Game Theory and Economics

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Module No. # 05

Extensive Games and Nash Equilibrium

Lecture No. # 04

Sub game Perfect Nash Equilibrium

Welcome to the fourth lecture, of fifth module, of this course called game theory and economics. Before we start this lecture, let me take you throughout we have been discussing so far. We have been discussing extensive games. Extensive games are those games, where the actions are taken step by step, not simultaneously.

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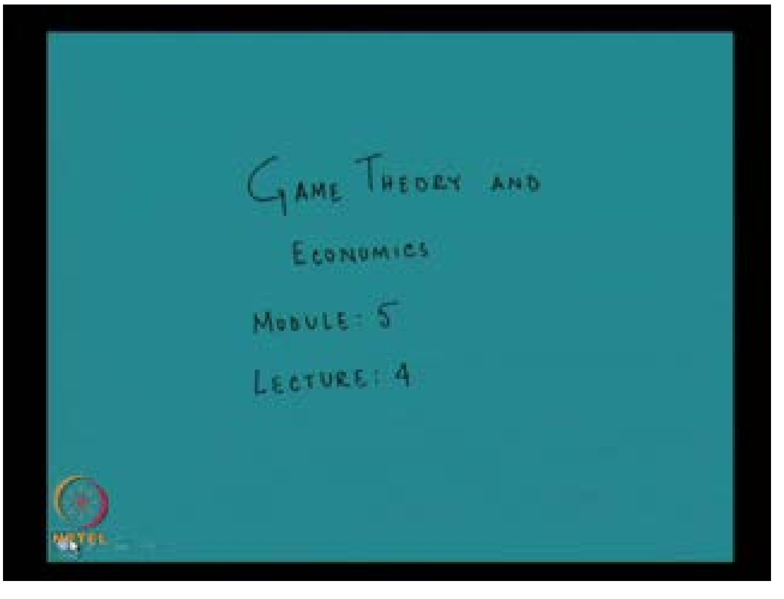

So at one stage, some particular player might be taking some action from a set of actions. In the next stage, some other player might be taking his or her action and that is how the game progresses.

It may happen that a player who took an action in stage 1 might be taking some action in stage 4 or stage 15, so it does not matter. There could be repetition of the same player, in different stages. So, this is the general setting of an extensive game. And we have been discussing not extensive game in general but, extensive game with perfect information, where every player knows the actions of the other players which have been taken before his action is required to be taken.

We have also defined Nash equilibrium in an extensive game. We have said that in a Nash equilibrium, in an extensive game, it must be the case that like in the case of strategic game, the strategy of every player adopts in that Nash equilibrium should be such that if any player deviates from the strategy in the equilibrium.

Her payoff will not go up and if her payoff does not go up, then there is no reason why any player should change his or her strategy from the Nash equilibrium strategy, so this was the general definition. What we shall do today is to just check how we have understood the idea of Nash equilibrium by solving one problem and then we shall try to go beyond Nash equilibrium.

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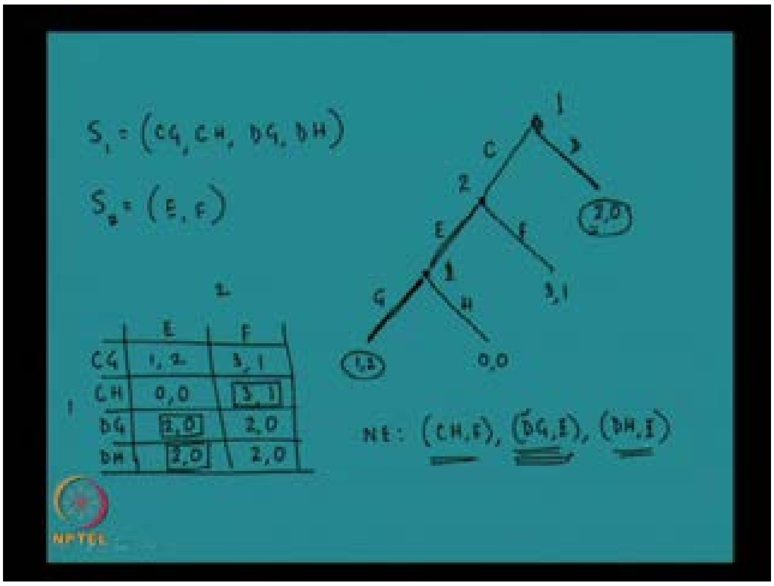

This is one exercise of finding Nash equilibrium; let me draw the game first. So this is the game tree as we call them (Refer Slide Time: 03:39), you can see that here player 1 moves in two stages. In the first stage, player 1 gets to move and he can take action C or D; if he takes the action D the game ends there, where he gets 2 and player 2 gets 0. If he takes the action C, then it is now player 2’s turn to move, who can either take the action E in which case the game progresses further again to player 1, but if player 2 takes the action F then the game again ends, in which case player 1 will get 3, player 2 will get 1. If E is taken by player 2 then, player 1 gets to move in stage 3 and the actions that he can choose from are G and H if G is chosen, then he gets 1, player 2 gets 2, if H is chosen both of them get 0 each.

Now suppose we want find out what are the Nash equilibria in this game. So, how shall we go about finding the Nash equilibria? First, I shall try to find out the set of strategies of player 1. Here the strategy set of player 1 is the following: this is the strategy set of player 1, the strategy set consists of four strategies. CG which means that in the first stage when the history is empty, he will take the action C and the meaning of G is that if the history is C E then he will take the action G, so that is the meaning of CG (Refer Slide Time: 05:41).

Likewise there is CH so at first stage, he is taking the action C and in the last stage he is taking the action H. There is DG and DH which means that, which has a problem of interpretation as we have seen before. What it means is that in the first stage, when the history is empty player 1 is going to take the action D and he is specifying that in case the history is C E, which is evidently ruled out by his own action D; nevertheless if the history is C E, then the action that he is going to take is G.

So that is the interpretation of DG likewise DH. So, these are the four strategies of player 1 and what about player 2, his strategy set consists of only two elements. He gets to move after the history C.

So, either he can take the action E which is one strategy of his or he can take the action F which could be the other strategy. After we have written down the strategy set of each player, the next step is to find the strategic form of this extensive game. That is we look at this game as just like a strategic game and if it is a strategic game, we can find out the Nash equilibrium by drawing the payoff matrix.

