Game Theory and Economics

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Module No. # 04

Mixed Strategy Nash Equilibrium

Lecture No. # 05

Dominated Actions and Iterated Elimination

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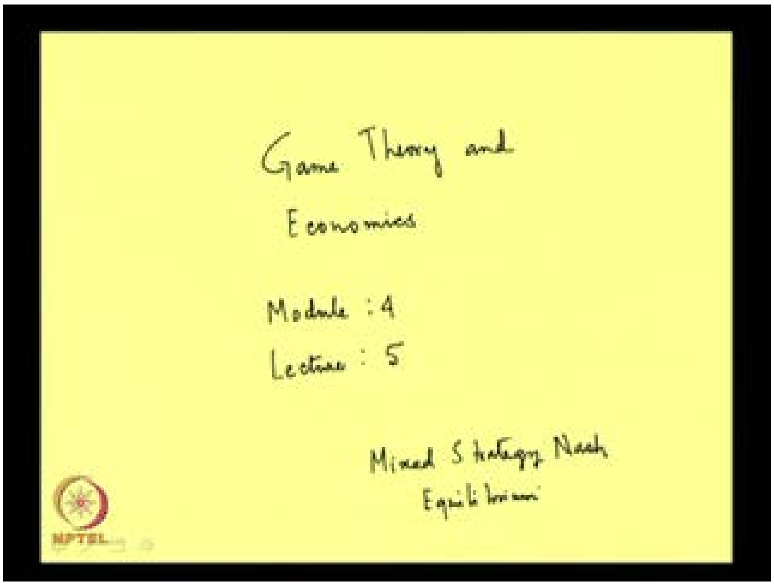

Welcome to the fifth lecture of module 4 of this course called game theory and economics. Before we start, let me just take you through what we have been discussing in the previous lecture. We have been discussing mixed strategy Nash equilibrium and in the previous lecture, we have been discussing one particular aspect or particular property of mixed strategy Nash equilibrium that in the proper mixed strategy Nash equilibrium, the actions to which positive probability is assigned, their expected payoff must be same and the actions for which the probability attached is 0, their expected payoff can be at most the value of expected payoff of those other actions. By using that property, we can solve many games and we can also check whether in a particular game, particular mixed strategy profile is a Nash equilibrium profile or not.

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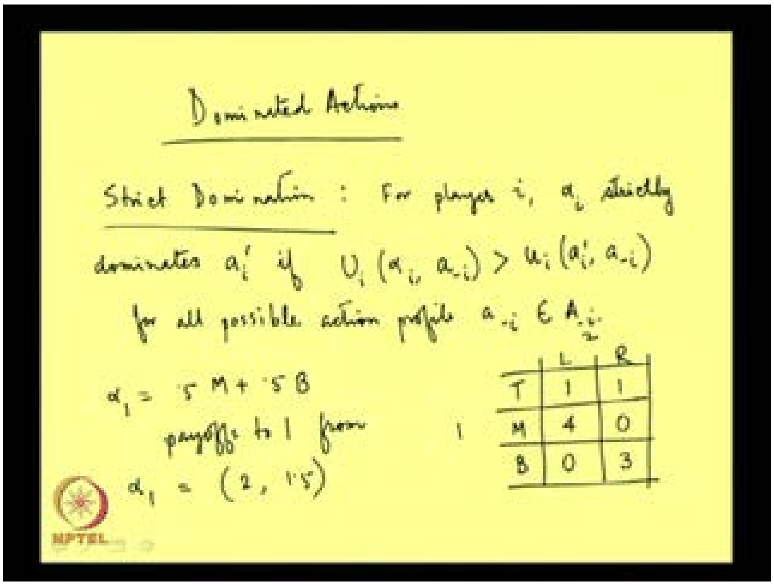

Today, we shall look into other aspect which is weakly dominated and strictly dominated actions, in case of mixed strategy. Remember, we have talked about dominated actions before also, but that was in the context of strategic games, where no randomization was allowed. Here, we are talking was strategic games, where randomization is allowed and the preference is Von Neumann preference.

So, here what will be the definition of dominated action and we know that there can be two kind of domination. So, first let me define strict domination. We say that for player i, alpha i strictly dominates a i dashed, if for all possible action profiles small a naught i belonging to capital a naught i, which means that given that the other players are playing a naught i, a naught i could be any kind of action profile, the expected payoff to player i from playing this mixed strategy alpha i is more than his payoff, if he plays a i dashed. Then we say that this mixed strategy alpha i is strictly dominating his action a i dashed.

So, one can give some examples. There are 2 players here; player 1 has three actions, player 2 has two actions and these numbers are representing the payoffs to player 1. So, I am not writing the payoffs to player two.

Now, if we apply the idea of strict domination what we discussed before, the idea that we discuss before where no randomization was allowed, then no action of player 1 dominates her other actions. For example, if L is played. So, if I compare T and M, if L is played M is better, but, if R is plate T is better.

So, neither T is dominating N or M is dominating T. Likewise, you can take any other pair and we can see that no action is dominating any other action; no strict domination is there, but if we consider now mixed strategy of player 1, then we can in fact, see that T can be strictly dominated.

For example, let us take alpha 1 to be 0.5 M plus 0.5 B. So, if 0.5 M 0.5 B is the mixed strategy of player 1, then what is his payoff? It will be 0.5 multiplied by 4 plus 0. which is So, payoff to 1 from alpha 1, will be equal to 2, 1.5 payoffs because 2 can take either action L or action R. If we takes the action L, player 1 will get 2; if we takes the action R, player 1 will get 1.5 and we can see that no matter what player 2 plays, the payoff to player 1 is higher; 2 is greater than 1; 1.5 is greater than 1.

