Game Theory and Economics
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Module No. # 04
Mixed Strategy Nash Equilibrium
Lecture No. # 03
Mixed Strategy Equilibrium: Concepts and Examples

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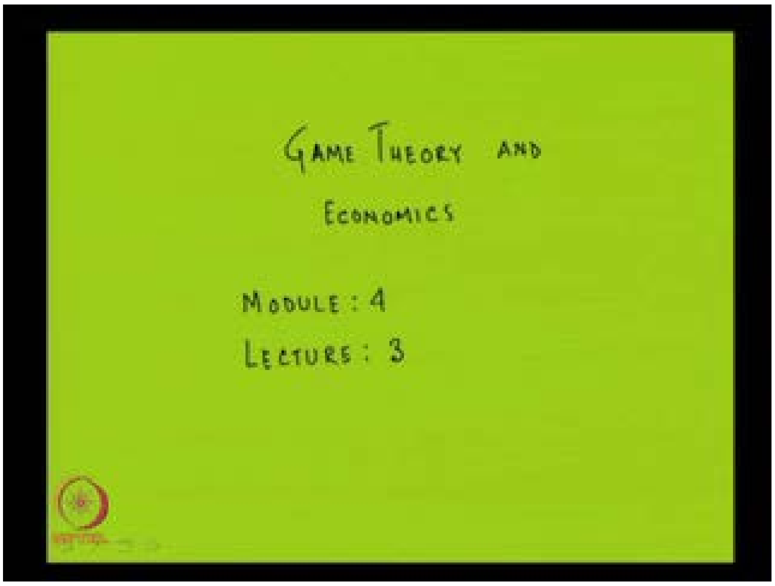

Welcome to third lecture of module 4 of this course called game theory and economics. So far, what we have done is that we have defined mixed strategy Nash equilibrium, in the course of the previous lecture. Let me go over it once again so that we can show how it can be applied to certain games. So, the following was the definition.

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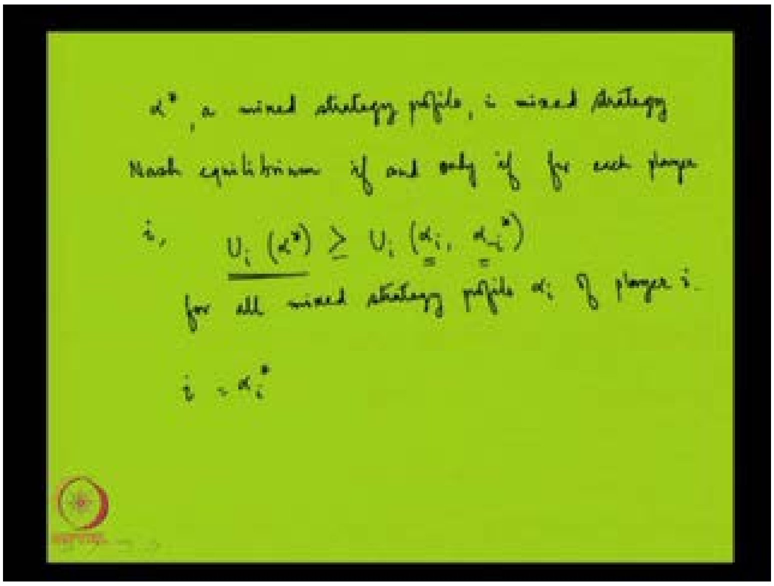

Alpha star, a mixed strategy profile, is mixed strategy Nash equilibrium, if and only if, for each player i... So, this is the definition. What we are saying is the following. Each player has a mixed strategy and for player I, this mixed strategy is alpha i star and what is a mixed strategy profile? A mixed strategy profile is a collection of the individual mixed strategies of different players.

So, a particular mixed strategy profile alpha star will be called mixed strategy Nash equilibrium, if by playing this alpha star, if everybody plays, star actions that is, alpha 1 star, alpha 2 star etcetera alpha n star, then for any player i, the expected payoff from that mixed strategy profile is either greater or equal to the expected payoff that player will get, if he plays some other mixed strategy which he is able to play and that representative mixed strategy, which he can play is denoted by alpha i.

Given the other players - the rest of the players are sticking to their star mixed strategy; that is, the other players are playing alpha not i star. If this is satisfied then we say that alpha star is a mixed strategy Nash equilibrium. So, this is the definition. What is to be remembered is that here, this capital U is the payoff function of players under Von Neumann-Morgenstern preference. So, these are the expected payoffs. These are not the kind of ordinal payoffs that we have seen before.

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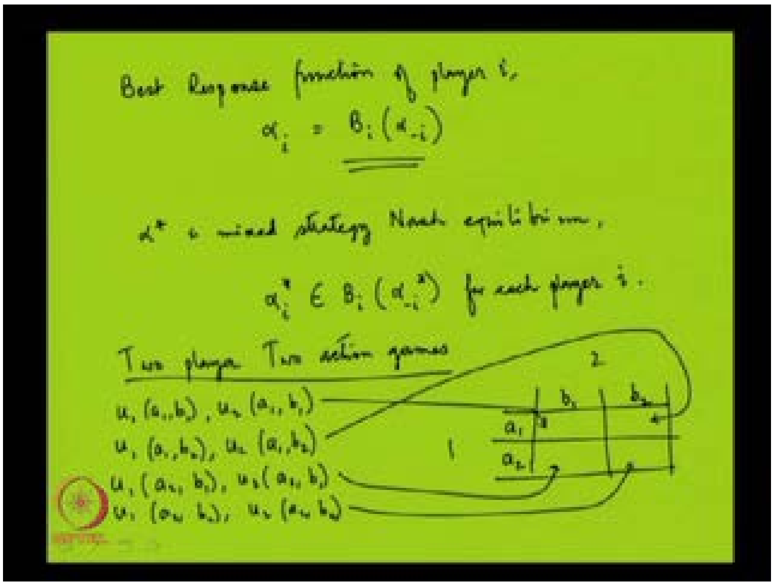

Now, this is as far as the definition goes, but while trying to find out the Nash equilibrium of a particular game or Nash equilibria of a particular game, it might be helpful to take help of what we have seen before, what is known as the best response functions.

So, here also you can use the idea of best response functions, but here the best response function of player i is given by this and this is given by an alpha i, which means that the best response functions of players are not giving us some particular action, but they are giving us a mixed strategy. We are not saying a i; this is not what we are saying; we are saying alpha i, which is basically a mixed strategy of probability distribution over the set of actions. This again is a function of alpha not i. So, it is not a function over action profiles, but it is a function over mixed strategy profiles; so that is it.

