Game Theory and Economics

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Module No. # 04
Mixed Strategy Nash Equilibrium
Lecture No. # 02
Mixed Strategy, Mixed Strategy Equilibrium

Welcome to the second lecture of module 4 of the course, called game theory and economics. We have been discussing mixed strategy Nash equilibrium.

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We have not yet defined, what is actually a mixed strategy Nash equilibrium. Instead, I have been trying to motivate you towards the problem. Here is a case where the players can randomize between their actions. It is possible that they do not take their actions for certainty – they may randomize over their action. They may take certain actions with some probability and some other actions with some probability. We want to find out if players are allowed to randomize; what kind of equilibrium we can get from such situation?

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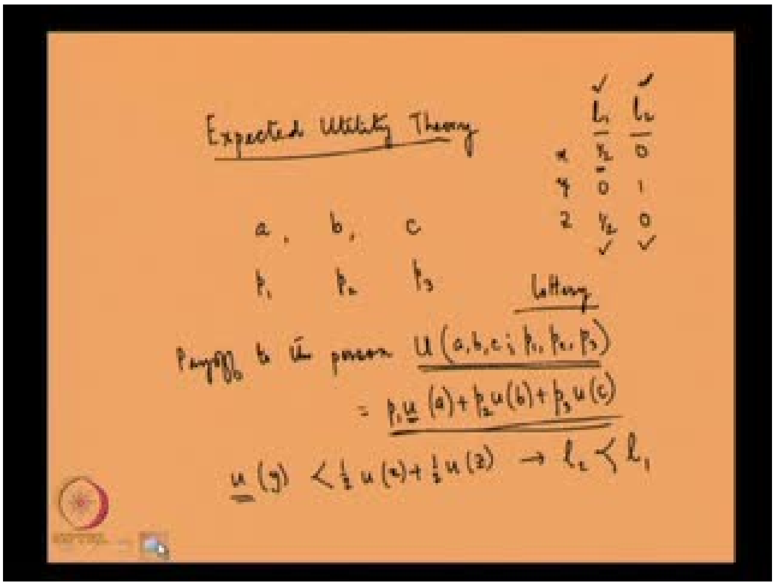

So far, we have seen that if the outcomes are uncertain then the payoff of the players can be represented by what is known as the Expected Utility Theory. That is, if there are three possible outcomes, each happening with some probability p_1, p_2, p_3, then the payoff to the concerned person, who is observing these three outcomes and who can get a utility out of each of these three outcomes, will be given by, let us call it, U. It is given by p_1.

This U is what is the total payoff that the player is getting, the person is getting, from these three outcomes, each of them is occurring with some probability p_1, p_2, p_3. This total utility can be represented as the expected value of these individual utilities. And these individual utilities are represented by u. So this is u and this is U. The u's are defined over certain outcomes. If a happens then this u represents the payoff to the player from this certain outcome. But we know that b and c can also happen, so the total payoff to the person is the expected value of these individual payoffs, and is represented by U. We can do so only if this preference of the player follows the Von Neumann Morgenstern property. This u is known as the Bernoulli payoff function or the Bernoulli utility function.

This Expected Utility Theory gives us a clue of how people evaluate their payoffs, when the outcomes are not certain or there are more than 2 or 3 outcomes. By knowing that, I can do this. I can represent the payoff of a player from this uncertain situation. This is called a lottery. From a lottery by this formula, I cannot be certain whether a player will prefer one lottery over another lottery.

For example - the example that I gave in the last lecture. If suppose x is preferred to y and y is preferred to z, and suppose there are two lotteries. These two lotteries are l_1 and l_2.

Then, I do not know whether the player is going to prefer this over this or this over this. It can happen that a player have u_b. This is possible. This tells me that this player is preferring l_2 over l_1. Or it may very well happen that this goes the other way, in which case, l_1 is preferred to l_2. In this case, the player is having this lottery over this lottery. In this first lottery l 1 the first outcome sorry this is x and y and z

In the first lottery l_1, what is happening is that – x is happening with probability half, y is happening with probability 0, and z is happening with probability half. So here, both x and z can happen with same probability. The player can like that lottery over this lottery, where only y happens with certainty.

Particularly, a player will like l_1 over l_2, if a player likes to take risk because there is a possibility that z will happen. In that case, he does not have any much utility. It does not get much utility out of the situation. But there is again a possibility that x happens, and that induces this person, who likes to take risk, to prefer l_1 over l_2. In l_1, you know, only y is happening with certainty, which is better than z but worse than x. So, to know whether a person will take l_1 over l_2 or a person will take l_2 over l_1, it depends on the individual characteristics of this small u functions.

What kind of shape of u will induce risk-taking behavior and what kind of shape of u will induce not risk taking behavior can be shown in the following example.

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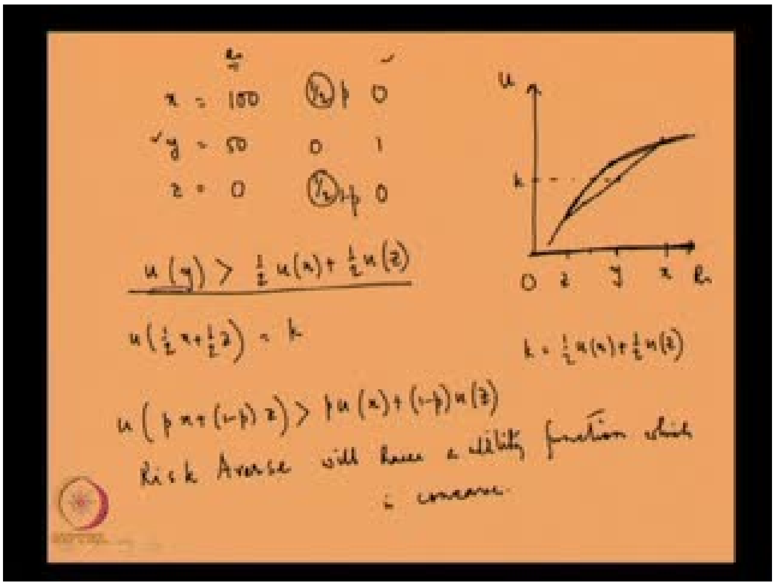

Suppose x is 100, y is 50 and z is 0. Now let us consider the following lottery – half 0 half 0 1 0. so these are the. Along this axis, we are representing the amount. You can think this 100 as the amount of money that the people are getting. So this is in rupees and this is utility that the player is getting, from having those amount or money.