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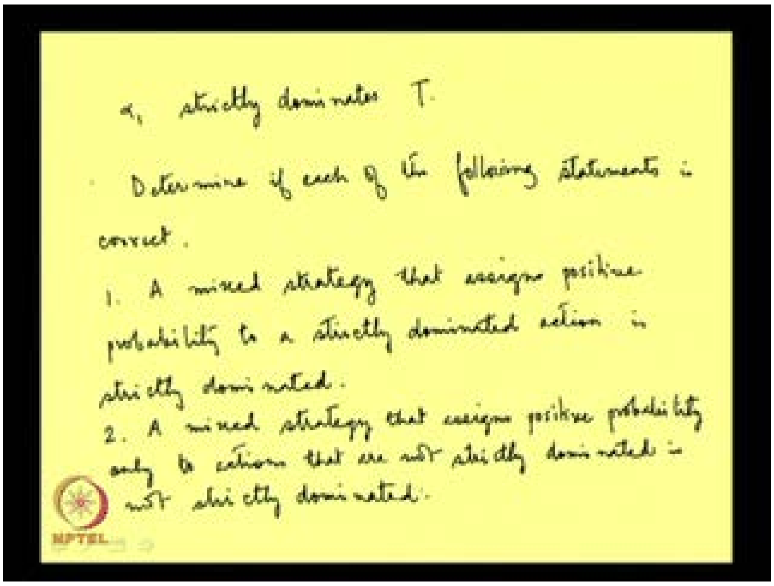

Here, in this case, alpha 1 strictly dominates T. So, this is the basic idea of strict domination. We can do some exercise from strict domination so that our idea of strict domination becomes more clear. So, this is an exercise.

I have to say whether this statement is correct or not. This is 1, number 2. So, these are the two statements. Let us first look at the first statement. A mixed strategy that assigns positive probability to a strictly dominated action is strictly dominated and second one is a mixed strategy that assigns positive probability only to actions that are not strictly dominated is not strictly dominated.

So, if I have to say that first one is correct, if it is a true statement then I have to have a proof for that and if I have to say that this is a false statement, then I have to give an example, why this is false.

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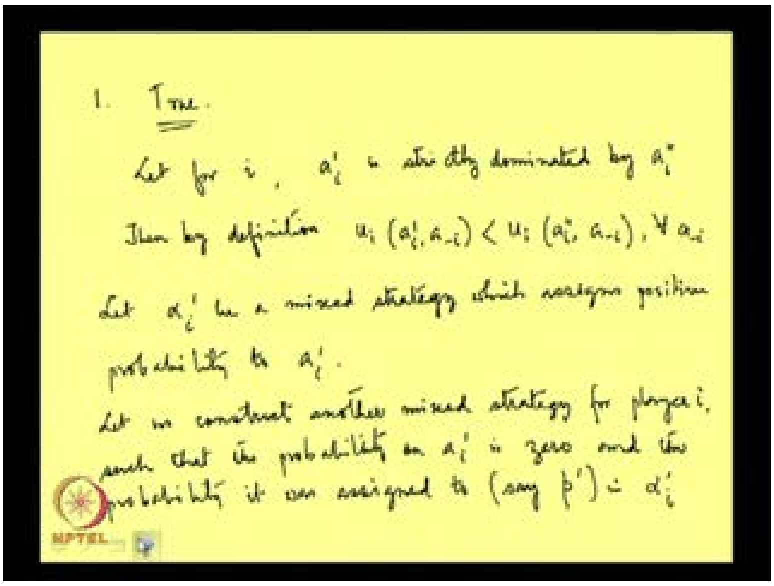

Let me start with the first statement. Our conclusion that it is true; it is not a false statement; it is a correct statement. Now, I have to say why it is true. I have to proof that so let for player i, a i dashed is strictly dominated by a i double dashed.

Now, if this is true then by definition, it must be the case that this must be true that the player i should be getting less payoff, if he plays the a i dashed rather than a i double dashed, given that the other players can take any action.

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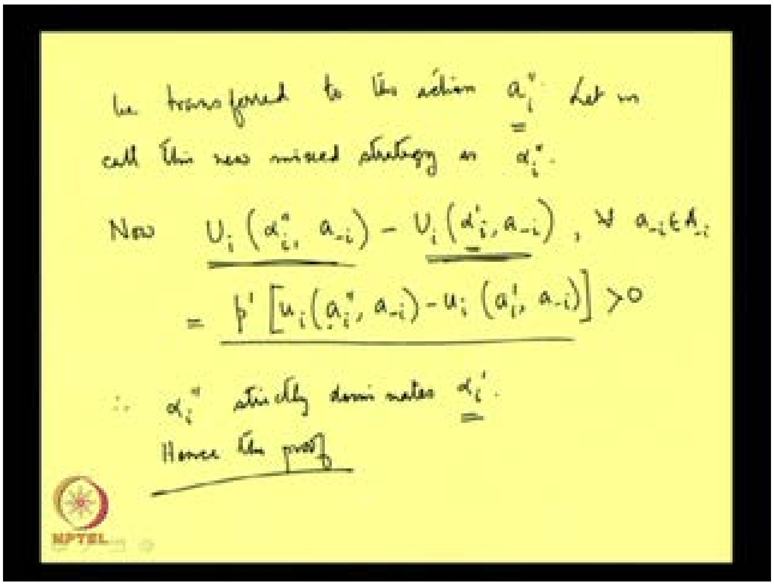

Now, let alpha i dashed be a mixed strategy which assigns positive probability to this action a i dashed and let us construct another mixed strategy for player i such that the probability on a i dashed is 0 and the probability it was assigned to say p dashed suppose, in alpha i dashed be transferred to the action a i double dashed.

(Refer Slide Time: 13:55) So, essentially what we are doing is that we are starting with a mixed strategy of player i, which is denoted by alpha i dashed and suppose, the probability that a i was having in this alpha i dashed was p dashed, which is positive.