Now, if I know the idea of best response functions, defined in terms of in the case of mixed strategy, then it might be easy to define Nash equilibrium in terms of mixed strategy, in terms of best response functions.

So, how do I define it? It is like before. So, alpha star is mixed strategy Nash equilibrium. The idea is as before; everybody must be optimizing, given the action of the others. So, given the actions of the others, if everybody is optimizing and people find a particular set of mixed strategies such that each of the mixed strategies is the best response to the rest of the mixed strategies, then that is a Nash equilibrium. So, in terms of best response functions, it can be written as the following.

For each player i, the action the mixed strategy which we can get from this alpha star must be belonging to his best response function given the other players are playing alpha not i star. So, this is the basic idea.

Now, we shall now try to find out in concrete situations, how mixed strategies Nash equilibria are derived or calculated. So, we shall start with the simplest situation of two players, two action games then we shall try to see whether we can use the intuition from this two player, two action games to games which are more complicated, where there are more than 2 actions or may be more than 2 players.

Let us take any general case. Suppose, there are these two players: player 1 and player 2 and suppose, their actions are a 1, a 2, b 1, b 2.

Now, in this case, if the players are adopting mixed strategies and their payoff functions are the following that u 1 a 1, b 1, u 2 a 1, b 1. So, this belongs to this cell u 1 a 1, b 2; this is here and this is here. So, we are not writing any numbers, just writing the general expression for each of the Bernoulli payoff functions that will be derived by this combination of actions.

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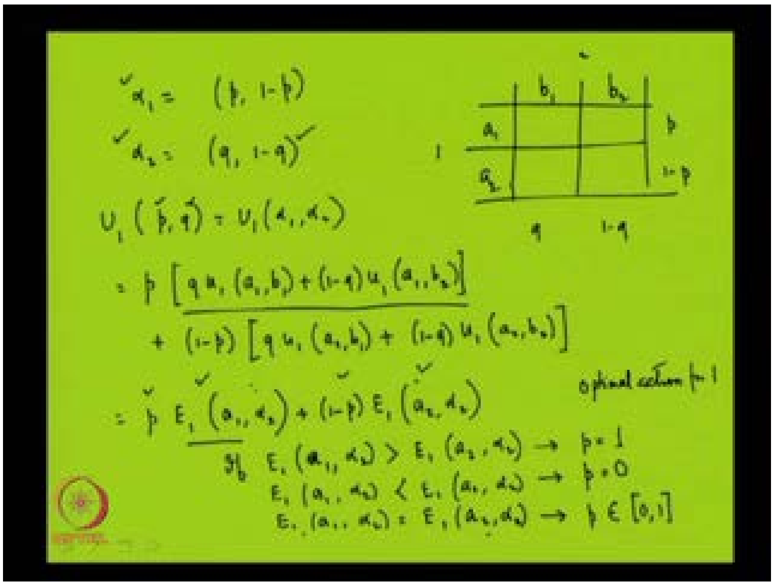

Suppose, alpha 1 that is the mixed strategy of player 1 is p and 1 minus p and alpha 2, the mixed strategy of player 2 is q and 1 minus q. So, in this case, if this is the mixed strategy of player 1 and this is the mixed strategy of player 2 then what is the payoff of player 1 from this set of mixed strategies? That we can find out. So, let us write it as the following.

Just p and q. In the sense that it is nothing, but alpha 1, alpha 2 and I know alpha 1 is equal to p and 1 minus p, but writing 1 minus p is redundant because there are two actions. If I know the value of p, I know that the other action is taking the probability 1 minus p. So, I need not write 1 minus p. So, that is why just p and just q. Now, what is the payoff to player 1 in this situation, if he is playing the actions with p and 1 minus p probabilities and the other player is playing the actions with q and 1 minus q probabilities? So, it will be the p Just to repeat the diagram once again.

This is player 2; this is player 1. This is played with q and 1 minus q; this is played with p and 1 minus p. Now, what is the payoff to player 1 from these mixed strategies? First p, the probability of playing this action multiplied by what the player 1 gets by playing a 1. If player 1 plays a 1 with q, player 2 will play b 1, in which case, his payoff will be a 1, b 1, but with 1 minus q probability, player 2 will play p 2 in which case he will get a 1, b 2.

If player 1 plays the second action, that is a 2, then again with probability q, player 2 will play b 1, in which case, the payoff is this much and if player 2 plays b 2, this is the payoff, alright.

Now, if I look at this expression, the expression in the third bracket - square bracket then what is this? This is giving me the expected payoff to player 1, if he plays the action a 1. So, I can write it in short form as expected payoff to player 1, if he plays a 1 and player 2 plays alpha 2, which is this one - this mixed strategy, plus the expected payoff to player 1, if he plays a 2 and player 2 sticks to his mixed strategy alpha 2.

So, this is the payoff to player 1 - the expected payoff to player 1, which is once again a weighted average it seems, the weights being p and 1 minus p. It is a weighted average of two expected payoffs - this E 1and this E 1and this E 1is telling me, what is the expected payoff to player 1, if he plays a 1 and this E 1is telling me, what is the expected payoff to player 1, if he plays a 2 and the weights being the probabilities with which player 1 plays a 1 and a 2.

Now, from this, it is easy to gauge what the optimal action of player 1 will be. It depends on whether E 1 a 1, alpha 2 is greater than or less than or equal to E 1 a 2, alpha 2. For example, if E 1 a 1, alpha 2 is greater than E 1 a 2 alpha 2 then what is the optimal action for player 1? In that case, this is higher and player 1 has control over p. So, in this case, player 1 will set the value of p is equal to 1 because that is the higher value attached to p. So, p will be equal to 1; so this is the optimal action for player 1.

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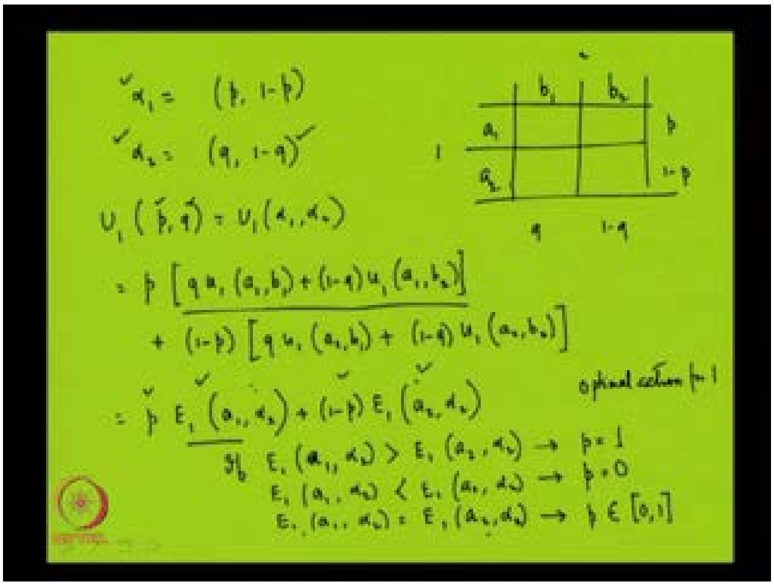

Similarly, if it goes the other way. So, this value is higher than this value. In that case, this 1 minus p will be set to 1 and 1 minus p will be 1, if p is equal to 0; so p is equal to 0. If they are equal then it does not matter; this value is equal to this value. So, it does not matter what value of p, one assigns to it. The total value of U 1 remains at each of these values.