So, suppose 0 is here, this is z, and this is suppose x. And I know that, in general, if a person is getting more money, he is getting more utility, which is why x is preferred to y.

Now, the question is, how can I know that the utility function passes through this point and this point and what kind of shape does it take?

Now suppose, if it is so happens that u y is greater than half of u_x plus half of u_z, which means that the person concerned is liking the certain outcome. This is being liked.

Y is somewhere in between. So I can write this as half x plus half z, because y is fifty, half of 100 plus half of 0.

Now, if this is true, then at this point corresponding to this point, if I draw a straight line what is the point in the middle of the straight line. This middle of the straight line point, if I call it, suppose this is y. Suppose this is k, then k is nothing but half of u_x plus half of u_z. From this information what I know is that u of y is somewhere above this line, above this point.

So the curve – the u curve – must pursue this point, this point, and this point. Now I can go on changing the value of this half and this half, in the sense, that I can take this to be p and this to be 1 minus p.

So if the person always likes a certain outcome, then I have this equation that p of x plus 1 minus p of z is greater than... this I know. Here p was half but p could take any value.

And for each value of p, the value on the curve is above then the value on the chord. So the curve will have a shape like this. It means a person who likes to take the certain outcome, which is in between in terms of preference, is a person who can be called as a risk-averse person. A risk-averse person, who does not like to take risk, who like to take the middle path, will have a utility function, which is of this shape. And this shape is called concave.

So this is a fundamental deduction that we have reached from this theory of expected utility. That is, if a person is a risk-averse person, his utility function is going to be a concave function.

And more risk-averse a person is, it may happen the degree of risk aversion changes, if the risk-aversion changes more, then the curve becomes more and more concave.

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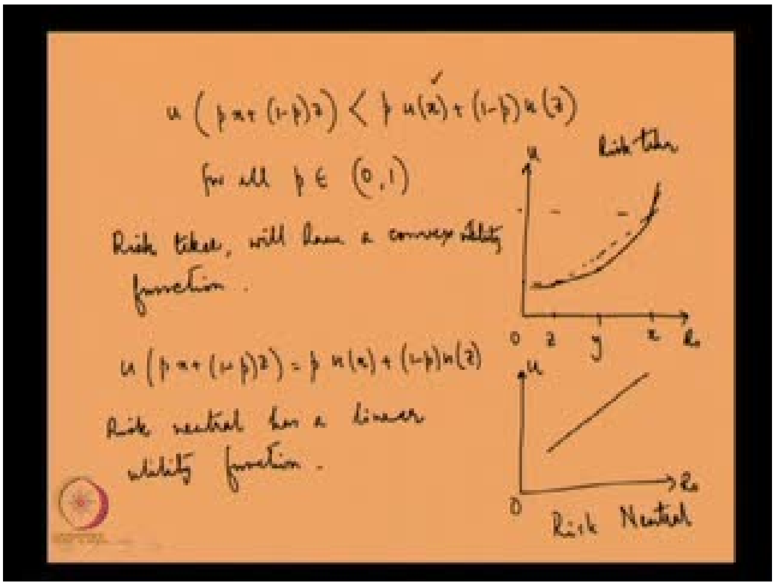

By the same logic we can also say the following that – u of. If it so happens, that if this is true for all p in the range 0 to 1, not all, excepting 0 and 1, then what is happening is that the person is taking the risk. He is thinking that, so here is rupees, here is utility. So suppose this is u_x, u_z and this is u_x. This is the point on the chord. Here, the point on the chord is this one, higher than the point on the curve, which is here, for example. And the function is of this shape – a convex function. This is a person who is a risk-taker or a risk lover. A function like this is upward rising and convex.

And what about the person, who is indifferent between whether he takes risk or not take risks? So it may happen that it is an equality. If it is an equality, then the function is neither convex nor concave. It is just an upward rising function. So this is the case of risk-taker and this is the case of risk-neutral, who is indifferent between taking risk and not taking risk.

So this is a way to characterize the risk-taking behaviour of different people that we just look at. The utility functions of people and from the shape of the utility functions, we can deduce whether the person likes to take risk or does not like to take risk.

But mind you, this is crucially based on our theory of expected utility. And expected utility theory is just a conjecture in laboratory experiments. It is seen that people may not follow the conjecture of expected utility theory.

Now, so in real life, it can be asked whether people are risk-takers or risk-averse. It varies from person to person and from situations to situation also.

For example, a certain person might like to buy a lottery ticket. Now if a person is buying a lottery ticket, what he is doing is that he is entering into a risky situation. If he had not bought a lottery ticket, there is no risk – he is not going into any uncertainty. But if he is buying a lottery ticket, he is entering into uncertainty.

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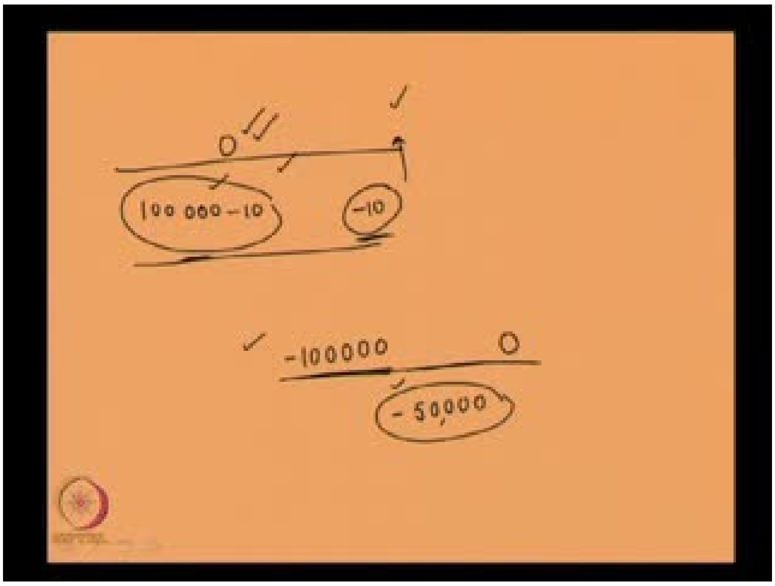

Let me tell you why? If he does not buy any lottery ticket what is his payoff? Suppose if he does not buy, his payoff is 0, he is neither making any loss nor making any gain. He is just retaining what he had before. So there is no addition or subtraction from his wealth.