Now, we want to have we want to prove that it is possible that another mixed strategy is there, which is strictly dominating this alpha i dashed and this is the mixed strategy, we are trying to construct. How we are constructing this new mixed strategy? What we are doing is that, we are now putting the probability on a i dashed to be 0 and the probability it had in alpha i dashed, which is p dashed is now being reassigned to the action, which was dominating it and which is a i double dashed. So, let us call this new mixed strategy as alpha i double dashed.

Now, what is the payoff to player i, if he takes the action this whereas, the other players are taking their different actions and minus this. What will the value of this? The value will be you see for other actions taken by player i, I mean this is an expected payoff and this is an also an expected payoff. How do I evaluate the expected payoff? For each action of player i, I find out what is the payoff that this player is getting and multiply that with the probability with which this action is attached. Now, in this case, in the double dashed case, player i is having no probability attached to the action as i dashed.

So, that action does not I do not have to write that particular component whereas, this action a i double dashed has now a greater value compared to this U i alpha i dashed, a naught i. So, in the net what we shall have is this p dashed u i of a i double dashed, a naught i minus this because this is the p dashed, which has been here now, which has been transferred to a double dashed here.

So, this is the only term which will remain because the other terms will cancel out and this I know is positive and this is true for all a naught i belonging to capital a naught i. Since this is positive, which means that alpha i double dashed strictly dominates and hence the proof.

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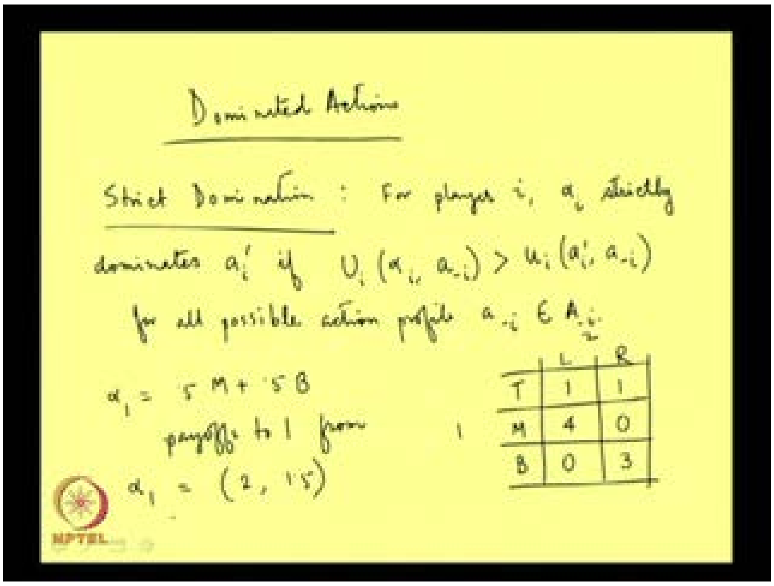

Now, one thing that needs to be specified is that here we are talking about strict domination of 1 mixed strategy by another mixed strategy, but in the definition if you remember, when we wrote the definition we were talking about strict domination of an action by a mixed strategy, but we are now, as you can see, we are extending it to a case where a mixed strategy is being dominated by another mixed strategy.

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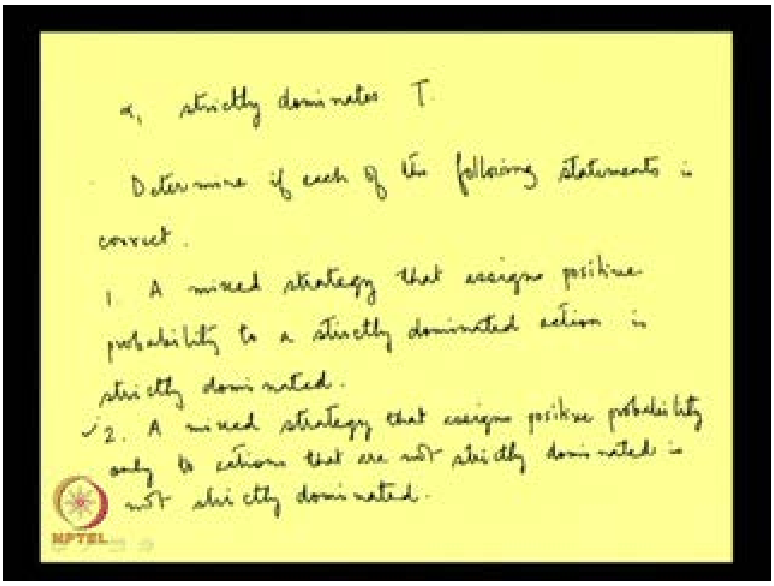

Now let us look at the second problem this one a mixed strategy that assigns positive probability only to actions that are not strictly dominated is not strictly dominated. Now, here, our conclusion is that this is not a true statement; it is a false statement.

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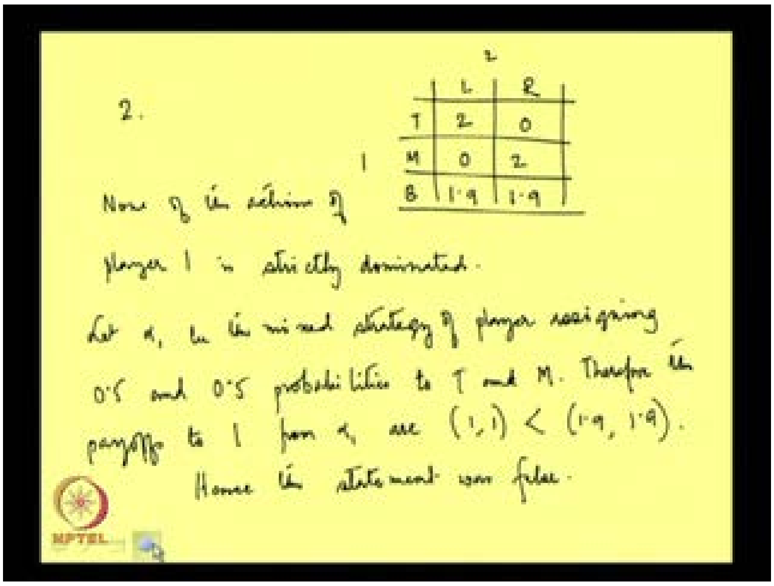

Now, when we are saying that it is a false statement, we have to give an example why it is false and that will be sufficient. So, let me construct an example. Like before, I have three actions for player 1 and two actions for player 2 and like before, I am just writing the payoffs to player 1 in each of this action profiles.