So, in that case, p can take any value; it can take any value between 0 and 1. So, this is the this is one of the basic ideas that we are getting from here is that for any player, if I have to find out the best response function of that player to the mixed strategy the other player has devised, then the best response function of a player can be find out by comparing the expected payoffs of the actions of this player.

If the expected payoff of the first action is higher then the probability attached to the first action will be equal to 1; If the expected payoff from the first action is less then the probability attached to that will be 0 and if they expected payoffs of these two actions are same then p can take any value from 0 to 1. This kind of calculation can be done for player 2 also and if I have these two calculations - these two best response functions so as to say, then I can plot this best response function and find out what is the Nash equilibrium. So, that is more or less the way, we are going to solve the two player, two action games.

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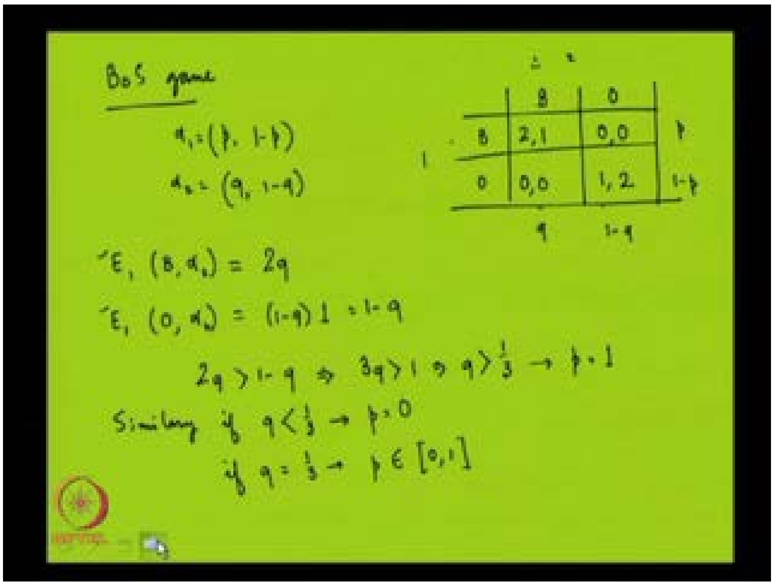

So, let us start with the battle of sex game, which is a familiar game and let us try to see what are the Nash equilibria in this game. So, this was the BoS game. Like before, the method that we have just. What we are going to assume is that these players are playing the actions with some probabilities and the mixed strategy of player 1 is p and 1 minus p, for player 2, it is q and 1 minus q. Now, what we have seen before is that it depends on how the expected payoff to player 1 from these two actions compare.

So, expected payoff to player 1 from action B given that alpha 2 is being played is what? It is 2 multiplied by the probability with which player 2 is playing B, which is 2 q plus 0 because here this payoff is 0. So, I need not bother about that. If player 1 plays the action O whereas, player 2 is playing the mixed strategy alpha 2, then from this, he is getting 0.

So it does not matter. From this, he is getting 1 minus q multiplied by 1; so, this is just 1 minus q.

Now, as we have just seen that if this value is greater than this value, p will be equal to 1. What does that mean? 2 q is greater than 1 minus q, which means 3 q is greater than 1, which means q is greater than one-third.

So, if q is greater than one-third, then player 1 is going to play B with probability 1 and similarly, we can show that if it goes the other way, that is, if q is less than one-third, then the expected payoff from O is going to be higher than expected payoff from B and in that case, p will be equal to 0. If q is equal to one-third p can take any value between 0 and 1 and these are some of the results that we are getting.

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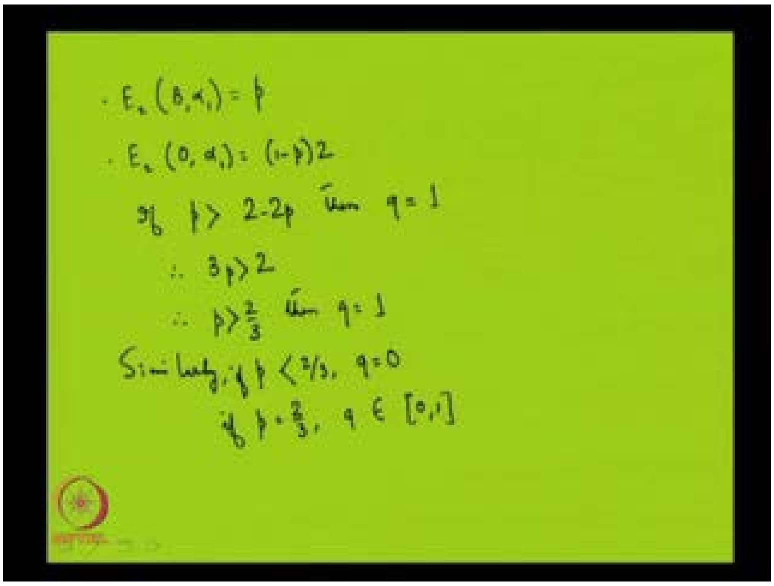

What about player 2? Player 2, I have to look at this value. The expected payoff to player 2, if he plays B and player 1 plays alpha 1. (Refer Slide Time: 23:20) If player 2 is playing B, he is getting 1 with probability p and if he plays from here, he is getting 0. If he is playing O then with 1 minus p probability he is going to get 2.

So, if this is higher than this, p is higher than 2 minus 2 p, then q is going to be equal to 1. This expected payoff is higher than this expected payoff and so, the person player 2 is going to attach probability 1 to the first action which is B; so, q is equal to 1. What does this mean? It means 3 p is greater than 2 or p is greater than 2 divided by 3; if this happens then q is equal to 1. Similarly, I can deduce that if p is less than two-third, q will be equal to 0. If p is equal to two-third, q can take any value between 0 and 1.

