Game Theory and Economics
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Module No. # 05
Extensive Games and Nash Equilibrium
Lecture No. # 05
Backward Induction

Welcome to the fifth lecture, of the fifth module, of this course called game theory and economics. So, what we have done so far in this course is that, in this particular module, we have been discussing sub game perfect equilibrium, in case of extensive game. In the last lecture, we have looked at the definition of sub game perfect equilibrium and how it can be applied to games - extensive games - with perfect information.

While doing that, we have seen that, this idea is important and in many cases we can apply it. It has some interesting properties also; for example, if I have got a sub game perfect equilibrium, it is necessarily Nash equilibrium also in an extensive game. So, every sub game perfect equilibrium is Nash equilibrium, because what happens in sub game prefect equilibrium is that, it induces equilibrium that is Nash equilibrium in every possible sub game of the game.

Now, the game itself is a sub game of it. Therefore, in the game also there is equilibrium if we have a sub game perfect Nash equilibrium; so these are important properties. However, this procedure of finding sub game perfect equilibrium - how we did that? The idea that we invoked was the following. That suppose I have a strategy profile, then with respect to that sub game perfect equilibrium strategy profile, we shall get a corresponding terminal history and with respect to the terminal history, the players are getting some payoff.

What we say is that there might be many sub games, in that game, which are in consistent with that terminal history, which may not fall other sub games, may not fall in the path of the terminal history.

Never the less in the sub game perfect equilibrium, the strategies have to be specified by the players in all those sub games, which are out of the way of the terminal history also. In the sub game perfect equilibrium, all this strategies which do not fall in the terminal history described by the equilibrium strategy profile, all those strategies must also be optimum.

Now, just for conceptualizing this, for visualizing this, remember we do not need that players are rational. I know that the other players are rational, we do not need that assumption. We also do not need the assumption that the players know the structure of the game.

So, this things may not be common knowledge, I may not know what is the structure of the game, I may not know that the other players are rational or irrational, does not matter. What matters is that I go to each and every sub game of the entire game, I look at the actions of the other players in those sub games, if those actions are steady, if those actions are repeated over time, then I have got equilibrium strategy profile, that is all. I do not need that the players I have has some behavior assumptions about the preferences of the other player, they may not be rational, but their actions, if they are consistent, then I have a steady state.

So, this is the idea of sub game perfect equilibrium, but another way to look at sub game perfect equilibrium is to rely on rationality that the players know that they are rational. So, it is not only the fact that I know I am rational, I also know that the other players are also rational. They know the structure of the game, in the sense, what are the actions that are available to other players when it is their turn to move.

So, how many stages are there? If the structure of the game is known, if rationality is a common knowledge, then we can have another interpretation of sub game perfect Nash equilibrium, in the sense that they rely on this fact that other players are rational. Therefore, they can form a belief what action other players will take if they play an action.

So, in this case, players are not relying on experience, their they are relying on fact that the other players are rational and they also have information about the structure of the game.

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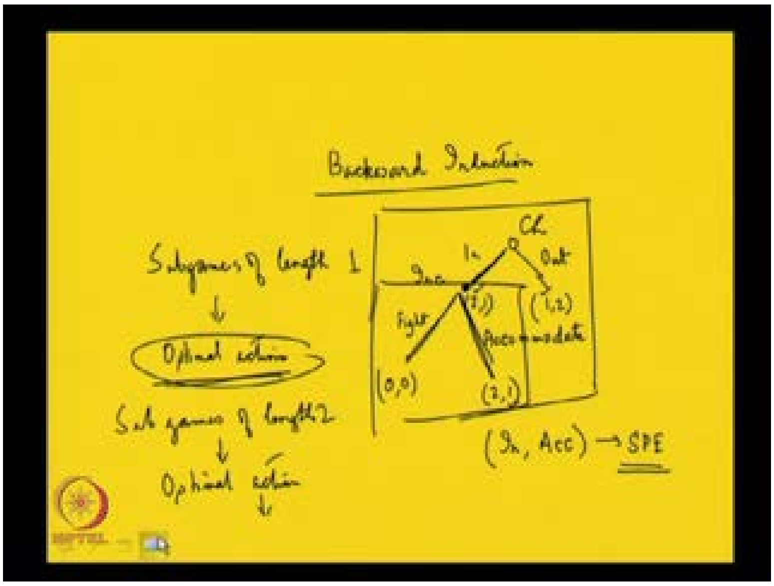

So, one example will be the following, let me give you an example. This idea, I am depending on the fact that the other players are rational, I have tried to find out what is the sub game perfect equilibrium, this method is called backward induction. Let us try to apply this idea of backward induction in the familiar game, the entry game. This was the game, suppose I want to invoke the idea of backward induction, then how shall I go about doing it.