So, payoff of player 2 is not written. Here, observe that none of the actions of player 1 is strictly dominated. Now, what we have to show is that if I assign positive probabilities to some actions, which are not strictly dominated then that mixed strategy can be strictly dominated because we have to show that the statement is false. Let us take the following mixed strategy. Let alpha i be the mixed strategy of player 1 assigning 0.5 and 0.5 probabilities to T and M.

Therefore, the payoffs to 1 from alpha 1 are 1 and 1 and which is less than 1.9 and 1.9. So, we have basically shown that I have two actions which are not strictly dominated and I construct a mixed strategy by assigning positive probability to them and that is getting strictly dominated. This new make strategy is getting strictly dominated by another action. So, hence the statement was false. Now, this was the case of strict domination.

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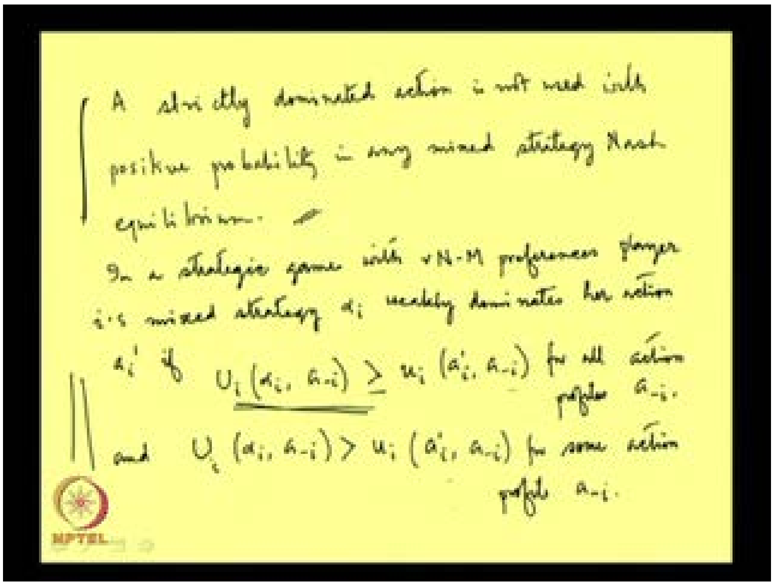

One important property regarding strict domination and Nash equilibrium is the following. That a strictly dominated action is not used with positive probability in any mixed strategy Nash equilibrium.

So, this is the fundamental result of mixed strategy and Nash equilibrium that if I have an action, which is strictly dominated, then it is never used with positive probability in mixed strategy Nash equilibrium. (Refer Slide Time: 23:28) This is basically coming from the first statement that we have seen before - this statement, because a mixed strategy which is assigning positive probability to a strictly dominated action is getting strictly dominated.

So, if it is getting strictly dominated, there is no reason why it should be played in Nash equilibrium. This is the reason why we are getting this result.

Now, we have talked about weak domination, when we talked about different kinds of domination, in case of no randomization. Weak domination can be there in case of mixed strategies also. The definition is the following. This should be true. So, in this case we are calling that alpha i is weakly dominating the action a i dashed, if for all actions by other players for all action profiles of other players, alpha i should be giving at least more than what a i dashed is giving and for some action profiles, at least one action profile of other players a naught i, alpha i should be giving strictly more payoff then what is given by a i dashed. In that case, we are saying that a i dashed is weakly dominated by alpha i.

Now, since we have seen this result that a strictly dominated action is not used with positive probability in any mixed strategy Nash equilibrium, it can be used - this result can be used to solve games also that a strictly dominated action is never played with any positive probability in Nash equilibrium.

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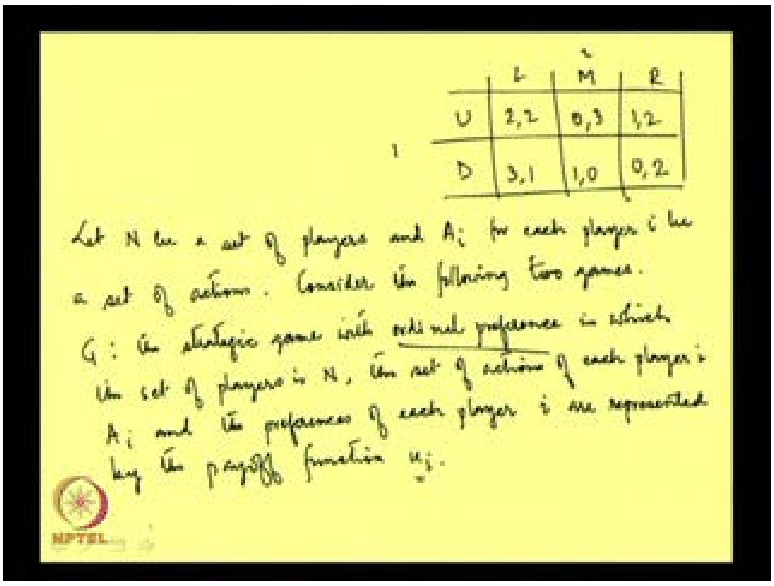

So, if I look at the following game. Here, one of the actions will be strictly dominated by a combination of other actions. I am not going to exactly say which action is going to be strictly dominated by a combination of other actions, but if an action for player 2 for example, can be found, which is strictly dominated by some other action or mixed strategy by the other player then this strictly dominated action can be thrown out and if this action is thrown out then this game becomes a 2 by 2 game.