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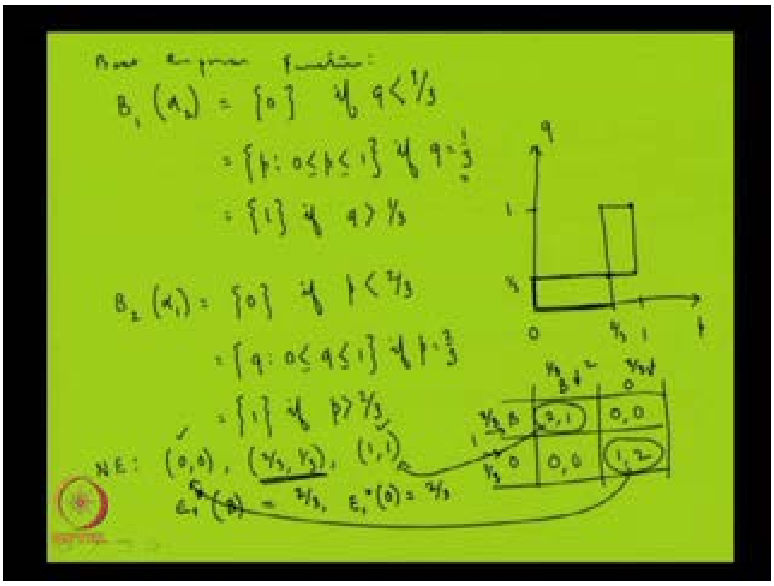

So, to sum up all these things, what we have seen so far, let me write the following. The best response function of player 1 to the action taken by player 2 is given by what? What I am going to write is, the probability the player 1 is attaching to his first action, that is the value of p. (Refer Slide Time: 25:36) Now, we have seen that if q is greater than one-third, p is equal to 1; if q is less than one-third, p is equal to 0; so, this is 0, if q is less than one-third.

So, this is the best response function of player 1 and similarly, the best response function of player 2 is given by the following. If p is less than two-third, q is equal to 0; if p is greater than two-third, q is equal to 1. So, let us try to see how it translates in terms of diagrams.

Let us suppose this is 1 and this is 1. Now, what is the critical value of q? Critical value of q is one-third. Suppose, this is one-third and if q is less than one-third, what we are having is p is equal to 0. So, this is the best response function.

If q is equal to one-third, p can take any value. So, I get all the way to 1 and if q is greater than one-third, sorry, this is should be 1, then p is equal to 1. So, this is the point 1, 1 and then we look at the best response function of player 2, if the critical value of p is two-third. If p is less than two-third, q is equal to 0; so, I am coinciding with the horizontal axis. If p is equal to two-third, q can take any value; so, I get a vertical part and if p is greater than two-third, q is equal to 1.

So, these are the best response functions and one can see that there are three Nash equilibria here, mixed strategy Nash equilibrium that is. One is occurring here, the other is here and the last one is here. So, the first one will be given by p is equal to 0, q is equal to 0, the second one is given by two-third, one-third and the third-one is given by 1, 1.

(Refer Slide Time: 30:05) Now, this 1 and this 1 are basically PO strategy Nash equilibrium. If you remember in PO strategy Nash equilibrium, the probability distribution degenerates to a case, where a particular action takes the value 1. So, in this case, that is what is happening. If you recall the game, these two Nash equilibria can be seen directly without taking recourse to mixed strategy and best response functions.

So, this was the game. Now, it is obvious that this is a Nash equilibrium and this is also a Nash equilibrium. Here, we are getting that Nash equilibrium of 1, 1. So, this is this. With probability 1, B is played by player 1 and with probability 1, B is played by player 2 and this relates to this. With probability 0, player 1 and 2 play B. So, these are the Nash equilibrium which we have seen before, but the new Nash equilibrium which is a proper mixed strategy Nash equilibrium is this two-third, one-third which is happening here. So in this Nash equilibrium player 1 is playing action B with two-third probability and player 2 is playing the same action - action B with one-third probability. So, that is that.

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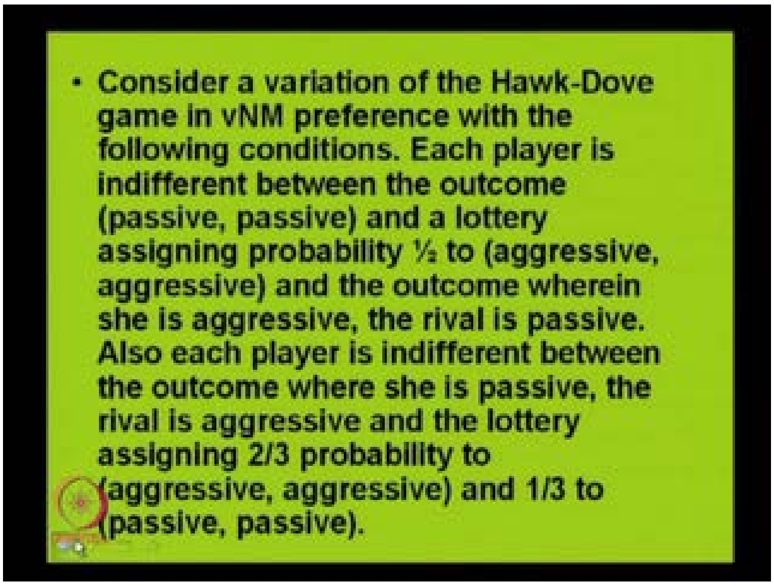

What we are going to do now is take another exercise and see how we can find Nash equilibria in a different situation.

This is the question. Consider a variation of the Hawk-Dove game in Von Neumann Morgenstern preference with the following conditions. Each player is indifferent between the outcome passive, passive and a lottery assigning probability half to aggressive, aggressive and the outcome wherein she is aggressive and the rival is passive. Also, each player is indifferent between the outcome where she is passive, the rival is aggressive and the lottery assigning two-third probability to aggressive, aggressive and one-third to passive, passive.