Now, here, what I shall say is the following that the challenger knows the incumbent is rational. Given that the challenger knows that the incumbent is rational, he knows - that is a challenger knows that if he takes this action in, challenger also knows the structure of the game, so he knows that after he has taken the action in, the incumbent will be left with two options, one is fight and the other is to accommodate.

Now, since the challenger knows that the incumbent is rational, he can figure out that the incumbent will never fight. So, this is sub optimal for the incumbent, because he is getting 0 here if he fights. Therefore, the only action that is possible or that is rational for incumbent to take is to accommodate, because he is getting 1 here.

So, this is what the challenger is figuring out by intersecting the game, by looking at the game, trying to figuring out what the other player will do if he takes a particular action. Therefore, the challenger in that case what he shall do? He now knows that if he gets in, obviously the incumbent will accommodate. In which case, basically the payoff can be thought of as being here. So, the challenger has to basically compare between 2 and 1. If he gets in, the incumbent will accommodate, so he will get 2. If he gets out, he is getting 1, therefore he will always get.

So, this is the way to apply backward induction, where people are depending on each other's assumption of rationality, which is basically the old assumption that the theory of rational choice people want to maximize their payoffs, people know the structure of the game and then we are trying to find out what is the sub game perfect equilibrium. Here, we are seeing that in accommodate is the - In fact, there is no other sub game perfect equilibrium; this is the unique sub game perfect equilibrium in this game.

So, this is the idea, we can use this notion of sub game perfect equilibrium, I mean this notion of backward induction to find out the sub game perfect equilibrium in other games also. But, before we go to that let me just tell you the rule of thumb, the way of doing this backward induction method, the way of applying this backward induction method. So, what we do is that we start with sub games of length 1, what is meant by length 1? Well, in any game the length of the longest terminal history is called the length of that game.

So, if I am talking about the sub game of length 1, by that I mean that the longest terminal history in that sub game is 1 - of length 1. So, I take each and every sub game of length 1, I look at the player who is to move at the beginning of those sub games. So, each of this players who are going to move at the beginning of this sub games, I try to find out what is their optimal action or in many cases, it can be actions, not action; there can be more than one action which is optimal.

So, I figure out what are the optimal actions of the players who are in the beginning of sub games sub length 1, then I go backward and that is why it is called backward induction, I go backward and I look at the sub games of length 2. So, I take the optimal actions, then sub games of length 2. When I go to the sub games of length 2, we take these optimal actions as given and go to the sub games of length 2. Then, we look at the players who are at the beginning of the sub games of length 2, we try to figure out what are their optimal actions. Given these optimal actions are of sub games of length 1.

So, we again have optimal actions in this stage, likewise we go on. So, we go on backwards from sub game of length 1 to sub game of length 2, backwards may be 3, till we reach the beginning of the game. By doing that the series of optimal actions that we trace out by this process, those series are nothing but the sub game perfect equilibria or equilibrium.

So, this is basically the method here also that is what we have done. We started out with this sub game of length 1, we have seen that the incumbent has the optimal action of accommodate, then taking this accommodate action as given, we went back and looked at this sub game, which is basically the entire game of length 2. Then, saw that with given optimal action accommodate player 1 who is to move in the beginning of the sub game of length 1 of length 2 must a get in. Therefore, accommodate is the optimal set of strategies, which is known as the sub game perfect equilibrium, so this is the idea.

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Now, let us step back a little bit, see how and which situations are conducive for the applications of this notion of sub game perfect equilibrium of this backward induction method. Now, obviously, if I have a game which is infinite horizon, what is meant by an infinite horizon game? If the length of the longest terminal history is infinite, then we have an infinite horizon game. If the length is finite, then we have a finite horizon game. Now, this is crucial, because in this case if I have to apply the backward induction method, I have to start from somewhere, somewhere of sub games of length 1. But, if the game is infinitely stretched, then there is no finality, there is no end of the game, therefore I cannot apply this method. So, this method can be applied only if the game is a finite horizon game - this method of backward induction that is one thing. But, this method that I described here rather loosely can be tightened a little bit by stating it more formally what is the method of backward induction, let us try to do that.