If he buys a lottery ticket then there are two possibilities – one is, he is getting the lottery. Now, if he is getting the lottery, then what is his payoff? Suppose the lottery is worth Rs 1 lakh. But from this I have to subtract the price of the lottery ticket, which is suppose, Rs 10. So this is the money that he gets if he wins the lottery. If he does not win the lottery, what is the amount that he is left with? He is left with minus 10 rupees. I am just talking about the addition or subtraction to his wealth. So, the addition to his wealth is just minus 10 rupees if he does not get a lottery, and if he gets the lottery, it is 1 lakh minus 10.

This is like the situation that we have discussed before. 0 is like d. It is happening with certainty. You are not buying a lottery ticket and whatever getting is for certain – you are not entering into any risk. But if you are buying a lottery ticket, then there are two possibilities – either you get the money, a lot of money, this much amount of money, or if you do not get the lottery then you make a loss, which is of minus 10 rupees.

So some people will buy this lottery ticket, they will prefer this over this. Or some people will not buy a lottery ticket. Every person does not buy a lottery ticket.

For them, this situation is better than this situation. So risk-taking behavior varies from people to people. Similarly, think about an insurance policy. What is happening in an insurance policy? It is just like the opposite of a lottery ticket. By buying a lottery ticket, you are getting into an uncertain situation; whereas in case of insurance policy, we are already in an uncertain situation but by buying an insurance policy, we want to get out of that uncertain situation.

How? Here the payoffs are like the following – Suppose one does not buy any insurance policy and suppose there is a burglary in his house. If there is a burglary in his house, he is losing some money. Suppose the amount of money that he loses because of the theft is Rs 1 lakh . This is a loss, if he does not buy an insurance policy and there is a theft. But there is a possibility that he does not buy an insurance policy and there is not theft. There is no burglary. In this case, he will be left with no addition to his wealth or no subtraction to his wealth.

So his payoff is 0. So this is an uncertain situation to begin with. There is a possibility that burglary might happen, there is also a possibility that it might not happen.

Now why people then buy an insurance policy? Well, if you buy an insurance policy, may be, you are going to pay an amount of money, which is suppose Rs 50,000. The total amount of premium that you are going to pay for this insurance.

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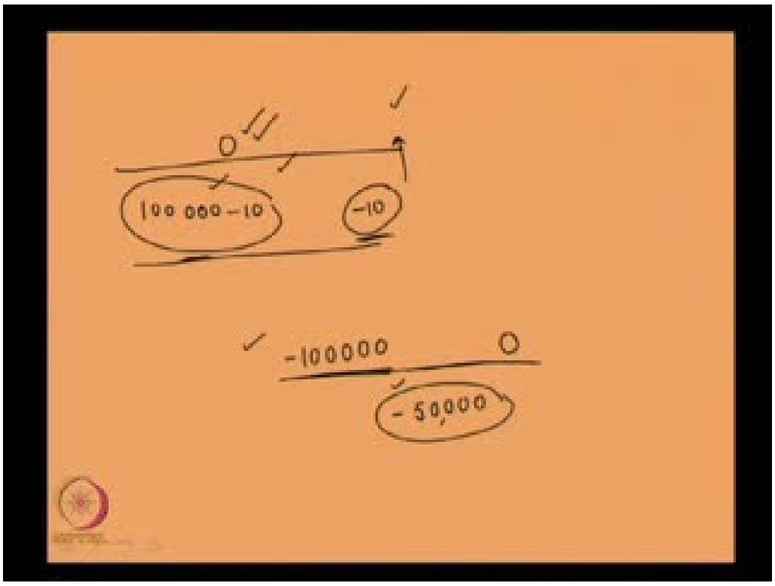

Now if you buy the insurance policy, it is certain that you have to pay this premium, there is no uncertainty. But if you pay the premium what is your gain? The gain is that if you have the burglary, you are not losing any money. This Rs 1 lakh that you could have lost because of this theft, that will be compensated by the insurance company.

So here some people will like without going, some people will not like to take the Insurance policy, in which case they are preferring this kind of lottery – minus 1 lakh rupees and 0. But some people are risk-averse, and they will like to buy this insurance policy and go for this certain outcome of minus 50000 rupees.

And it may so happen that the person who is buying a lottery ticket, who is getting into an uncertain situation, may be the same person who is also buying an insurance policy, because this risk-taking behavior may differ according to the amount of money involved.

So if the amount of the money involved in the lottery is different from the amount of money involved in insurance policy, he may show a risk-aversion in some case and risk-taking behavior in other case.

Now, so this is more or less a digration from the mixed strategy Nash equilibrium discussion that we were having so far. To carry forward the discussion of mixed strategy Nash equilibrium,

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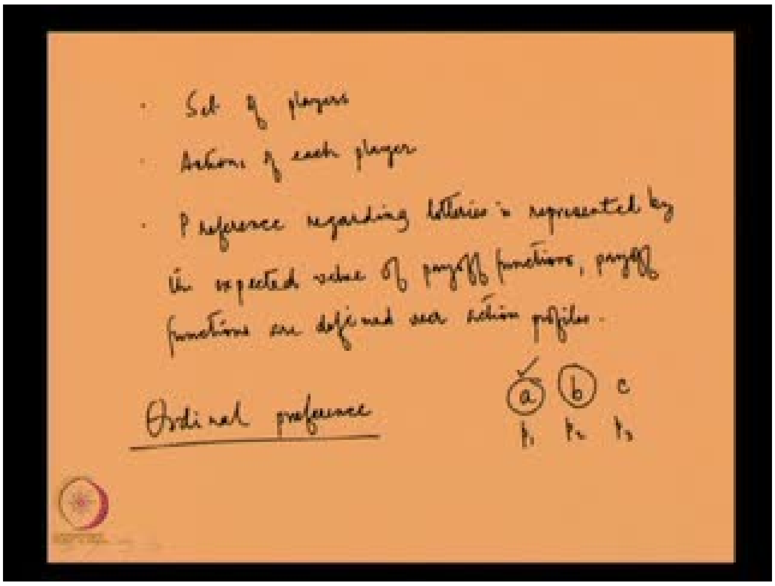

We can translate what we have discussed so far about the setting of uncertainty and randomization of actions by players, in terms of language of game theory. So, in game theory, what we need to specify is a set of players. How many players are invoked that I have to specify here, in case of mixed strategy Nash equilibrium setting, and the action set of each player. And thirdly, I have to tell something about the preference pattern of the players.