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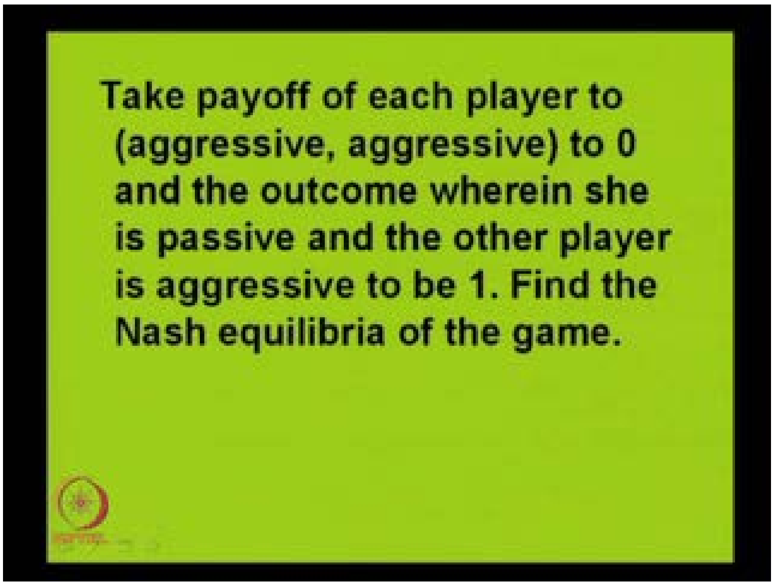

Take payoff of each player to aggressive, aggressive to 0 and the outcome wherein she is passive and the other player is aggressive to be 1. Find the Nash equilibria of the game. There are two stages of this exercise. First we have to find out what are the numbers, which represents this preference - this Von Neumann Morgenstern preference and then we have to find out the Nash equilibria or the mixed strategy Nash equilibria of the game.

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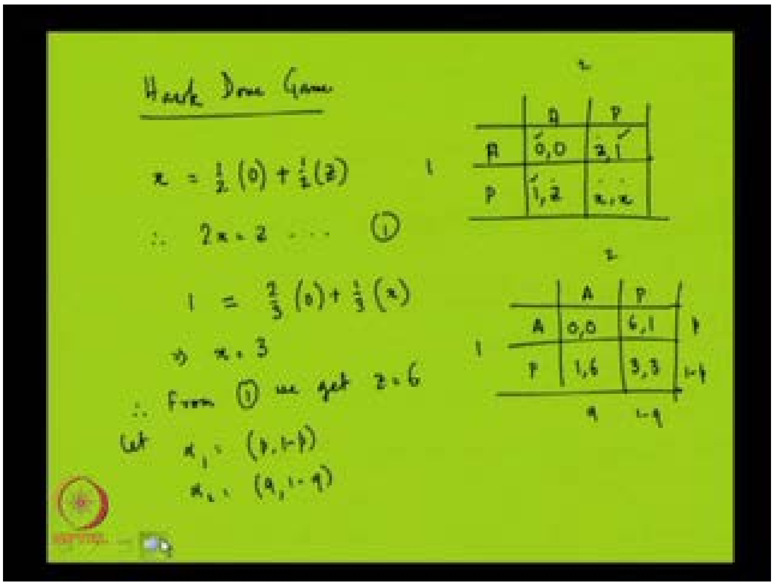

Now, this is the Hawk-Dove game. To refresh your memory, the Hawk-Dove game that we were familiar with, look like the following. So, this is aggressive, aggressive of the Hawk action and this is the passive action. If both of them are aggressive, they get 0 each; if I am aggressive, I get 3, if the other person is passive and the other person gets 1, in that case and if both of them are passive, they get 2 each. So, this was the game and these were the numbers that we have seen before in the ordinal preference game.

Now, in this Von Neumann preference variation what is say it what the condition that has been given is that each player is indifferent between the outcome passive, passive, but this two may not be there and what numbers can we give to these outcomes. One clue is given in the game itself - in the exercise itself. What is said is that each player's payoff to aggressive, aggressive is 0. So, this is already there; it is 0, 0 and each player's payoff to the outcome in which she is passive and the other player is aggressive to be 1; so, this is also there.

What is not there and what we need to find out is this number 3 and this number 2? So, they may be unknown and these numbers 3 and 2 may not be there given the preference that is specified in the question.

So, since these are not this may not be valid, let us represent them by some symbol. Let us call it z and this symbol, let us call it x. Now, the condition that is given in the game is that each player is indifferent between the outcome passive, passive. If each player is playing passive, each player is getting x and x should be equal to the lottery that assigns probability half to aggressive, aggressive. If aggressive, aggressive is the outcome then each player is getting 0.

So, with probability half, a player is getting 0. So, this is the payoff and with probability half to the outcome in which she is aggressive and the other player is passive, when she is aggressive, the other player is passive, one gets z. So, with half probability, this occurs. So, I am taking the expected value. We get 2 x is equal to z. So, this is one condition. What is the second condition? Each player is indifferent also between the outcome in which she is passive and the other player is aggressive

So, if I am passive and the other player is aggressive, I get 1. This is equal to the lottery that assigns two-third to the outcome aggressive, aggressive which is 0 and one-third to passive, passive which is x. So, from this condition itself, I can find out the value of x which is 3 and if I substitute it in 1, we get z is equals to 6. So, x and z are known now and the game therefore, looks like the following.

So, this is the game. What is now needed to be found out is that, what are the Nash equilibria of this game - the mixed strategy Nash equilbria of this game. Now, the method that we are going to adopt is something which we have just discussed. We shall suppose that alpha 1 is p 1 minus p and alpha 2 is q 1 minus q. So, these are the probabilities attached to these actions and we have seen that it depends on the expected payoff to each player from each of the actions.

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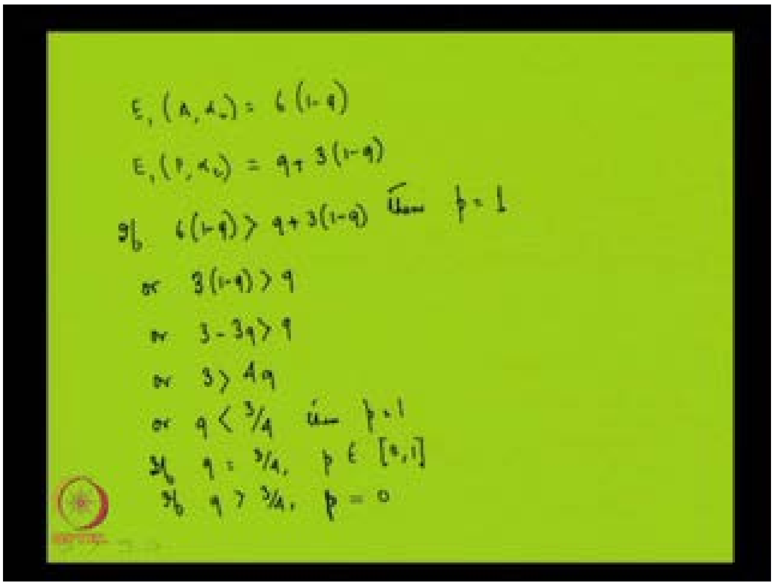

He will compare the expected payoff from the actions and decide what is the optimal p or what is the optimal q. (Refer Slide Time: 40:05) Now, expected payoff to player 1 from action aggressive is given by, if he plays aggressive, from this he gets nothing, from this he gets 1 minus q multiplied by 6; if he plays passive then the expected payoff is q plus 3 multiplied by 1 minus q. So, these are the expected payoff and if the first value is greater than the second value then p is going to be 1 and the other cases follow from there.