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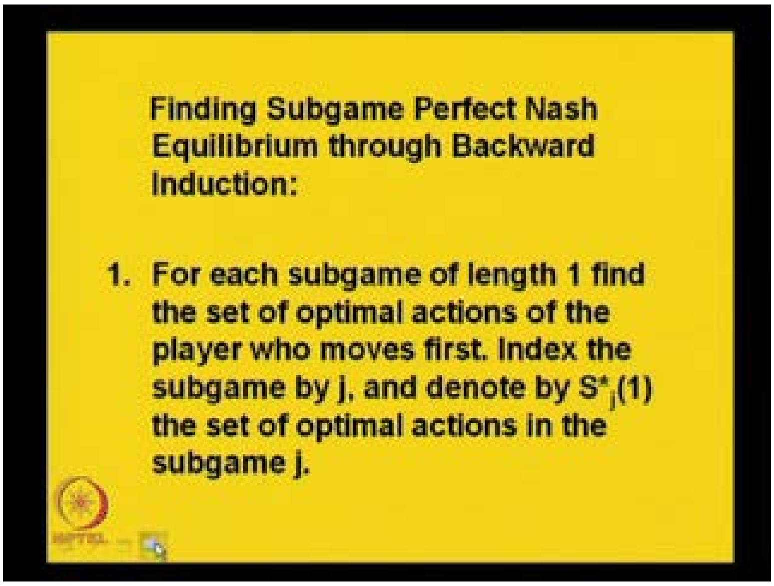

This is the method formally stated. Finding sub game perfect Nash equilibrium through backward induction, there are some steps. First step, for each sub game of length 1 find the set of optimal actions of the player who moves first. So, at the beginning of the player, at the beginning of the sub game, there is a player, we have to find out what is his optimal action or optimal actions.

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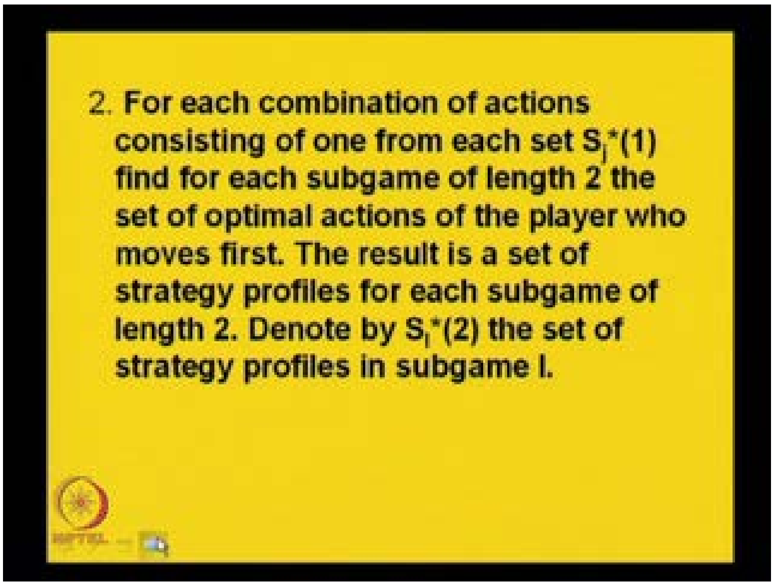

Index the sub game by j, denoted by S j star 1 the set of optimal actions in the sub game j. So, there might be many sub games of a length 1, they can be given index j. By capital S j star 1, I mean the set of optimal actions in that sub game of length 1, which is index by j. This is the second step, for each combination of actions consisting of 1 from each set s j star 1 find for each sub game of length 2, the set of optimal actions of the player who moves first.

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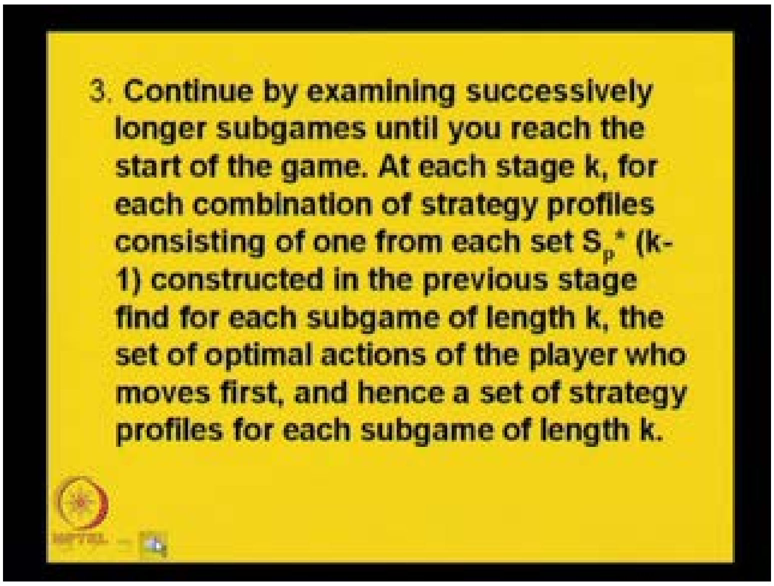

So, this is just what I have said before, but more formally from sub game of length 1 then we go back to sub game of length 2, the result is a set of strategy profiles for each sub game of length 2 denote by S l star 2, the set of strategy profiles in sub game l, l is just an index. This is the third stage; continue by examining successively longer sub games until you reach the start of the game, this is just induction.