I write the strategies as actions and I have written them down, I can write the payoffs also. If I combine CG with E, the payoff is 1 2; CG with F, basically the game is coming from this node to this node and going to this terminal history, so this is 3 1 (Refer Slide Time: 08:50).

CH E is 0 0, CH F is again 3 1 if the strategy of player 1 consists of the action D obviously, the game is ending in this 2 0 itself, this terminal node itself.

So all this payoff for this strategy profile will be 2 0 and looking at it, if I have to find out the Nash equilibria from this strategic form then I know how to find it, I have to look at the possible deviations from any action profile. We have to see if those deviations are profitable or not, if they are not then we have Nash equilibrium.

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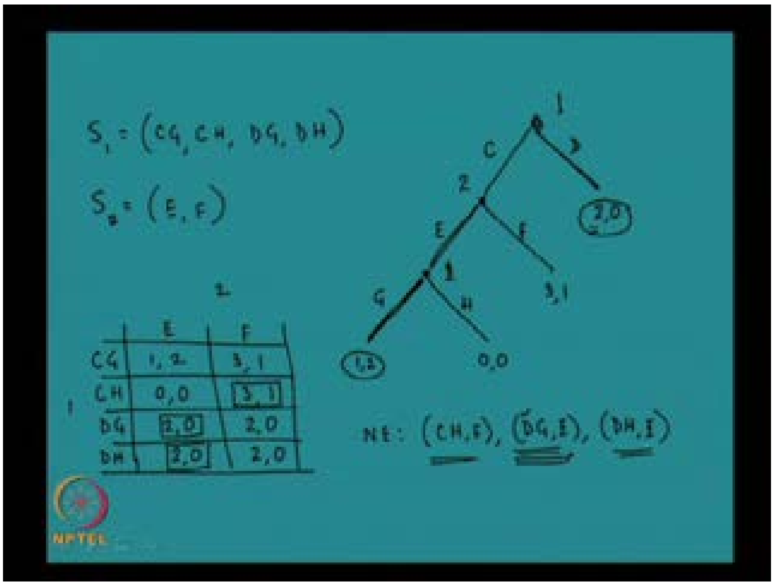

So by doing that I can see that this is a Nash equilibrium, this is one and this is another (Refer Slide Time: 10:10). So Nash equilibria at the following, CH F DG E DH E and what are the justification for these Nash equilibria? How we are certain that they are Nash equilibrium? We have to look at the deviations and if the deviations are not profitable then they are not Nash equilibrium. For example, let us take this strategy profile DG E.

DG and E now, why is this a Nash equilibrium? If player 1 is playing D and G, then from player 2's point of view if he deviates - How can he deviate? - He can deviate by playing F instead of playing E. But by deviating and by playing F, he is not improving his payoff because the payoff is remaining at 2 0; so deviation is in this case not profitable.

Therefore from player 2's point of view, he is doing the optimal thing. From player 1's point of view, there could be two sorts of deviations: one could be after that empty history. Here if he deviates and chooses C instead of D, then given the other actions specified by this strategy profile we shall reach this terminal history C E G. Therefore, he will get 1, 1 is less than 2; therefore there is no point for 1 to deviate as far as the action after empty history is concerned.

What about this action G, if he changes his action and takes up the strategy DH instead of DG then, his payoff is not improving because payoff remains at 2. So it does not matter if he changes this action from G to H and thus changing the strategy of his payoff remains at 2. So we can talk about other sorts of deviation of player 1 also. He can deviate to CH for example but, CH is unprofitable because in that case he will get 0 whereas, now he is getting 2.

So this is a Nash equilibrium strategy profile; likewise, these two can also be shown to be Nash equilibrium. This was the definition and application of Nash equilibrium in extensive game.

But we have also seen in the previous lecture, the idea of Nash equilibrium in case of extensive game is a little problematic. Problematic in the sense that is not robust, it is not a very robust concept because their might be Nash equilibrium where if a player deviates from the strategic that is specified in the Nash equilibrium strategy profile. Then those deviations will generate a non optimal action or non optimal strategies on the other players.

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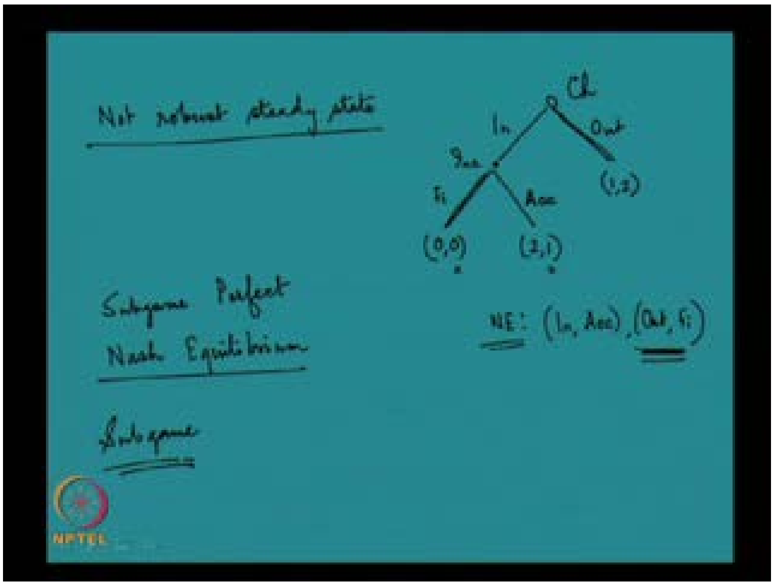

This is something which we have seen before in the last class (Refer Slide Time: 15:00), so this was the familiar entry game. This was the game, there were two players: one is the incumbent, they may be the monopolies firm in the industry and there is a possible challenger, a possible new entrant who will be called the challenger.

If the challenger stays out, then it is best for the incumbent he gets 2, the challenger gets 1. If the challenger gets in, then the incumbent might like to fight with the challenger in which case, it is worst for both of them they could get 0 each or the incumbent can choose to accommodate; in which case, the incumbent gets 1 because he is no longer the monopolist, challenger gets 2. In this setting what we have seen that there were two Nash equilibria, I am just repeating the conclusions that we have drawn in the last lecture.