Now, in case of no mixed strategy Nash equilibrium, preference was defined over the action profile. So there were different action profiles of people. A player was supposed to tell which action profile he prefers over other action profiles and what is the order of preference.

In case of mixed strategy Nash equilibrium, what a person will be required to tell, is that which lotteries are preferred to him and which lotteries are less preferred to him.

So here the preference is not defined over action profiles but it is defined over lotteries because the events are themselves uncertain. And if the events are uncertain, then there are probabilities attached to each event.

So a person is asked, which distribution of these probabilities do you prefer more over other distribution of the probabilities, which are defined over set of events?

So preference, regarding lotteries, is represented by the expected value of payoff functions. And this expected value of payoff function, these payoff functions are defined over action profiles.

So it may happen that a can occur, b can occur, c can occur, and there are probabilities attached to each of these a and b and c are p_1 p_2 p_3. And this p_1 p_2 p_3 may vary, giving us different lotteries.

I have to know for a particular player, how does he order these different lotteries, which are defined over this action profiles? a is coming from a particular action profile because if the players are taking some particular actions, then a particular event is taking place. That is event is called a.

If they are taking some other action, the different action profile, b, can occur, which is another event. And behind the occurrence of each of these events, there are some probabilities attached to the distribution of the probabilities, is called lottery.

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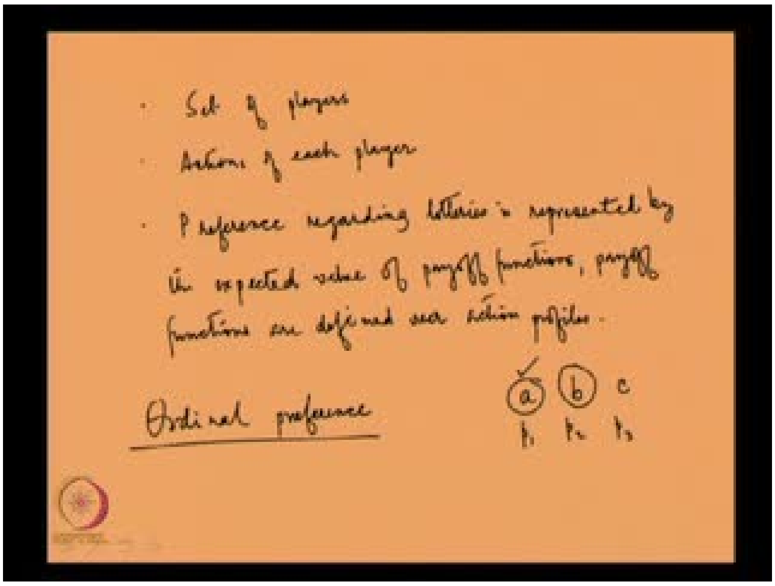

And what is equate is that a person’s preference must be known over this set of lotteries.

Now, once we have defined the setting in terms of the language of game theory, it becomes a little tricky to compare this situation to the situation that we have been discussing so far, in the sense that so far in the setting of strategic game. We say that the preference of the players was ordinal. This is something which we have assumed so far. By that we mean that it does not matter about the absolute value of the numbers, which represent peoples' preference. As long as the numbers relative value remains the same, the persons preference remains the same. That is what is known as ordinal preference.

But here once I have defined preference, not over action profiles but over lotteries, then we shall see that this ordinarity does not remain so far.

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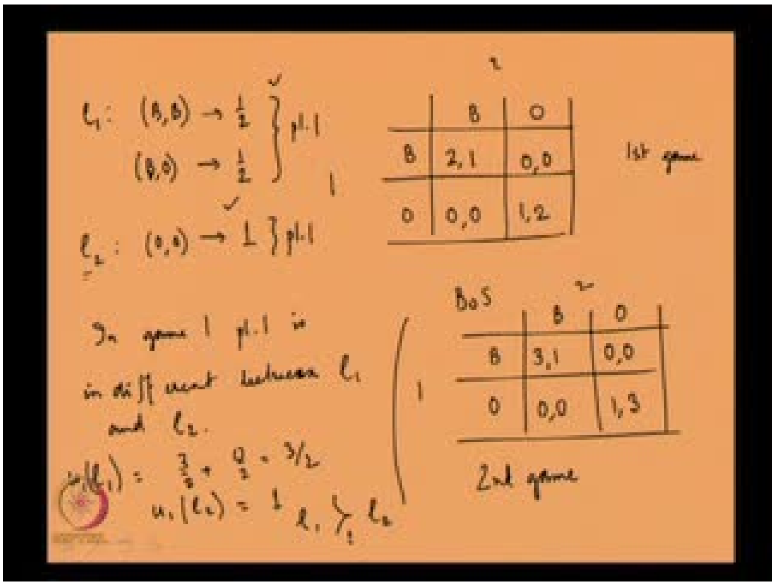

For example, let me give an example. Let us take the case of battle of sexes. So this was the original setting – 2 1, 1 2. The point was that if two players go to the same venue, like they go to the boxing match together, then both the players are better-off compared to the situation where they go to different venues. But the benefit to the first player is more because he likes to watch the boxing match more than his partner. Then vice versa for the other player it is the o o which is better.

Now ordinality will demand that the same game can be represented by the following numbers – 3 1, 0 0, 0 0, 1 3. Because here, what I am doing is just raising the value from 2 to 3, and by doing so, I am not tinkering with the qualitative difference between the numbers, which represent a particular player's preference.

For example, player 1 still likes going to the boxing match with his partner better than going to the opera house with his partner, which is better than going to each of these venues separately. So that quality remains the same. But if these numbers have to represent the preference of the players over certain outcomes, whose expected value represent their preference over lotteries, then there is a problem. For example, let us take this first game first.

So this is the first game. This is the second game. In the first game, let us consider one lottery. So lottery one. This lottery is the following that b b is occurring with probability, suppose, half; and b o is occurring with probability half.

And another lottery consider the following – o o is occurring with probability 1.