So, if p is equal to 1 and now, we can derive this. As 3, 1 minus q is greater than q or 3 minus 3 q is greater than q or 3 is greater than 4 q or q is less than 3 by 4. So, if q is less than 3 by 4 then p is equal to 1 and from here, it means that if q is just equal to 3 by 4 then p can take any value; if q is greater than 3 by 4, p will take a value 0. So, this is the best response function of player 1. I will write it in a more neat way this thing in the next page may be, but let us now try to see what is the best response function of player 2.

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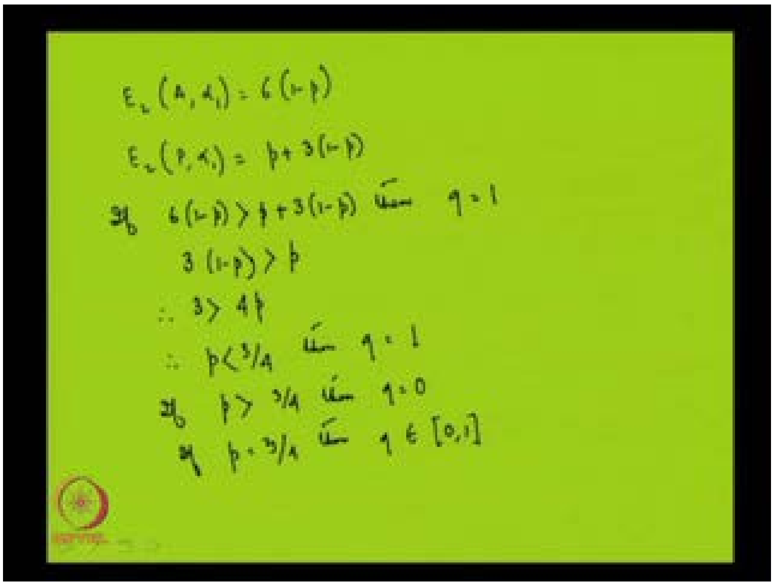

(Refer Slide Time: 42:34) Now player 2's, from his point of view, if he plays A, the expected payoff to him is 6 multiplied by 1 minus p and if he plays passive, given that player 1 is playing alpha 1, then he is getting p plus 3 multiplied by 1 minus p. So, like before, if this happens then q is equal to 1; all the probabilities will be attached to the first action the A action and this can be simplified as and the rest of the formula as shown in the slide. So, if p is less than three-fourth then q is equal to 1. Similarly, if p is greater than three-fourth then q will be equal to 0. If p is just equal to three-fourth then q can take anywhere between 0 and 1.

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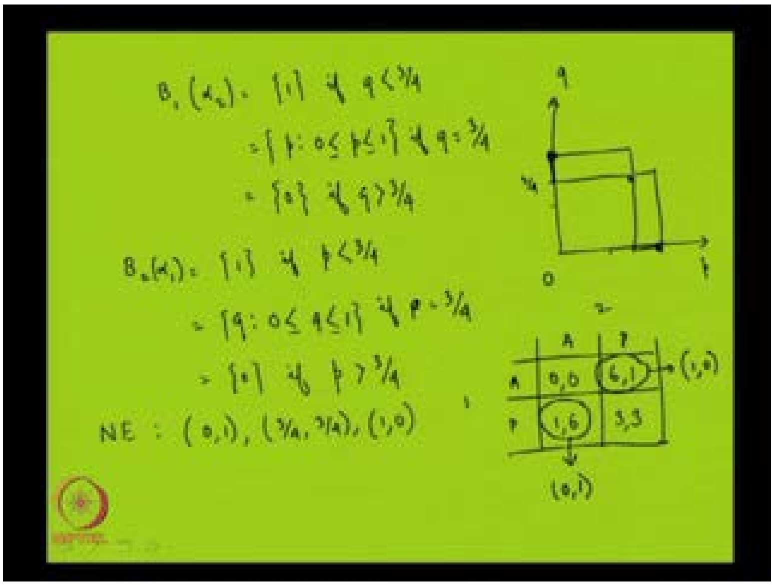

So, the best response functions will be the following. If q is less than three-fourth, p is equal to 1; if q is greater than three-fourth, p is equal to 0. So, let us draw the diagram, draw the best response function. So, the critical value for q is three-fourth. This is suppose, three-fourth, if q is less than three-fourth, p is equal to 1; if q is equal to three-fourth p can take any value; if q is greater than three-fourth p is 0. The best response function of player 2: if p is less than three-fourth, q is equal to 1; if p is equal to three-fourth, then q can take any value between 0 and 1 and q takes the value 0, if p is greater than three-fourth.

Let us suppose this is three-fourth value of p then if p is less than three-fourth, q is equal to 1; if p is equal to three-fourth, q can take any value and if p is greater than three-fourth, q is equal to 0.

So, once again here, what we have are three Nash equilibrium. One is here, one is here, third-one here so they are given by 0 1 this point is three-fourth three-fourth, the last point is 1 0

Now, recall the game. There it was 3, 3, 6, 1, 1, 6. Now, it is obvious that this is a Nash equilibrium because given the first player is playing P, the second player will not deviate and similarly, given the second player is playing A, the first one will not deviate. So, these two are Nash equilibrium and this is nothing, but 0, 1 and this is nothing, but 1, 0 and these are the Nash equilibrium. We have seen here and here.

The third Nash equilibria, which we have not seen with the aid of pure strategy Nash equilibria is that this Nash equilibrium, where three-fourth is the probability attached by each player on the action A; so, that is it.

So, this was the case of two players and two actions. Now, it might be asked that what happens if suppose, the number of players is more than 2 or the number of actions is more than 2, then how do we find out the Nash equilibrium. Now, to deal with such situations, what we are going to use is a very useful characteristic of mixed strategy Nash equilibrium and this characteristic is something which we have seen before.