So that we can link up to the discussion of this lecture, one was in and accommodate, the other was out and fight. There were no problem as far as in and accommodate is concerned but, what we have seen is that there was a problem as far as out and fight is concerned. Out and fight is a Nash equilibrium, there is no doubt about it because given that the challenger is choosing out the incumbent can choose either fight or accommodate, he do not make a difference to his payoff which remains at 2.

So fight is absolutely ok, from the point of view of the challenger also as long as the incumbent is choosing fight, the challenger is better off by choosing out because if he gets in there will be fight, in which case the challenger will get 0. So it is better for him to choose out and get 1.

However there was a problem of interpretation with this equilibrium. The interpretation problem was the following: in Nash equilibrium the idea that we are invoking is the idea of a steady state, which means that I have been observing the behavior of the other players. By observing the behavior of the other players, I form a belief what their actions will be even I get to play with them and those beliefs are proven to be true when we actually play the game and that is how the game progresses.

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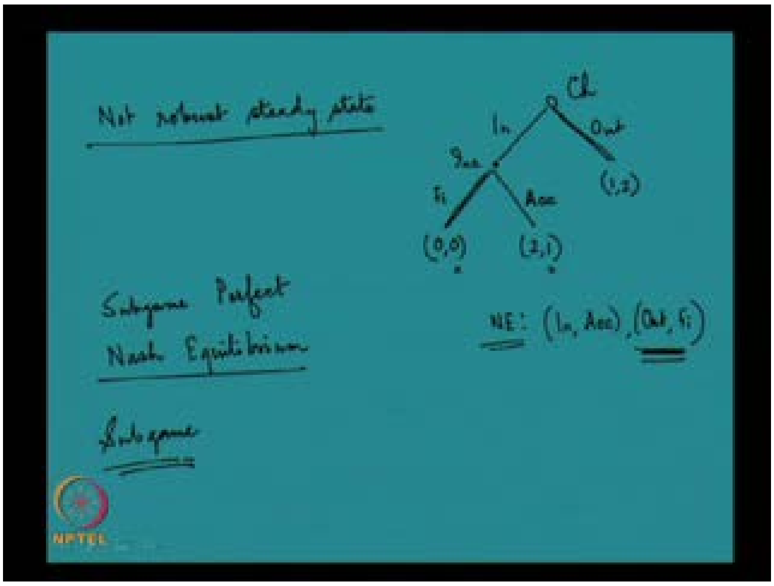

But here, when the game is structured in a sequential way, it is an extensive game; out fight is a strategy profile where the challenger never gets to see the action of the incumbent because he is staying out, because the challenger is staying out.

So though the incumbent is telling him I will fight with you, that action of the incumbent is never observed if the challenger stays out. If there is no experience as far as the challenger is concerned, he cannot form a belief, a credible belief regarding the action of the incumbent.

So there is this problem of interpretation that you cannot see the action of other players where those actions are included in the strategy profile. One way to rescue the Nash equilibrium concept in such a dilemma is to say that, I know that out and fight is the equilibrium when the challenger will choose out but, it may happen that from time to time the challenger does some experiments. He sometimes chooses to go in, so in those cases when he does the experiments, the incumbent chooses fight. So, these are real life observations.

There are cases where the challenger gets in and the incumbent has fought with him therefore, the challenger or the player who is in the place of the challenger forms a belief that if he gets in, the incumbent will fight with him that is the strategy of the incumbent.

The strategy of the incumbent is fight which means that, if you get in I will fight with you, which the challenger gets to see by experimenting from time to time.

So that could be one way of rescuing the idea of Nash equilibrium in extensive games. But, in this case that path that way out is not open to us, why? Because if the challenger indeed gets in by experimentation or whatever may be by error also, the incumbent will find it sub optimal to fight with him.

Because if he fights he get 0 whereas, if the incumbent accommodates, he gets 1 which is more than 0. So, even if we try to interpret this Nash equilibrium where the players can do some experiments can make some errors and reach to the actions, takes the actions which are not the equilibrium actions.

Then the actions specified by the other player is not optimal, which means that this idea itself that this is Nash equilibrium but, this idea this Nash equilibrium itself is not a robust equilibrium, it is not a robust steady state. Because if some player deviates then the action of the other player, which is specified by this Nash equilibrium profile is not optimal that is why it is not a robust steady state.

So, we have to have more robust, more reasonable concept of equilibrium if we have to deal with extensive games with perfect information; a notion of equilibrium, a notion of steady state, which is more reasonable, more logical and more robust than the idea of simple Nash equilibrium.

The idea that we shall develop now is what is known as sub game perfect Nash equilibrium. To define this new concept of equilibrium, which is a little bit more than the idea of Nash equilibrium, let us first define what a sub game is. We have to define what is a sub game of a game and once we have to define what is the sub game of a game, then we have prepare the ground to define what is a sub game perfect Nash equilibrium. So let me go to the definition, this is the definition.

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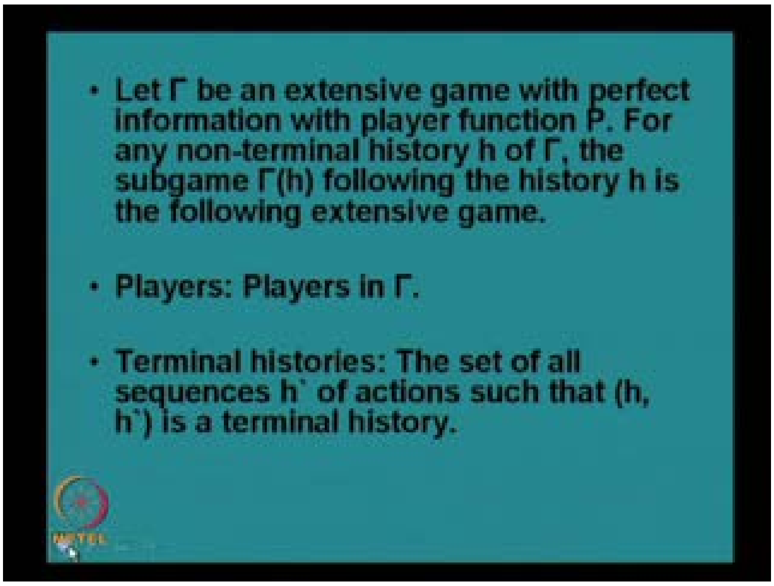

Let gamma be an extensive game with perfect information with player function P. If you remember P is a player function, which is applied over non terminal histories and if I applied P over h, h is a non terminal history there, then it is give the identity of the player whose turn it is now to move to take an action.