Now, these are two the lotteries I am considering. In l 1, the first lottery, these two players are going to the boxing match with probability half, and any player goes to the boxing match with probability half. And his partner goes to opera house and that he is going to boxing match, occur with probability half. And this is for first player – player 1. Here, I am just considering the preference of player 1.

Now, if this is the case, and if I consider this first game, then this player will be in different between these two lotteries, because with half probability they are going to the boxing match together, from which he is getting two.

With half probability he is going to the boxing match, his wife goes to the opera house, and if that happens, he gets 0 and this is equal to 1.

And if this lottery occurs – o o happens – then he is getting again 1. And I know these two are equal, which means this person – player 1 – will be indifferent between these two lotteries, this lottery and this lottery.

Now, is player 1 indifferent in this game between these two lotteries? The answer is no. So this is game 1, but what happens in game 2?

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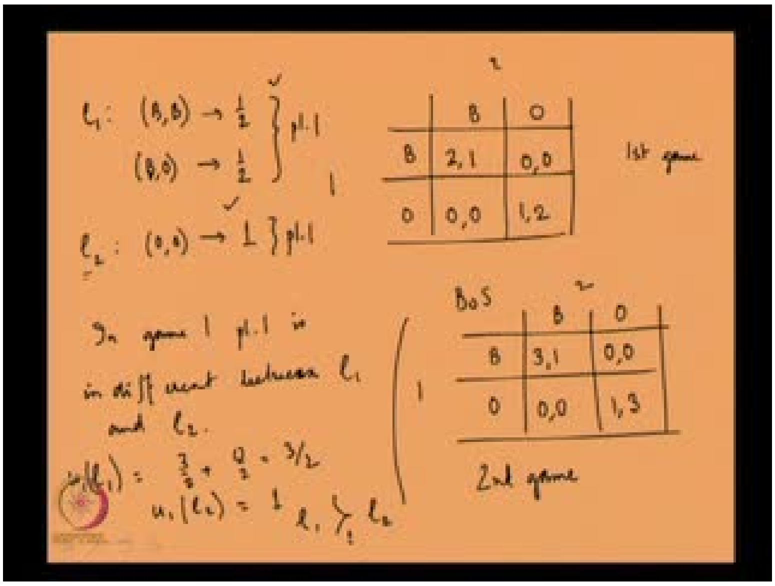

In game 1, player 1 is indifferent between l_1 and l_2. In game 2, if I consider lottery 1, what is the payoff of the player 1 from lottery 1? Here b b is occurring with half probabilities, so 3 by 2, plus B; o is occurring with half probability, so 0 divided by 2, which is 3 by 2. What is his payoff from lottery 2? sorry In lottery 2, 0 0 occurs with certainty, so he is getting 1. So in this case l_1 is preferred by player 1 to l_2.

If I change the numbers, the absolute value changes, whereas qualitatively the numbers do not change. Then the game does not remains the same. The preference of the players are no longer is represented by this game as it was represented by this game.

So one has to be careful about the absolute value of numbers, when one is considering randomization by the players.

So ordiality does not remain if we consider randomization. Now, this is well and good but let me do one exercise, so that the idea becomes more clear.

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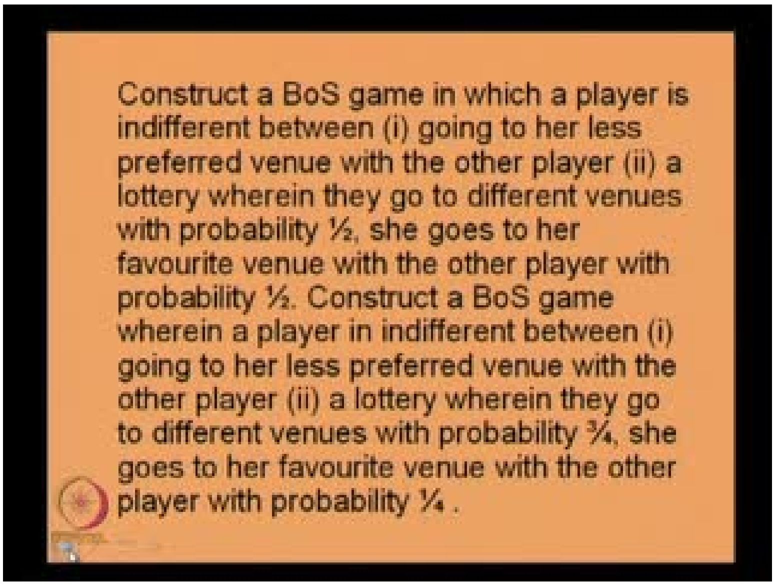

This is the exercise. Construct a BoS game, that is, Battle of Sexes game, in which a player is indifferent between going to her less preferred venue with the other player, and a lottery wherein they go to different venues with probability half, and she goes to her favorite venue with the other player with probability half.

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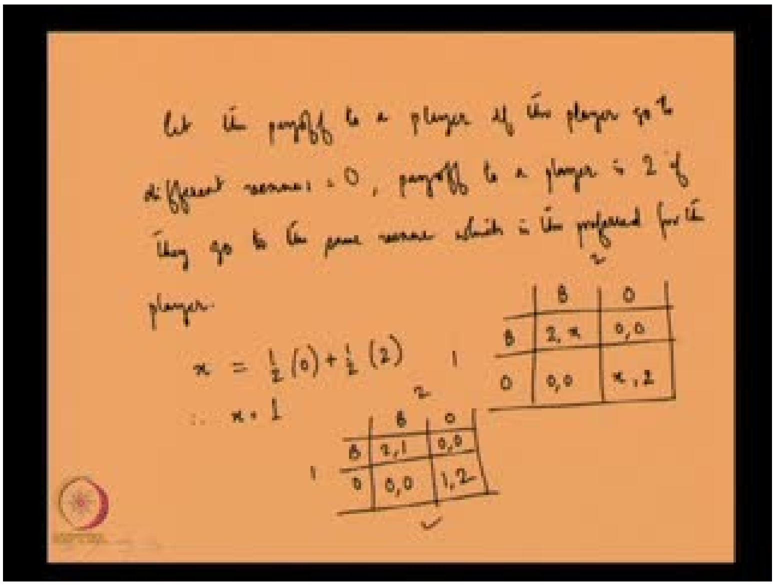

This is the first part and then there is this other part. To solve this problem, let us suppose the payoff to a player, if the players go to different venues, it is equal to 0. The payoff to a player is 2, if they go to the same venue, which is the preferred venue for the player.