I mean, two actions is to be considered for each player and I know how to solve this 2 by 2 games. I just have to construct the best response functions of each player and find out where is the intersection point. So, that will give me the Nash equilibrium. The idea that an action which is strictly dominated is not played in Nash equilibrium is helpful, in order to solve games.

Now regarding Here, we have basically borrowing the idea of strict domination and weak domination from what we have seen before, where we had discussed the games with no randomization and where there was no Von Neumann preference and here, we are considering the cases where there is Von Neumann preference.

Now, one may wonder, what is the relationship between these two kinds of games? I mean, if I have Von Neumann preference games and if I have games with no Von Neumann preferences, but the numbers of players are same, the action set remains the same, then what are the relationship between this two sorts of games?

So, the following results will be able to tell us more about this. Let So, this is how we are defining the game capital G. It is a strategic game with ordinal preference, where the set of players is capital N, the set of actions of each player is given by capital A i and preferences of each player is represented by the payoff functions, small u i

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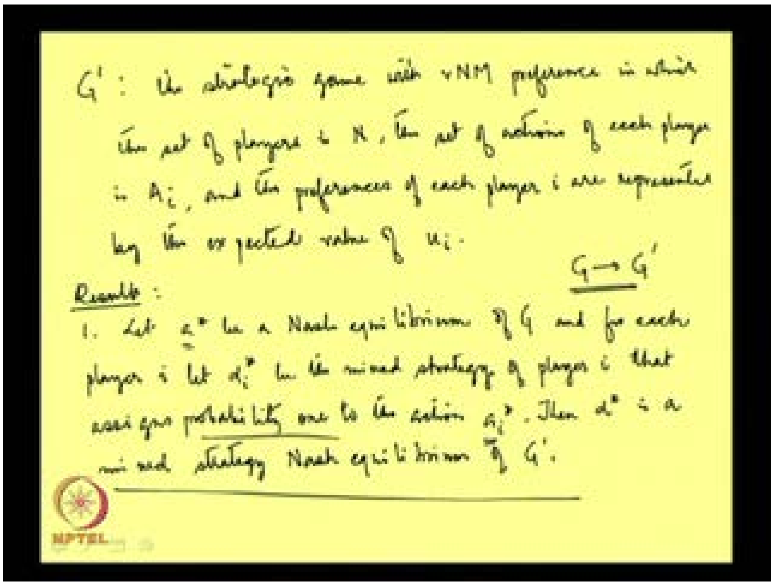

Let us define another game G dashed. This is also strategic game, but with Von Neumann Morgenstern preference.

So, we have basically defined two games with similar characteristics, but there are some subtle differences. In both the games, the set of players is the same which is capital N, the number of the action set of each player is also same. In the previous case, it was capital I; here, also it is capital I for player i. The preferences are different. In the first case, it was ordinal preference, which means the players were not randomizing. Here, the preferences are Von Neumann preference and preference of each player therefore, is given by the expected value of u i; u i is the payoff function that we had in the previous game.

So, if this is the scenario then what kind of result that we can get from here. I am not going to proof the result just stating these two results. So, number 1. This is the result; this is very intuitive. What is being said is that suppose, we consider the game G, where there is ordinal preference and suppose in that game there is a Nash equilibrium at a star, which means that player i is taking the action a i star; i can be any player.

Now, we construct a mixed strategy for each player and suppose for player i, this mixed strategy is called alpha i star. In that mixed strategy, it is just the case that this action a i is having the probability 1, which means that the other actions of this player i is having the probability 0. Then we are saying that this mixed strategy profile, which is alpha 1 star, alpha 2 star etcetera alpha n star will be a mixed strategy Nash equilibrium in the game G dashed, where G dashed was the game with Von Neumann preference, remember.

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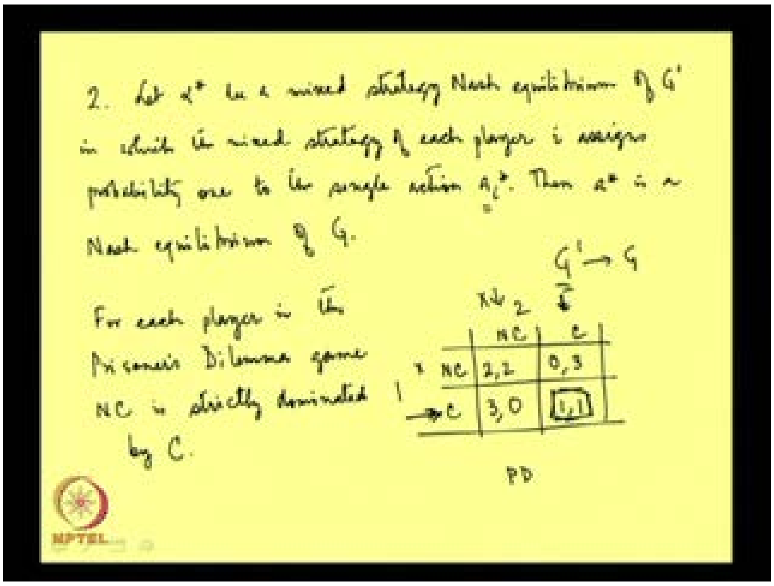

So, here we are having a result which relates G dashed to G and the second result with tell me the other way. How to go to G from G dashed? Then a dashed is a Nash equilibrium of G.