For any non terminal history h of gamma, the sub game gamma h following the history h is the following extensive game. Suppose there is a sub history or what is known as a non terminal history h of gamma, then corresponding to this non terminal history h there is a sub game, which we shall call as gamma h and it is defined as the following.

Now this gamma h itself is an extensive game. So it has to have 4 elements, which we have to specify. The first is the player set, so this is element number 1; the players in this gamma h are the set players in the original game that is the original game gamma. Second element that has to be specified is set of terminal histories. How the terminal histories of gamma h are defined? The set of all sequences h dashed - this h dashed the notation has not come really properly - so how is it defined, the set of terminal history the set of all sequences h prime of actions such that h h prime is a terminal history.

So h is a non terminal history of the original game gamma, from h if there are may be n number of sequences of actions, h prime such that h h prime is a valid terminal history. Then every such h prime will be a terminal history in the sub game gamma h.

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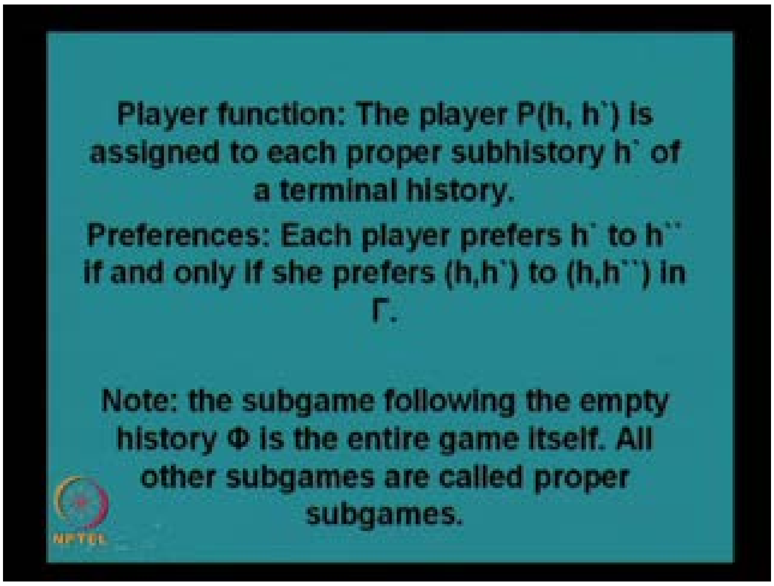

Thirdly, I have to specify the player function, it is very simple. The player p h h prime is assigned to each proper sub history h prime of a terminal history. Fourthly the preferences this is simple, each player prefers h prime to h double prime if and only if she prefers h h prime to h h double prime in the game gamma.

So in that sub game, we are talking about which we have defined as gamma h, suppose there are two terminal histories h prime and h double prime, then any player will like h prime to h double prime, if in the original game h the same player prefers h h prime to h h double prime.

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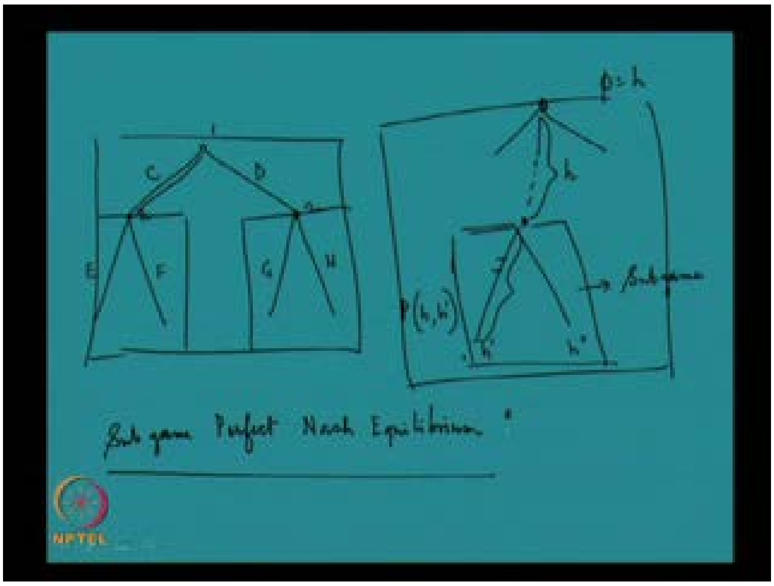

So these are quite intuitive ideas that we are talking about. Basically what if I have to draw a diagram it will look like the following. Suppose this is the original game, here the game is starting, after some point of time, suppose the game reaches here and this is h after h has happened, the part of the game that is left after h is happened is called the sub game of the original game (Refer Slide Time: 28:28).

That is the idea so that is why the player set remains the same. In general - this is a generalization - the players that are there in the sub game can at most be the original player set of the original game. So we can say that the player set of the sub game is same as the player set of the original game. Terminal histories as we have just said that all sorts of terminal histories were h h prime is a valid terminal history in the original game, then h prime is a terminal history in this new game that we are defining which is we have calling as sub game.

Thirdly, the player function for any terminal history h prime, the player function p h h - I can apply this player function over this p h h prime over every service tree of this terminal history h prime and those that player functions will be same as the player function in this new sub game.

Finally the preferences, if there are two terminal histories h prime, h double prime and if any player likes h h prime to h h double prime then he prefers h prime to h double prime again this is very intuitive. So this how the idea of sub game is defined.