If this is assumed, then we have a structure to the game. What we know is that if they go to different venues, the payoffs are 0 0. If they go to the same venue and that venue is the preferred venue of that player, the player is getting 2. So this is 2 and this is also 2. What do we need to know is this value x. x is the payoff to a player when they go to the same venue but that venue is not preferred by that player.

To find out x, what is the hint that is given? What is the clue that is given in the question? The clue is that going to her less preferred concert in the company of the other player, so if you go to the concert which is not preferred to you, you get x. This is indifferent. The person is indifferent to this outcome with the lottery in which, with probability half, she and the other player go to different concert. If they go to different concerts, she gets 0. And with half probability, they go to her more preferred concert. If they go to her more preferred concert, she gets 2. So this is the equation that needs to be solved and obviously from this, we get that x is equal to 1.

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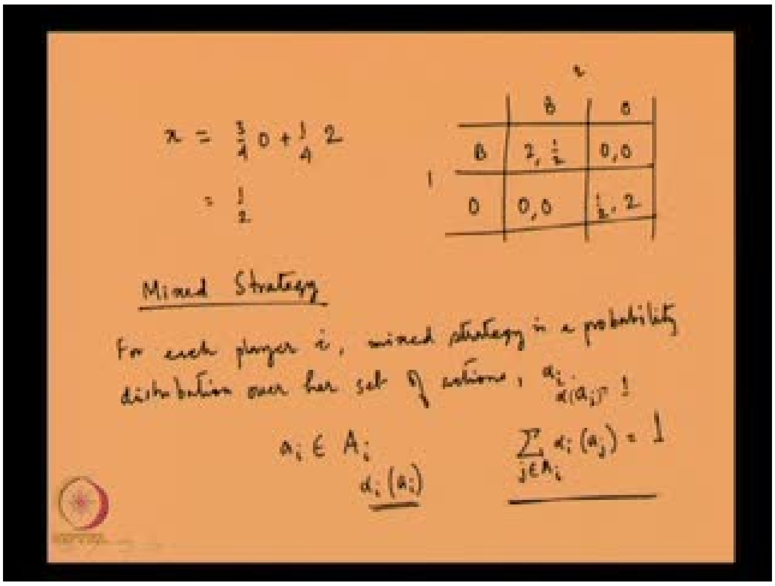

So the game is just the standard Battle of Sexes game that we have been seeing from the beginning itself. This is the game – 2 1 0 0 0 0 1 2. There is another part to this question. So, we retain that this is 2, this is x. Here the clue is the following – do the same in the case that each player is indifferent between going to her less preferred concert in the company of the other player. So this is x. He is indifferent between this event, and the lottery in which with probability three-fourth. She and other player go to different concerts, so 0. And with probability one-fourth, they go to her more preferred concert. If they go to her more preferred concert, the player is getting 2. So this is half. So the game is now a little bit changed in the sense that x is now half, not 1. So this the game that we were supposed to find out. Now, this was the discussion about the basics of the mixed strategy equilibrium analysis.

Now, let us try to define what is known as the mixed strategy Nash equilibrium? Because that is the thing that we are trying to get. To do that, let us first define what is a mixed strategy? Mixed strategy is defined for each player. Mixed strategy is a probability distribution over her set of actions. So it is denoted by, I am going to denote it by, alpha i.

So, alpha i is the mixed strategy of player I. It is nothing but a probability distribution over her action set. So if her action set is A_i, alpha i tells me what are the values of probabilities that are going to be attached to each element of this A_i. So suppose a_i belongs A_i, then alpha i should tell me what is the value of probability attached to this a_i. And this will be represented by alpha i a_i. And obviously, since the summation of the probabilities must adopt to 1, it should happen that this should be equal to 1. If I take all the actions, then the probabilities attached to all these actions must sum up to 1. Notice that this is a probability distribution.

Now, if I degenerate this probability distribution into a distribution, where only one action, let us call it a_1, or say it a_i is equal to 1, and the player has more than one action in his action set the,n this this becomes what is known as pure strategy. Here there is no randomization. He is saying that I am going to take this action a_i for certainty. So, pure strategy is something that we have seen before, where players were taking a particular action. Each player was taking a particular action and there was no randomization. So that was a special case of this general case, where randomization is allowed.

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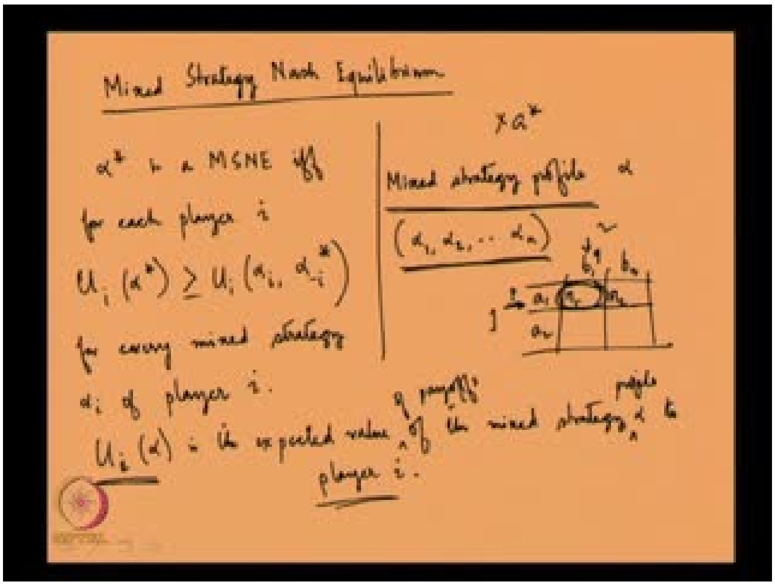

Now since, we have defined what is known as mixed strategy of a player, let us now look at the mixed strategy Nash equilibrium. Well, to tell you before, I tell you the idea of the mixed strategy Nash equilibrium. Let me begin by saying that in mixed strategy Nash equilibrium, the idea is more or less similar to the idea of Nash equilibrium that we have seen before. In the previous case, there were many action profiles. A particular action profile a star was a Nash equilibrium, if given the actions taken by other players in that action profile. Action taken by player I, which is suppose a_i star, is the best possible action that I can take. And this happens for each player and we call that a star the action profile as the Nash equilibrium. Here, players are not taking actions as such. Players are deciding on the mixed strategies that they can take and this mixed strategies are defined over the set of actions.