So, this is telling me how to go from G dashed to G. So, just we elaborate what is being said. Suppose in G dashed, that is in the Von Neumann preference game, there is a Nash equilibrium which is alpha star and this Nash equilibrium is such that suppose I take player i player i has this mixed strategy alpha i star. In that alpha i star, only one action is having probability 1 and the other actions are obviously having the probability 0. Suppose that action is given by a i star.

Now, if that action Suppose, I consider that game G, where randomization is not allowed. Then in the game G, a star is going to be a mixed Nash equilibrium, which means that player 1 is going to play the action a 1 star, player 2 is going to play the action a two star etcetera and that is going to be Nash equilibrium.

So, these results are intuitive. Now, we have seen the result that if I have actions, which are dominated, then those actions can be left out. They are not considered for Nash equilibrium because those actions will not be played with any positive probabilities in the mixed strategy Nash equilibrium.

So, this idea can be elaborated further and this idea can be seen from the following example that we have seen before. This is the familiar prisoner's dilemma game.

So, he has two actions. This was the game. Now, the interesting thing is that in this game each player has one action, which is strictly dominated. So, for each player in the prisoner's dilemma game, N C is strictly dominated by C. You can see this from the game itself directly. If player 2 is playing N C, for player 1, C is best because N C is giving him 2, C is giving him 3. If player 2 is playing C, even then C is best because C is giving him 1 and N C is giving him 0.

So, C is strictly dominating N C and for player 2 also, it can be seen that C is strictly dominating N C. Now, since I know that a strictly dominated action is not going to be played in Nash equilibrium with any positive probability, I can now forget about this action and this action.

So, the actions which are remaining are C and C and so, we are getting this as the profile by combining the actions, which are not strictly dominated.

So, this is known as deletion of strictly dominated actions. By deletion of strictly dominated actions, we are getting some profiles, which seem to be very reasonable kind of profiles that can occur.

Now, in this case, this profile is Nash equilibrium, but it is not necessary that if I delete the actions which is strictly dominated, then the profile that I will be left with are going to be Nash equilibrium; that is not necessary.

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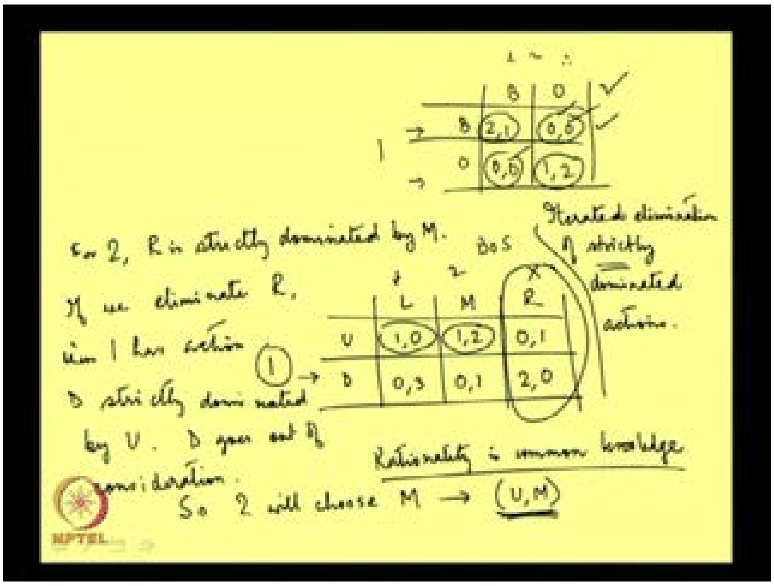

For example, we have seen such games before. Let us take the case of battle of sexes. This is the game - 2, 1, 0, 0, 0, 0, 1, 2. In this game, no action is strictly dominated. If player 2 plays B, player one’s best action is B; if player 2 plays O, player one’s best action is O.

So, no action is unequally better than the other action, which means that we are left with these actions and none can be deleted and we know that these profiles - all of the profiles are not Nash equilibrium. This is not Nash equilibrium; this is not Nash equilibrium.

So deletion of the actions, which are strictly dominated, does not necessarily give me profiles which are Nash equilibrium. They give me big set, large set and within that set, subset is the set which is containing the Nash equilibrium profiles.

So, here these four profiles and if I consider the mixed strategies, then there will be infinite number of mixed strategy profiles, all of them are left out after I have deleted the strictly dominated actions and we see that a very few of them, in fact, three of them are in fact, in equilibrium and the other are not.

(Refer Slide Time: 45:54) If I consider pure strategy, only two of them are in equilibrium and two are not and this idea of deletion of action, which is strictly dominated can be taken one step further in the sense that I can see that for me, let us take the previous case, this is strictly dominated. So, I am not going to play this, but suppose I do not have any action, which is strictly dominated, but I know that my rival, the other player has an action which is strictly dominated.

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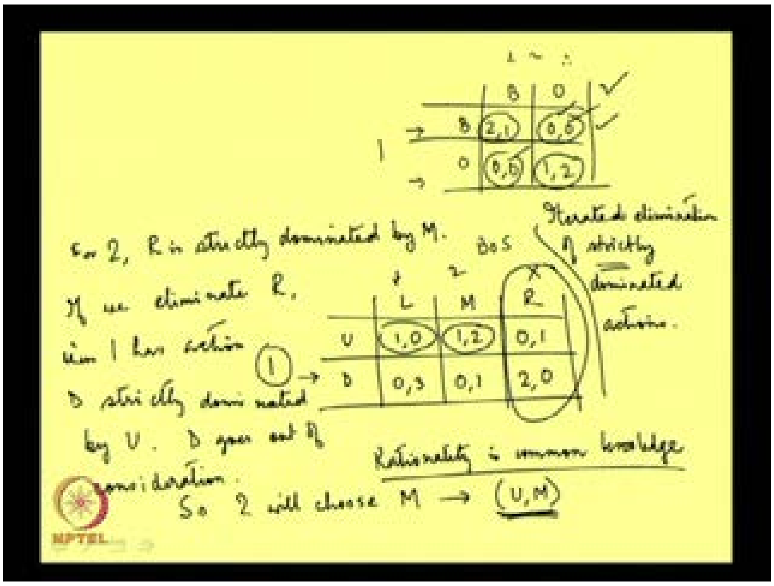

Now, that information can be used by me to pin point on those profiles, which are more rational, which will be played and not the profiles which contain actions, which are strictly dominated. So, let me give you one example.