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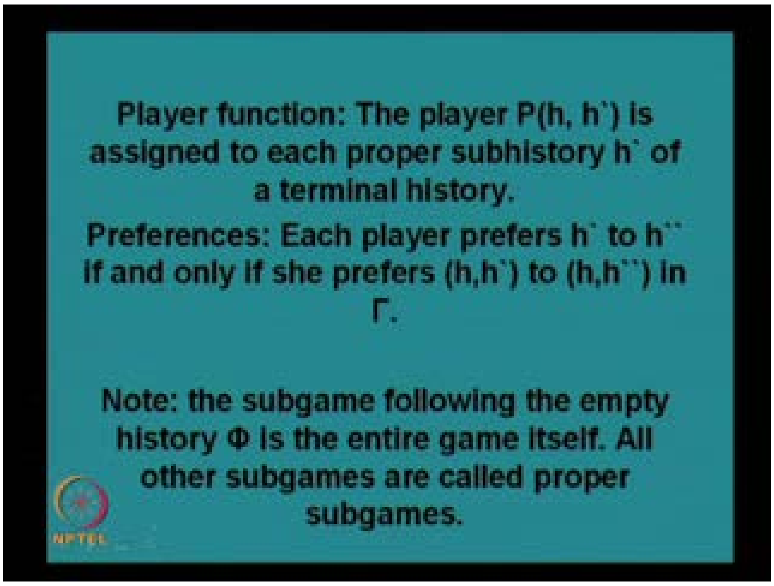

Note: first the sub game following the empty history phi is the entire game itself .This is again very easy to see and all other sub games are called proper sub games. So if I take this history to be h, that is this then we are talking about the entire game itself, the entire game is coming after the sub game after this history phi. So the game itself is a sub game of its own but, there are other sub games in a game also and all those other sub games will be called proper sub games.

It is like a set and it is a sub set, the set itself is a sub set of its own that we know but, if I take that subset away then the subsets that are remaining with me are called proper subsets. So that is how it is defined; now let us take some examples and see how we can find out the sub games from any game. So let us take this game (Refer Slide Time: 32:38). Here one has two actions and I am not writing the payoffs of the players because they are not irrelevant.

How many sub games are there? There are three sub games here. One sub game comes after the empty history itself which is the entire sub game that is one sub game. If take the history C, then this is what is remaining with me, the part of the game that remains with me; so this part is another sub game and this is a proper sub game not unlike the sub game that we have just specified before.

After the history D, we have this other sub game of the original game, which starts with 2 making the first move and after 2 makes the move so whatever move it is, the game ends there. So this game has three sub games but, it may happen that these two proper sub games are sub games of each other. Here these two proper sub games are not sub games of each other but, it may happen that the proper sub games are also sub games of each other.

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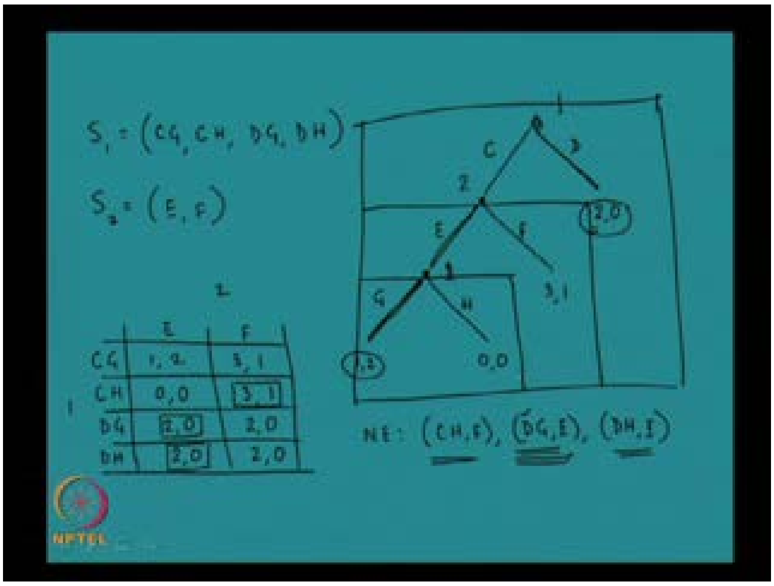

Let us take this game, in fact I have drawn this before. Let us take this game (Refer Slide Time: 34:40), how many sub games are there? Now I claim that here also there are three sub games: the first sub game is very easy to see the game itself, which comes after the history phi, then there is sub game which comes after the history C, which is this and there is a sub game, which comes after the non terminal history C E, which is this; so here I have three sub games.

So I have defined sub game and the next step is to define what is known as a sub game perfect Nash equilibrium. Let me first try to motivate the idea of a sub game perfect Nash equilibrium and then, I shall try to define it more concretely, more methodically.

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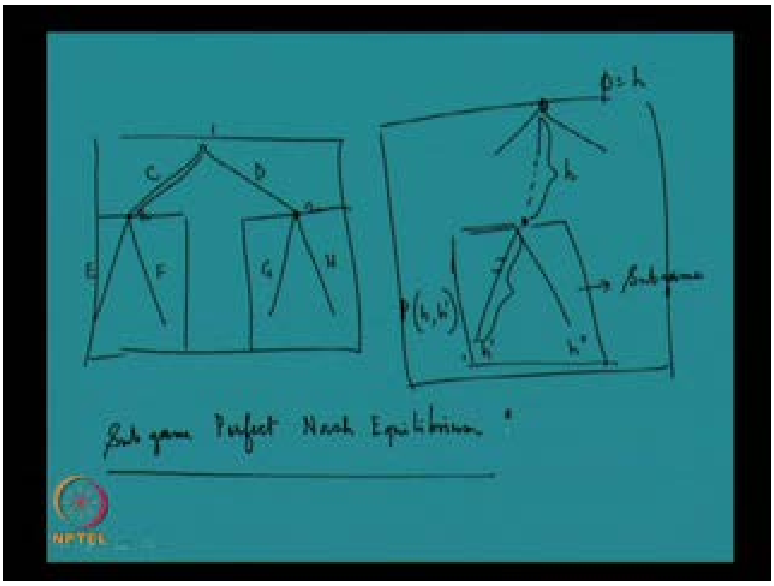

The idea of sub game perfect Nash equilibrium is this, it is like the idea of Nash equilibrium in an extensive game. It is also a strategic profile but, the specialty is that this strategy profile should be such that it generates equilibrium in every possible sub game.

In particular also in the sub games, which are not included in the terminal history, are generated by the strategy profile. So what I exactly mean by this can be illustrated by the following example.