So here, the crucial thing is not a star, but what is known as a mixed strategy profile, in state of an action profile. So we shall call it as alpha. Alpha is a collection of different mixed strategies, adopted by different players. So player 1 has this mixed strategy alpha 1, player 2 has the mixed strategy alpha 2…, and player n has the mixed strategy alpha n. Alpha is the collection of different mixed strategies and it is called a mixed strategy profile.

Now, we shall call a particular mixed strategy profile alpha star is a mixed strategy Nash equilibrium, if and only if for each player i Ui alpha star, this for every mixed strategy alpha i of player I, and here U i alpha is the expected value of the mixed strategy alpha to player i. So let me just take you through the definition once more. Alpha star is a particular mixed strategy profile, so it is a collection of different mixed strategies. It is called a mixed strategy Nash equilibrium, if an only if, the payoff to each player from alpha star is never less than the payoff to that player, if he takes some other mixed strategy, which is their, which he can take, which is a feasible mixed strategy. And this happens for each player.

This U_i star is the expected value of the payoffs to any player. For example, here U_i, so it is the expected value of the payoff to player i from the mixed strategy profile alpha. So, the story is the following that behind alpha I know that there is all these mixed strategies. Now if, I know these mixed strategies of different players, which is basically the probability distributions. And in each probability distribution, each player is saying that what are the actions, what are the probabilities with which he is going to play his actions.

Now any particular event will happen when the players will take a certain kind of action. For example, suppose this is a_1 and a_2, this is b_1 and b_2, this is player 2, this is player 1. This event a_1 and b_1 happens, if player 1 takes this action and player 2 takes this action. So what is the probability of happening this event. It will be given, by suppose, this is p, this is q, then the probability that this event happens a_1 b_1, happens is p multiplied by q.

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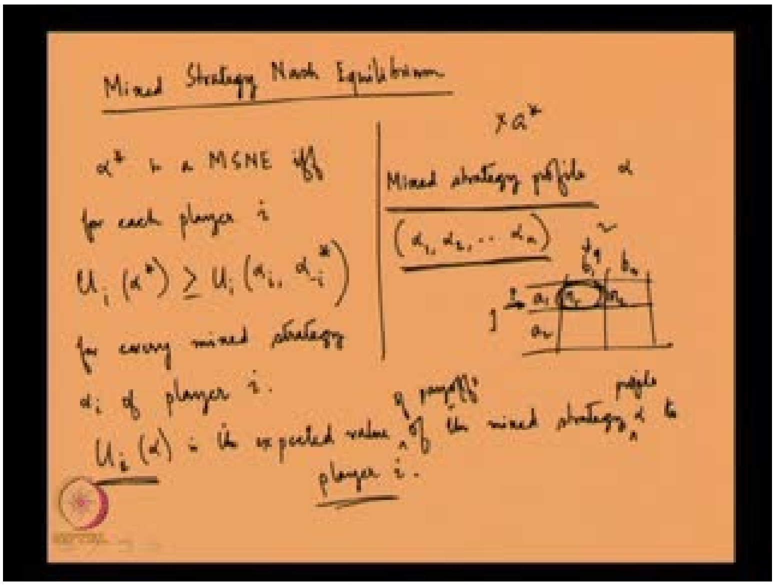

So this is just an illustration to demonstrate the fact that for every event, I have to know what are the probabilities that players are attaching to the actions, which are responsible for that event. And this event, if this happens, then player 1 will get some payoff out of it, let us call it pi_1. And so the payoff to player 1 will be pi_1 multiplied by p q. Similarly, here there will be some payoff to player 1 and behind that payoff, there is this event, and behind this event, there are these probabilities.

So p multiplied by 1 minus q multiplied by, suppose, pi_2. So that is again has to be taken into account. Now by adding such terms what we are going to get is this. That is why I have written that U_i alpha is the expected value of the payoffs of the mixed strategy alpha, which player 1 is getting.

If I look at the mixed strategy profile, alpha i can figure out what are the probabilities, which are responsible or which are behind a particular event. So for each event, I can calculate the probability of that event. And if I do that I can find out the expected value of the payoff to each player.

And if I do that, I can get these numbers. I can compare these numbers and if this is satisfied for each player, then this alpha star is going to be known as mixed strategy Nash equilibrium. So I shall stop here in this lecture. Before we finish what we have discussed in this lecture

We have talked about the expected utility theory and we said that by applying the rule of expected utility theory, we can get an idea about the risk-aversion or risk-loving nature of a person by looking at his or her utility function. How the utility function is shaped? And then we talked about the fact that in mixed strategy Nash equilibrium, where players are randomizing between actions, we are no longer having ordiality of preference. And thirdly, we define what is known as a mixed strategy, and we have defined what is known as mixed strategy Nash equilibrium.

In the next lecture we shall talk about how do find out mixed strategy Nash equilibrium in different kinds of games. Thank you.

<u>Question and Answers</u>

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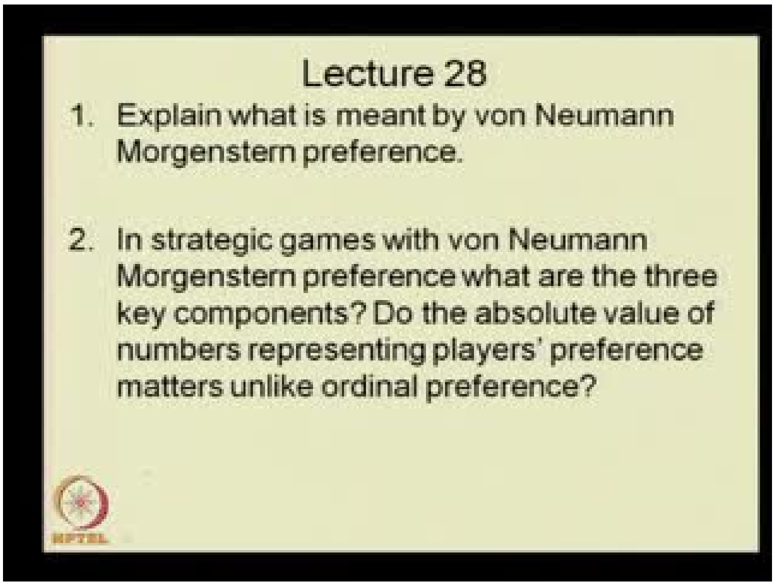

Q: Explain what is meant by von Neumann Morgenstern preference?