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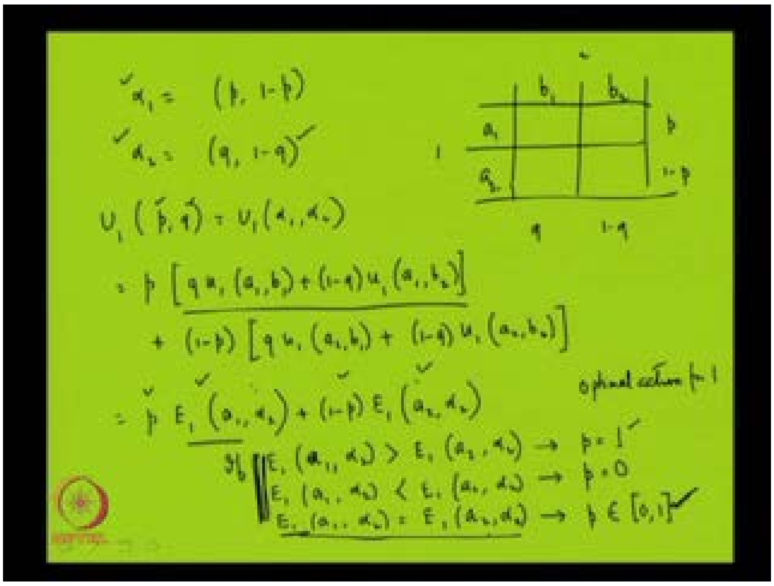

Let me go back a little bit. Here, this was the general case of two player, two action. Now, let us look at this part. From this part, it seems that player 1 attaches positive probability to both his actions, only in this case; in the other cases, one of the action is having a probability 0 and the other action is getting the probability 1; only in this case, a particular player, it might be player 2 also, is attaching positive probabilities to both of his actions and what is the special thing about this case.

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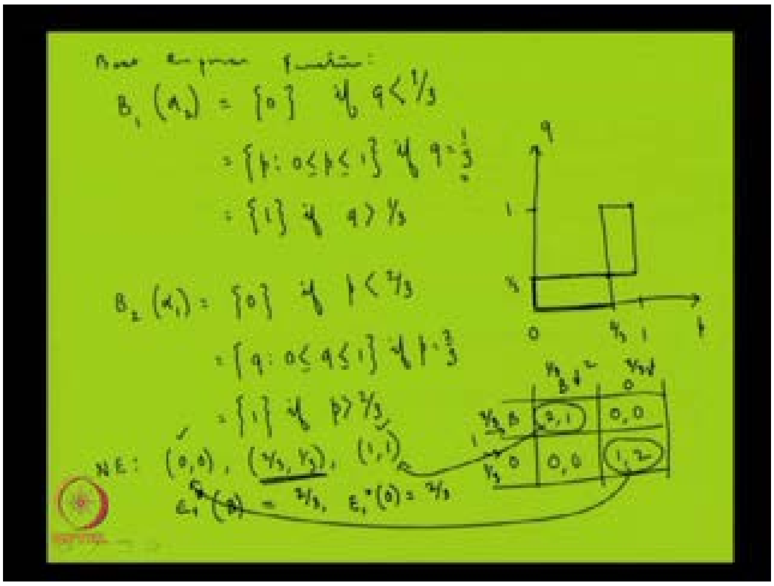

This case’s speciality is that here, the expected payoff from both the actions are equal. So, this is an important clue that if the expected payoff of more than two actions, because remember, we are talking about more than two actions, if it so happens that the expected payoffs of more than one action, they are equal, then the probabilities attached to them might be positive. If you remember in the Nash equilibrium also, if I take any Nash equilibrium, let us take this Nash equilibrium, the proper mixed strategy Nash equilibrium is this one - two-third, one-third. Now, if player 1 plays this action with two-third and he plays this action with one-third, then obviously, the expected payoff to player 1 from these two actions will be equal.

Let us look at this. What is player 1’s expected payoff in Nash equilibrium from the action B? This is in a Nash equilibrium; so, I am attaching a star here. It is given by 2 divided by 3. An expected payoff to player 1, if he plays O, is again 2 divided by 3. So, they are equal. Similarly, we will see that the expected payoff to player 2 in the mixed strategy Nash equilibrium from these two actions will be equal; in this case, it is going to be again two-third. So, this expected payoff and this expected payoff will be same. So, the point I am trying to make is the following.

In a proper mixed strategy Nash equilibrium, where the probabilities attached to the actions are positive, then their expected payoff of these actions must be equal and this is the characteristic of any mixed strategy Nash equilibrium and the reason is the following.

That if the expected payoffs are not equal, if some expected payoff is higher than the other expected payoff of two actions suppose, then there is no reason why the action which is giving me less expected payoff should have any positive probability. It would have then a 0 probability and the other action which has a higher expected payoff, if it is a single action which is having a higher expected payoff, it will get the probability 1 or it may happen that the two actions are having same expected payoff and the third action is having an expected payoff less than the value of other expected payoffs.

In that case, these first two actions which have more expected payoff may have positive probabilities not equal to 1 and the last action which is having a less expected payoff will have a 0 probability attached to it in the equilibrium; that is the optimal decision of the concerned player. So, this is an important characteristic of Nash equilibrium which we are going to explore in the next class and which will enable us to check whether a particular mixed strategy profile of different players is indeed a mixed strategy Nash equilibrium or not.

So, that we shall take up in the next class. Thank you.

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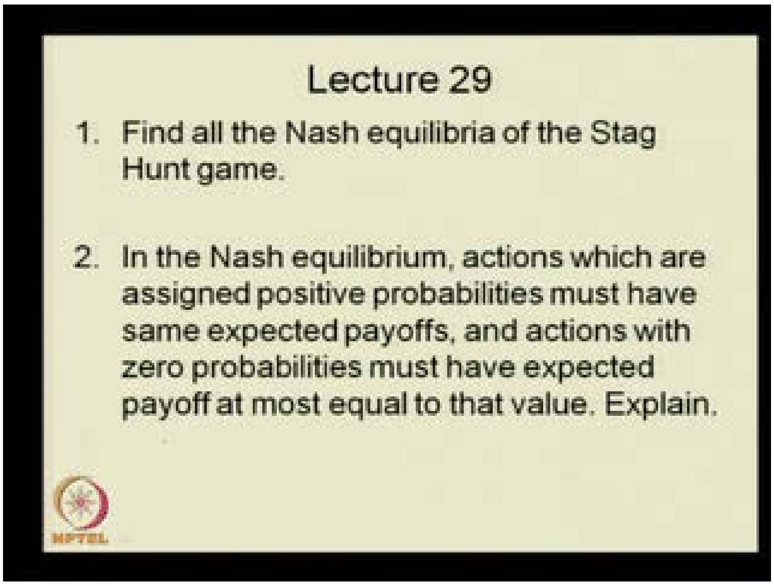