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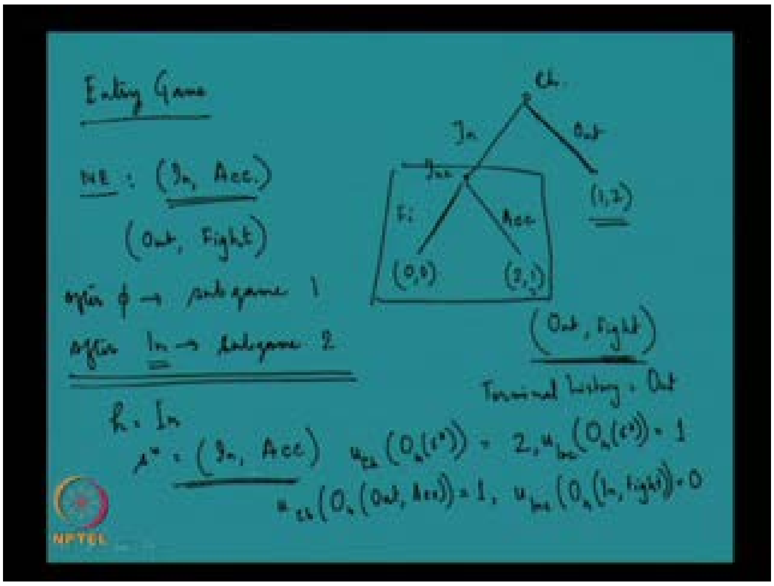

This is the familiar entry game once again and in this game, we have seen that out fight was a Nash equilibrium, the question is this a sub game perfect Nash equilibrium and the answer is no. The reason is that here what is the terminal history which is generated by this strategy profile, it is this out.

But this game has a sub game which is not reached by the terminal history which is this sub game and this sub game is occurring after the history is in and what is the action or what is the strategy specified by this equilibrium profile in this sub game?

In this sub game, the equilibrium profile is telling us that the incumbent will fight and the point is this. This action of fight is not equilibrium in this particular sub game, so in this particular sub game if the incumbent has to move and he has to choose between fight and accommodate, accommodation is better than fighting because accommodation gives him 1, fighting gives him 0. Therefore this is not an optimal action, it is not an optimal strategy for the incumbent player 2.

So that is why we say that this is not a sub game perfect Nash equilibrium in the sense that this strategy profile is not generating equilibrium in this sub game following the history in. Likewise, if I have to consider any strategy profile as a possible candidate for sub game perfect Nash equilibrium, I have to look into every possible sub game of that entire game and check whether the strategies of the players in that strategy profile generate equilibrium in those each and every sub game.

So that is what one means by sub game perfect Nash equilibrium, but remember, what I have done so far is just to give an illustration, vague definition.

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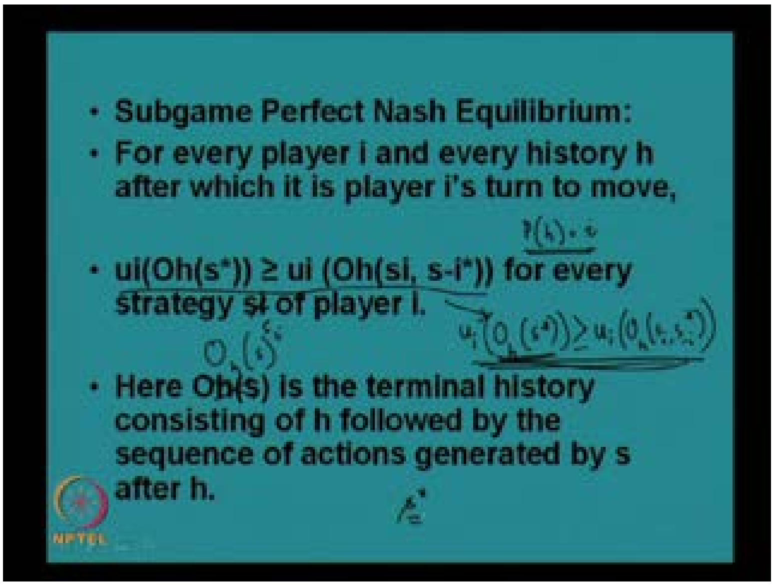

But now, I shall go to more concrete and methodical definition and here is the definition. So sub game perfect Nash equilibrium is defined as the following: for each for every player i and every history h after which it is players i's turn to move this has to be satisfied, let me write it down separately, it is subscript creating a problem.

This should be the proper notation similarly, this should be O h s. Now what we are saying is the following is that suppose s star is a sub game perfect Nash equilibrium, then s star must satisfy this condition for each player and for each player and for each non terminal history h.

Suppose, I take any non terminal history h and by applying this player function, I see this is the case that is it is now the player i's turn to move. Then it must be the case that this should happen, which means that if player i take the action s-i star and other players are taking their action s 1 star, s 2 star etcetera then the game will reach some terminal history after h has happened.

That terminal history is indicated by this o h s star, so what is this saying? Here O h s is the terminal history consisting of h, followed by sequences of action generated by s after h.

So if I take any strategy profile s and any non terminal history h, then O h s giving me that terminal history which is generated by this strategy profile after this non terminology h has happened. Here we are talking about this equilibrium strategy profile s star. I take any non terminal history, not necessarily on the path of the equilibrium terminal history which is important.

I take any non terminal history h and after h has happened I see now, it is i's turn to move, then it must be the case that i is taking an action s i star, which is optimal for him in the following this sense. If he changes his action, this should be s i; if it changes his action to s i, then his payoff cannot be more, it can either be less or equal and that is why taking this action s i that is optimal. This is true for every player i, whose turn it is to move after every h which is just an arbitrary non terminal history.

So this is the definition, precise definition of sub game perfect Nash equilibrium, just to repeat the main point of the definition. We have this s star which is the equilibrium profile, strategy profile and I know that every strategy profile generates a terminal history.

What is important about this sub game perfect Nash equilibrium is that I do not consider only those sub histories on the path of this particular terminal history generated by s star alone. I take into account every possible non terminal history h and I look at the player whose star it is to move after h has happened.

I look at the terminal history generated by the s star, after that h has happened and I try to see whether the action chosen by i is optimal or not; if it is optimal, then I have got the sub game perfect Nash equilibrium.