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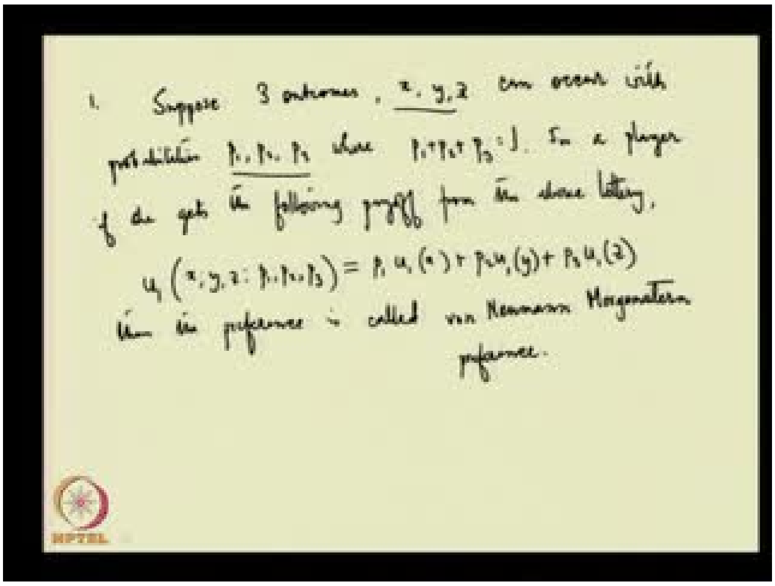

A: n von Neumann Morgenstern preference, what is important is that we are dealing with uncertain situations. So suppose three outcomes x y z can occur with probabilities p_1 p_2 p_3, where p_1 plus p_2 plus p_3 is equal to 1.

Now for a player, if she gets the following payoffs from the above lottery. This is a lottery that three outcomes are occurring with each with some probability. This is called a lottery.

Now what is the payoff of the player from this lottery. Remember it is not the case that any particular outcome is occurring, so we cannot talk about outcome and the related payoff to that.

What we can say is that what is the payoff the player is getting from the lottery itself, not from each outcome? For a player, if she gets the following payoff from the above lottery given by U_1 and we can write it as the following.

So before the colon, the outcomes are there; and after the colon the probabilities that respective probabilities are there. Then the preference is called preference von Neumann Morgenstern preference, which means that the payoff from the lottery itself is the expected value of the payoffs from individual outcomes. If that can be said to be occurring, then we can say that the preference is von Neumann Morgenstern preference.

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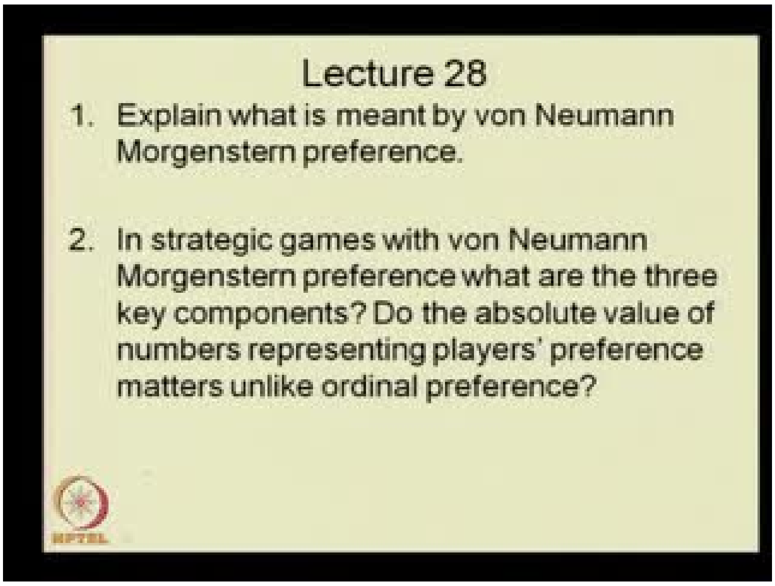

Q: In strategic games with von Neumann Morgenstern preference, what are the three key components? Do the absolute value of the numbers representing the players preferences matter, unlike ordinal preference?

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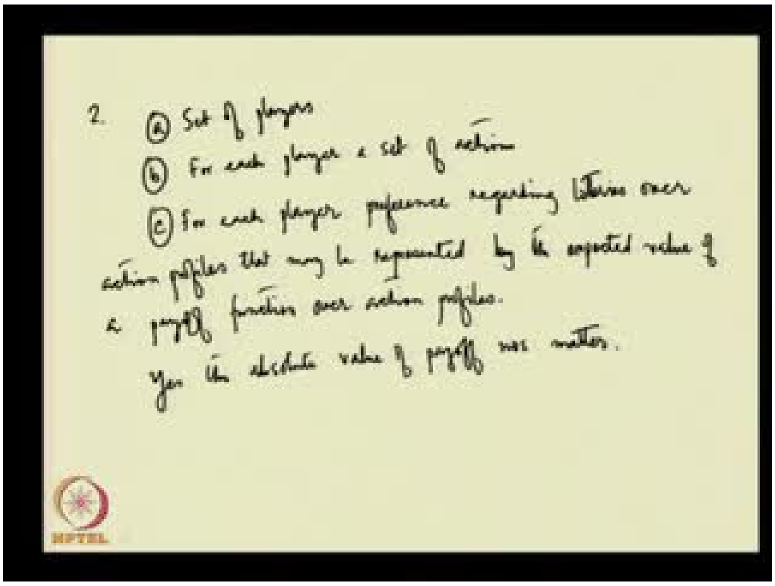

A: So the three components one like before the set of players, say b, for each player set of actions and c for each player preference regarding… ((no audio 60:16 to 61:02)) For each player, we have to specify the preference and this is specified by the following that the preference regarding lotteries over action profiles that may be represented by expected value of payoff functions over action profiles. And the last point is that do the numbers matter? Yes, the absolute value of payoff numbers matter. Thank you.