So this is the game. Here, for player 1, he does not have any action, which is strictly dominated because if L is played, U is better, but if R is played D is better. So, no action is strictly dominated by any other action. So, he cannot delete any of his actions, but if he looks at player 2’s actions then obviously there is one action, which is strictly dominated.

So for 2, R is strictly dominated by M because if 1 is playing U, M is better. If 1 is playing D, even then M is better; so, M is strictly dominating R.

Now, this fact is known to 1 also. Here, we are depending on the fact that the players are rational, in the sense, that they want to maximize their payoff.

Now, it is not only the fact that I am rational, I also know for sure that my rival is also rational - the other player is also rational. So, if I know my rival is rational, This is how we write it that rationality is common knowledge. If I know my rival is rational then I can figure out by looking at this game that he is not going to play this. So, R is out of the question, it is not been considered by player 2. Now, if this is going out of the question then however player 1 has action, which is strictly dominating other action. So, if we eliminate R, then 1 has action D strictly dominated by U because R is going out of consideration. If 2 is playing L, U is better; if 2 is playing M also, U is better. So, for 1, he is not going to play D. So, D goes out of consideration and if D goes out of consideration, now think of this game from the point of view of player 2. Player 2 can now see that he basically has to choose between this and this U, L and U, M and for him U, M is better; 2 is greater than 0.

So, 2 will choose M. Therefore, the only outcome that we are reaching is U, M. Let me go over this entire procedure once more. This is called iterated elimination of strictly dominated action.

So, step by step, we are eliminating one action after another, the actions which are strictly dominated. First, in this elimination process, what is important is that the players are rational and not only are the players rational; they know that the other players are also rational. So, here we are starting from player 2. Player 2 has this action R, which is strictly dominated by M.

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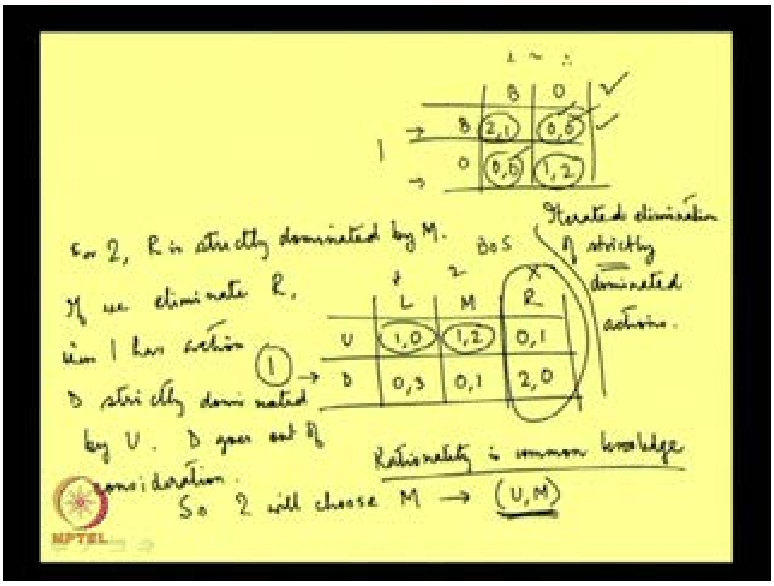

Now, this can be figured out by player 1 also. Player 1 knows that player 2 under no circumstances is going to play R, if player 2 is rational. Now, banking on this rationality therefore, R is going out of question. Now, player 1 has to consider between U and D and if R is out of the question, he can see that U is strictly dominating D.

Now, if U is dominating D, this is known to player 2 also. Player 2 knows that player 1 knows that player 2 is rational. Therefore, player 1 is going to play only U and if player 1 is only going to play U, then player 2 is going to play M because if U is played, M is better than L.

So therefore, we reach the outcome of U, M, which is the profile obtained by iterated elimination of strictly dominated actions. Now, observe that in this case 1, 2, this payoff at U, M - this is a Nash equilibrium. If U, M is reached then there is no reason why any of the players will like to deviate. So, iterated elimination of strictly dominated actions in this case gave us a profile which is a Nash equilibrium, but obviously as we have seen before that in many cases, the actions cannot be eliminated, for example, in this game itself.

We cannot eliminate any of the actions and if we cannot eliminate any of the actions that does not mean that the action profiles, you are left with are all Nash equilibrium; obviously, this is not a Nash equilibrium.

So, by iterated elimination, some cases we get some very definite idea about what action profile is going to be played, but in many cases our predictions will be very vague like this one. So, this is the case of iterated elimination of strictly dominated actions. In the next lecture, we shall take up the case of iterated elimination of weakly dominated actions because this was a strict domination case, but actions can may eliminated on the basis of weak domination also. In that case, we shall see that the result that we shall get after elimination of weakly dominated actions may vary from case to case. So, we shall talk about that in the next lecture. So, this is where we are finishing this lecture. What we have done in this lecture is that we have talked about dominations in the case of mixed strategy equilibrium and we have also talked about iterated elimination of dominated actions. Thank you.