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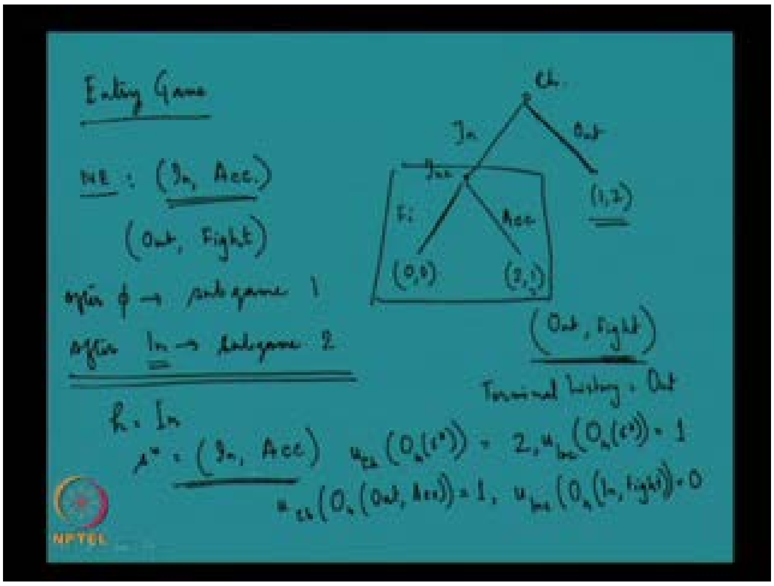

So that is the basic idea. Now I can apply this idea as I told you, in case of different games and try to see whether the Nash equilibria that we are talking about are sub game perfect or not. In this entry game, there were two Nash equilibria: one was in and accommodate, another was out and fight.

Let us see whether they are sub game perfect Nash equilibrium or not. I try to give the impression that out fight is not a sub game perfect Nash equilibrium but, before going to that let us do more systematically. Let us look at in and accommodate is this sub game is perfect Nash equilibrium. I know this, in this game there are two sub games, one is after h one sub game and after in, this is sub game 1 and this sub game 2.

So there are two histories to be considered after phi and after n. One thing which is interesting is that remember that after the phi itself the sub game that we are considering is the game itself.

In that game which is just we are looking at the game from the point 0. That sub game itself is the game and we know that in that game the Nash equilibrium is already in and accommodate. So after phi I need not check whether in this sub game, there is Nash equilibrium or not because I already know that, in the game the entire game as a whole in accommodate is a Nash equilibrium.

The only thing I need to check is that, after the history in that is in this sub game are we getting a Nash equilibrium or not. If we are getting a Nash equilibrium there then, we have got Nash equilibrium in every possible sub game and with through. Here and forget about this, so this is the crucial thing that we need to look at. Now here h is in and what is the equilibrium strategy profile? It is in and accommodate. What is the payoff of the first player? That is the challenger.

So let us call this as u ch, the challenger in the sub game following the history in. It is given by in and accommodate, which means he is getting 2 and what is the payoff of the incumbent in equilibrium which is 1. So this is what they are getting in equilibrium, point is this optimal if they deviate will they get better.

While if player 1 deviates and changes his strategy to out, so I am out now and the player 2's action strategy remains the same which is accommodate then what does player 1 get he gets 1 because this is what is happening now. So it is suboptimal for player 1 to deviate what about player 2 that is the incumbent.

Here he can deviate and he can choose to fight, that is the only deviation that he can do. If he chooses to fight he get 0, so which is less than 1. Therefore, we have the result that in accommodate is a sub game perfect Nash equilibrium because it is generating equilibrium in every possible sub game, after every possible non terminal history.

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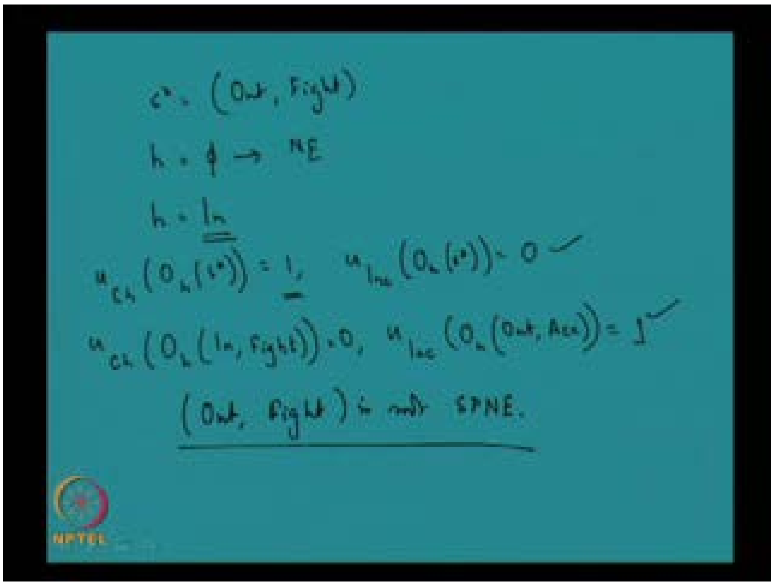

What about out fight? If I am talking about out fight, this is the thing that I am considering history I know if the history is phi then it is Nash equilibrium. So that is what we have seen before but, if the history is in, then I have to check whether this is equilibrium or not. So, if the history is in how much are the players getting? He is staying basically out here, so he is getting 1; the incumbent in equilibrium is getting 2.

If he deviates how can he deviate? He can choose to come in, so in and fight and if he comes in and there is a fight, he will get 0. So, it is not optimal for him to come in, so that is all right. So for the first player, the challenger it is in fact optimal for him to stay out but, now let us look at the incumbent now he can choose to accommodate.

Now here if he accommodates, he will get 1 if h has happen, then he will get 0. If h has happened that is the in action has taken place, then if he fights he gets 0 but, if he accommodates alright then he gets 1. So for player 2 that is for the incumbent after h has happened, so after in has happened, if he sticks to his strategy he get 0 but, if he deviates and accommodates he gets 1 which is better.

For player 1 that is for the challenger, if he sticks to his strategy which is basically out he gets 1 and if he chooses to get in, then he get 0; so, it is optimal for the challenger to stay out but, it is not optimal for the incumbent to fight and therefore, out fight is not sub game perfect Nash equilibrium.

