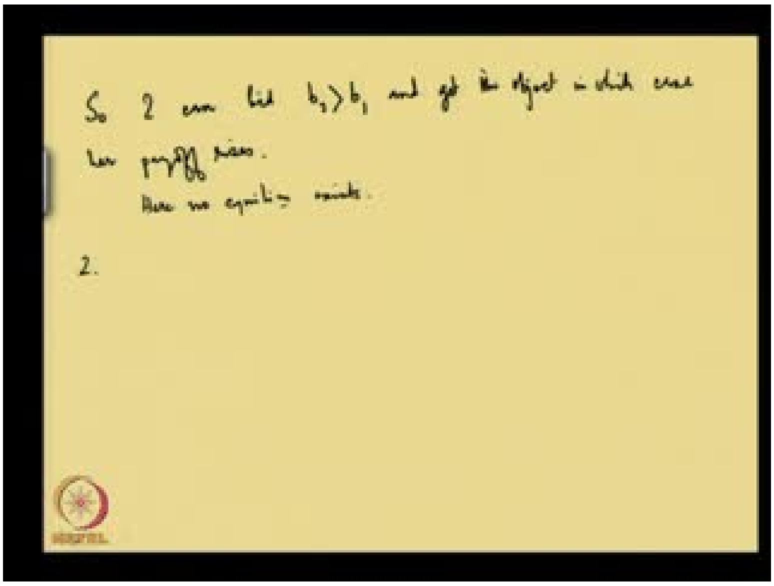
(Refer Slide Time: 53:48)



Now, remember, in first price all pay auction how is it characterized? Everyone pays equal to her bid and the highest bidder gets the object. Now, here, we have to show that there is no Nash equilibrium, how? Let us say b 1, b 2; we are talking about two players. Suppose, there are two players, there is a bid for b 1 and b 2, can there be Nash equilibrium? Suppose, there are the bids such that b 1 is not equal to b 2, so this is case 1, there could be a case 2, where b 1 is equal to b 2.

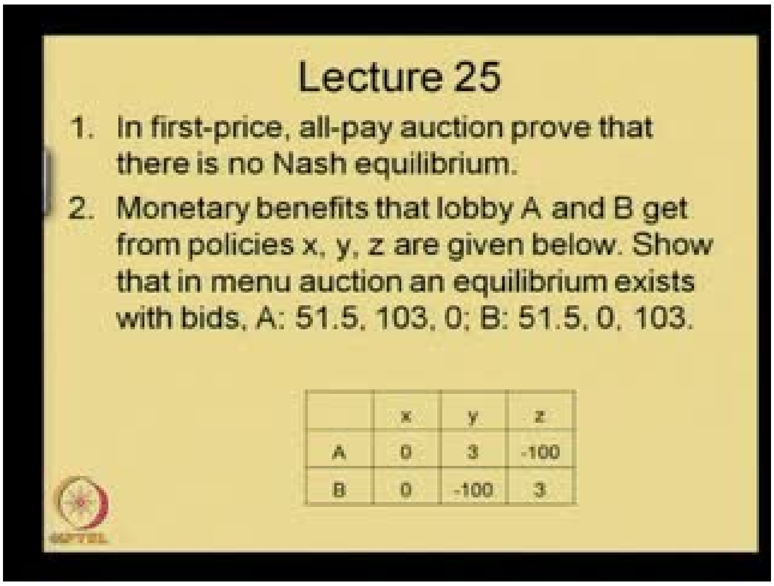
Suppose, when b 1 is not equal to b 2, let b 2 is greater than b 1. Now, in this case, obviously player two will get the object and pay b 2, but which means that there is a profitable deviation for player two, she can bid less than b 2 and still win the object. So, this thing, b 2 greater than b 1 cannot be Nash equilibrium, so this case is then ruled out if the bids are different, then they cannot be Nash equilibrium. If bids are equal, then is there profitable deviation for one player? Here again, we shall see that two has profitable deviation. This is because, in the case, where b 1 is equal to b 2, one is getting the object and two is paying b 2 without getting the object.

(Refer Slide Time: 57:50)



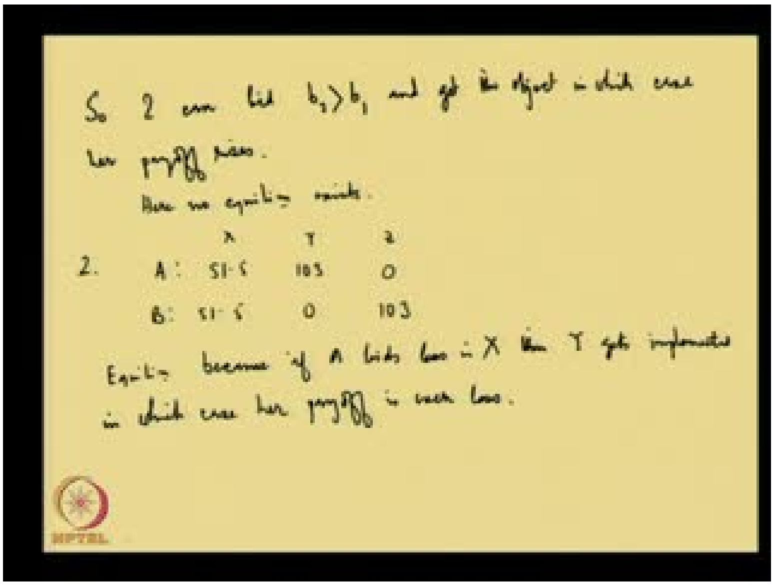
So, 2 can bid b 2 greater than b 1 and get the object. In which case, her payoff raises, so we find that none of these two cases – first, when b 1 is equal to b 2 or b 1 is not equal to b 2, in none of the cases, there is equilibrium, hence here no equilibrium exists.

(Refer Slide Time: 58:44)



Question number 2: so in question number 2, here given a matrix, where the numbers are showing the benefits that these two parties A and B get from each of this three policies x y and z. We have to show that in a menu auction, the equilibrium exists with the following bids, 51.5, 103, 0 for A and for B: 51.5, 0, 103.

(Refer Slide Time: 59:21)



Here, these are the bids that we have to prove that they are the equilibrium bids. Why they are equilibrium? Because, if A bids less in x, then y gets implemented, in which case her payoff is even less. Whereas, now what the player one is getting, he is paying

51.5 and getting 0, which means he is getting minus 51.5, whereas, if y gets implemented, he will get minus 100, so bidding less for x is unprofitable.

Similarly, we can show that bidding less for y is also unprofitable, if both x and y are bid less by A, then z gets implemented, which means in that case you will get minus 103 as the payoff, so deviation by A is unprofitable, similarly we can show the deviation B is also unprofitable, so that is it.