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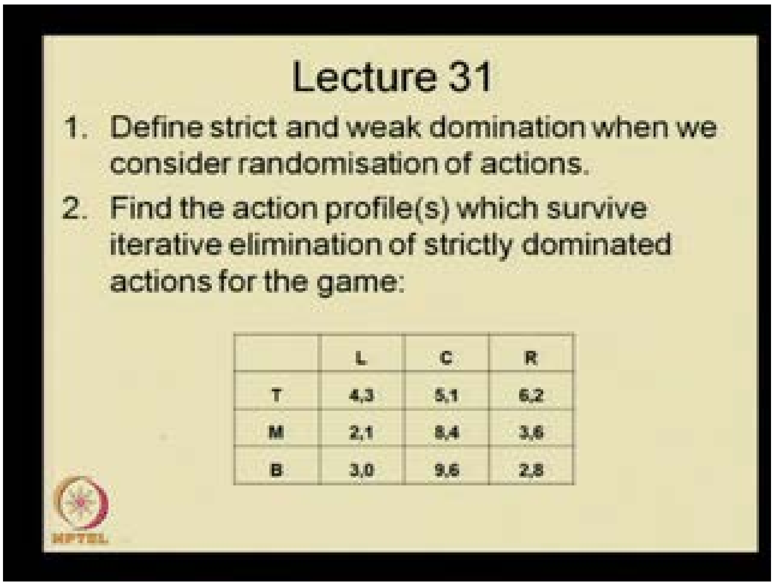

Define strict and weak domination when we consider randomization of actions.

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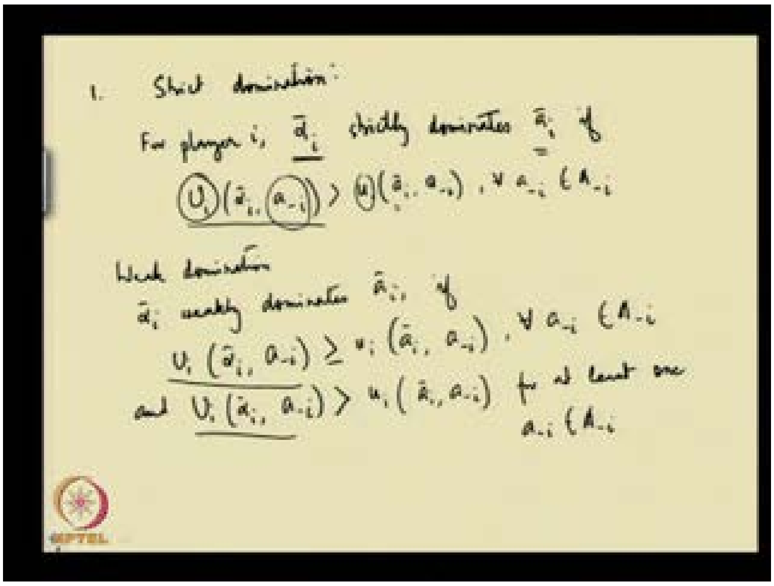

So, let us first define a strict domination. The idea of strict domination here, when randomization is allowed is similar to the idea of strict domination, when randomization was not allowed. What we are saying is that this mixed strategy of player i alpha i bar is strictly dominating an action of his, which is a i bar. If the expected payoff from alpha i, bar given that the other players are playing whatever action that they are playing, but these actions are not mixed strategies, but pure strategy action - that is, particular actions not randomization, the expected payoff from this alpha i bar should be always greater than the expected payoff from a i bar. and weak domination.

One more thing to note here I am using two u's here. I am using a capital and here small; capital stands for expected payoff and small, when I am using small u i, it is not expected payoff. So, for all the profile of actions of other players, the expected payoff must be greater from U i bar and for at least one action profile of other players, U i bar must be giving strictly greater than a i bar.

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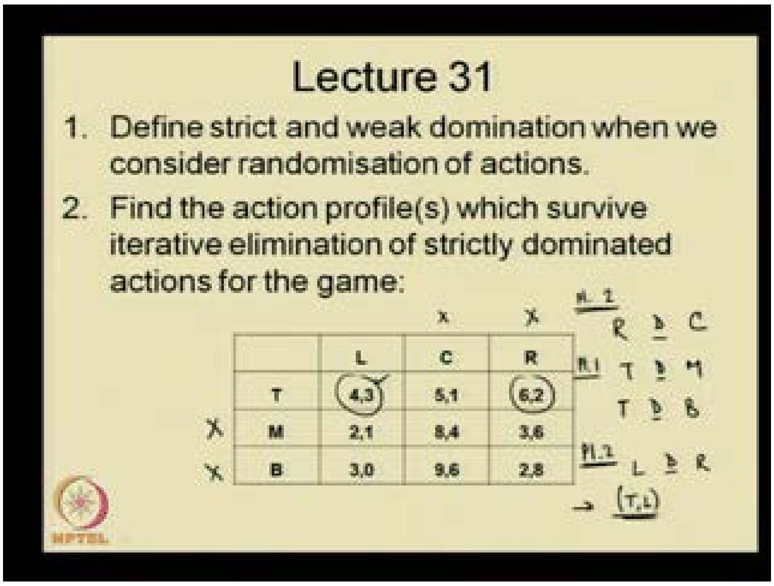

Find the action profiles - second question, which survive iterative elimination of strictly dominated actions for the following game. Let us write down which are the actions, which are strictly dominating other action. We find that in first round, player 2's payoff from R is always greater than player 2's payoff from C.

So, we write it as R strictly dominates - D under bar, over C. We also see that once C is ruled out for player 1, T is dominating M. So, M is going out. T is also dominating B; so, B is also going out.

So, we are left with this and this and player 2 has to choose and player 2 chooses this because 3 is greater than 2, L dominates R. So, the equilibrium that we have is T, L. Thank you.