So this is one way to show that the sub game, the idea of sub game perfect Nash equilibrium can be applied two cases where the equilibrium, Nash equilibrium is not robust. The Nash equilibrium is generating outcomes where in sub games which are out of the terminal history generated by the equilibrium strategy profile, the actions of the players are now optimal and in though those cases, we can use the idea of sub perfect Nash equilibrium to rule out that Nash equilibrium.

So, let me conclude here itself and in the next lecture, we shall talk about other facts of sub game perfect Nash equilibrium, thank you.

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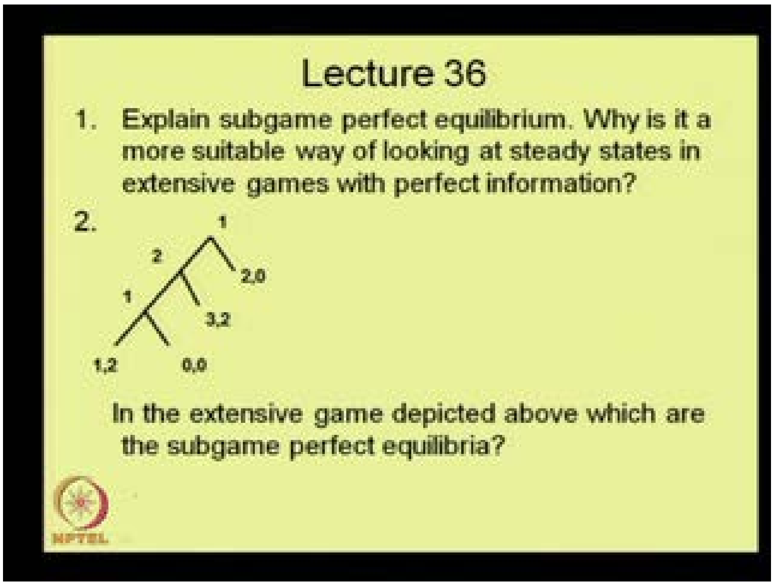

Explain sub game perfect equilibrium. Why is it a more suitable way of looking at steady states in extensive games with perfect information?

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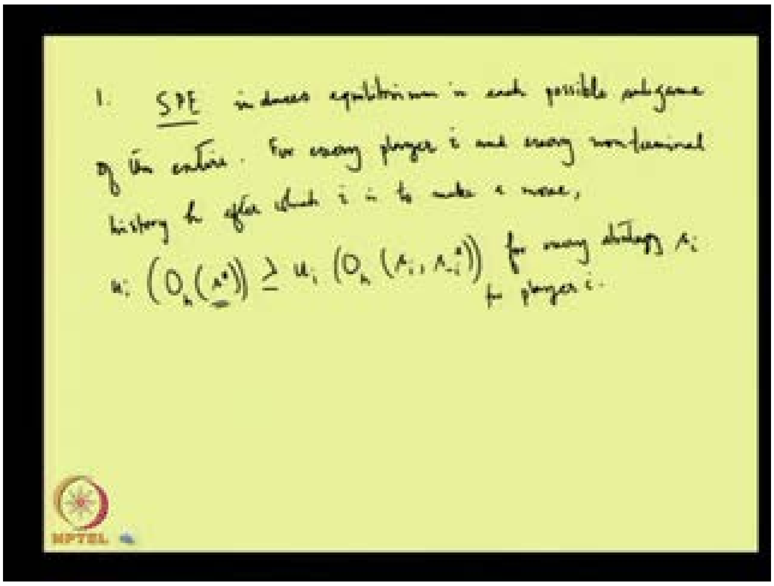

Sub game perfect equilibrium essentially what it means is that, it induces equilibrium in each possible sub game of the entire game. So technically, what we are trying to say is that for every player i and every non terminal history h, after which i is to make more we must have u i O h s star.

This is the definition, what we are saying is that in sub game perfect equilibrium, it is not only the case that the game as a whole is in equilibrium, in the beginning of the game the strategy situation by players are optimal but, the optimal strategies are there for each and every possible sub game.

Even in those sub games, which sub games are not touched by the equilibrium strategy of each player. So there might be some sub game, which is not reached by small s star but, in those sub games also, the strategies of players must be optimal.

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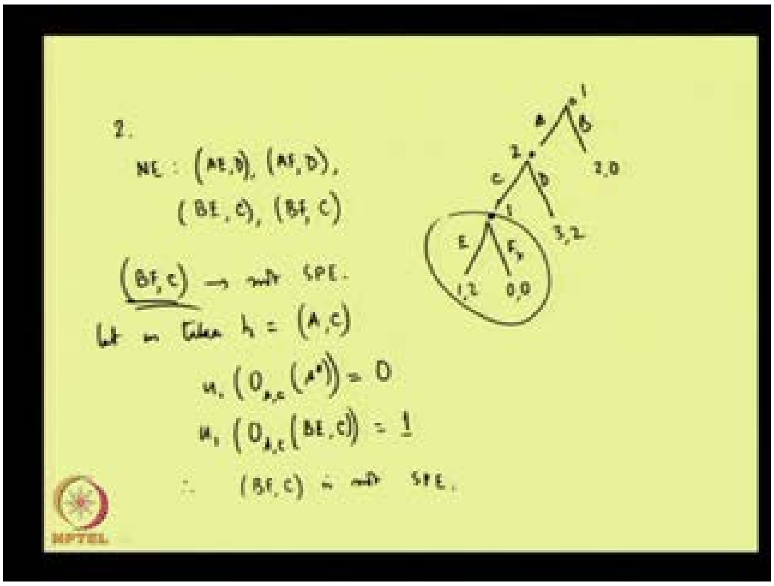

One example will make it clearer. In the extensive game depicted above, which are the sub game perfect equilibria, we have talked about this game before. So this was the game and we have seen that Nash equilibria in this game are 4, these are the 4 Nash equilibria. We have seen that in out of all these Nash equilibria, BF C was a little bit problematic; we said that, there was an interpretation of whether this is steady state. We are now going to show that this is not a sub game perfect Nash equilibria, not SPE why, let us look at the history A C.

Now after A C has happened, if the strategy of player 1 optimal that is the question that we are asking, after A C has happened what we are saying by this strategy profile BF C but, player 1 is getting if he plays according to BF C is 0 but, now suppose he deviates and he plays BE C, then he is going to get 1 therefore, BF C is not sub game perfect equilibrium.