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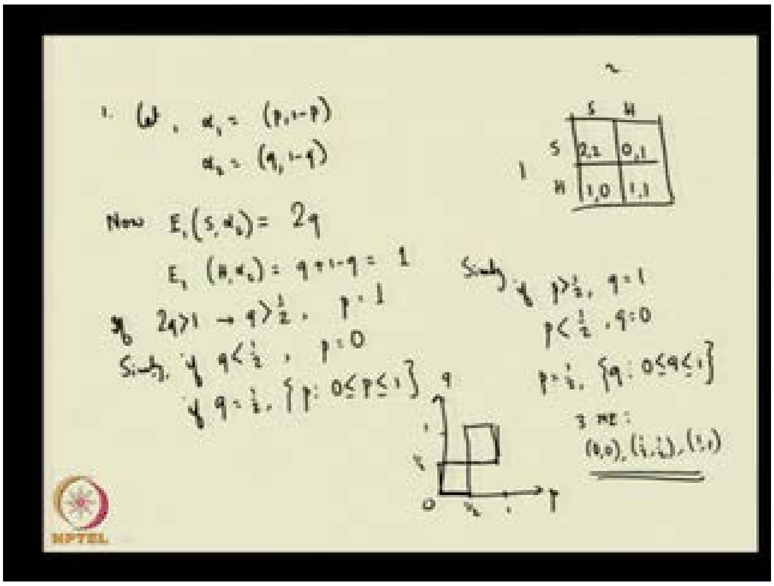

Find all the Nash equilibria of the Stag Hunt game. What was the Stag Hunt game, let us remember. So, this was the Stag Hunt game. We have to find all the Nash equilibria. Let alpha 1 that is the mixed strategy of player 1 be given by p and 1 minus p and alpha 2, mixed strategy of player 2 is given by q and 1 minus q. Now, the expected payoff to player 1, if she plays S and the other player is playing alpha 2, it is given by 2 multiplied by q and the expected payoff to player 1, if she plays H is given by this which is equal to 1. So, if 2q is greater than 1, which means q is greater than half, the expected payoff from S is greater than expected payoff from H. So, in that case, S will be played with certainty which means p is equal to 1.

Similarly, if q is less than half, p will be equal to 0; if q is equal to half, p can take any value - any value of p is optimum. Similarly, we can find out the best response function of player 2 also, which will be if p is greater than half, q will be equal to 1; p is less than half, q is equal to 0 and p is equal to half, then q can take any value. So, we have to basically plot these two best response functions and find the Nash equilibria. So, let us say, this is 1 and this is 1, this is half, this is half; if q is greater than half, p is equal to 1 and then it can take any value and then we are here and if p is greater than half, q is equal to 1, then we have this downward straight line. So, there are three Nash equilibria and the probabilities attached to p and q are the following 0, 0, half, half, 1, 1.

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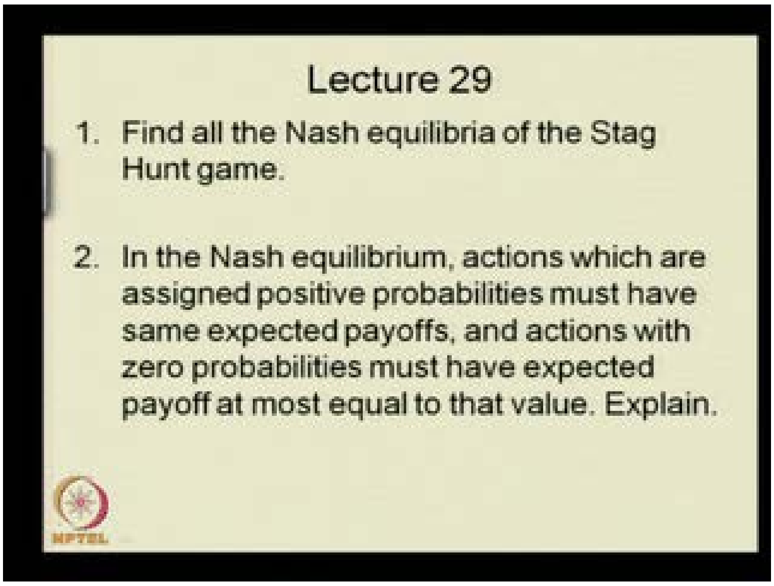

In the Nash equilibrium, actions which are assigned positive probabilities must have the same expected payoffs, and actions with 0 probabilities must have expected payoff at most equal to that value. Explain

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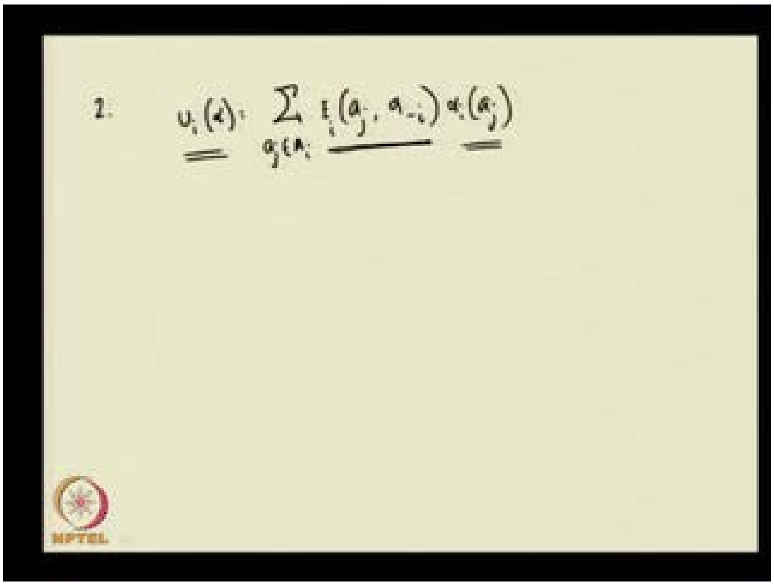

So, we know the following. The expected payoff of a player from any mixed strategy profile is given by this. Basically, we are talking about the expected payoff from each of the actions of that player and multiplying each of those expected payoffs with the probabilities with which that action is played by player i.

Now, here in equilibrium obviously, this has to be optimal. Now, if this is an optimal, suppose for some actions, the expected payoffs are there. The point is that if you assign positive probabilities to those actions, the expected payoffs have to be same because if one of the expected payoffs is more, then there is no question of assigning positive probabilities to other actions because all the probability will go to that action, which has a greater expected payoff.

So, players are randomizing in some actions; that is why, their expected payoff must be equal. Similarly, for actions which have lesser expected payoff, in that case, the probability will be equal 0, if expected payoff is less. So, that is it. Thank you.