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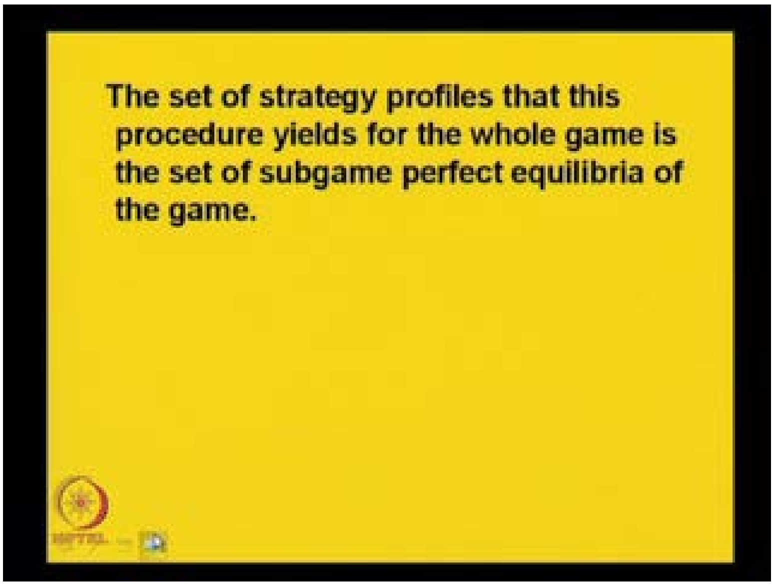

At each stage k, for each combination of strategy profiles consisting of 1 from each set S p star k minus 1 constructed in the previous stage find for each sub game of length k, the set of optimal actions of the player who moves first, and hence set of strategy profiles for each sub game of length k. So, just stating the induction, if you have found out what are the optimal strategy profiles of length k minus 1, then you go back to sub games of length k until you reach the start of the game.

The set of strategy profiles that this procedure yields for the whole game, is the set of sub game perfect equilibria of the game. So, this is the formally stated procedure of sub game perfect equilibrium, some interesting properties of sub game perfect equilibria are the following. One is that if I have got a finite extensive game with perfect information and then there must be at least one sub game perfect equilibrium in that game. So, finite games will always have sub game perfect equilibria.

Another interesting property is that if at any stage the player has a unique optimal action, if at any sub game if you have got a unique action for a player, who is to move at the at that sub game, then there is a unique sub game perfect equilibrium and this is inductive.

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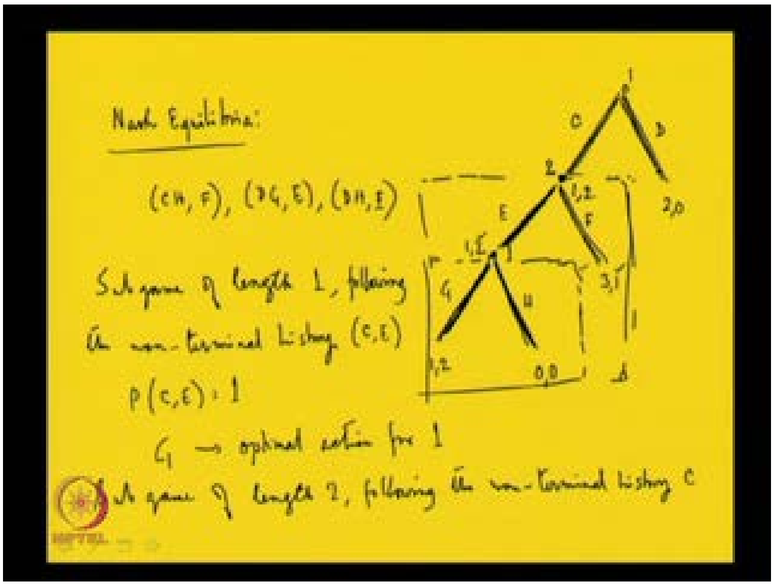

What we propose to do now is to take another example. Let us see if we can find sub game perfect equilibrium if there is indifference, in the sense that the game that we have just seen, the entry game, was a case where a player at the beginning of any sub game is not indifferent, he has an optimal unique action, but if there is indifference, then how do we deal with that.

But, before talking about difference, let us take a more difficult game where there is no indifference. So, a simpler, where there is no indifference game, where the actions are unique. This is a game which we have seen before, so this is the game, suppose we have to find out what are the sub game perfect equilibria or what is the sub game perfect equilibrium in this game, to do that what we need to do is to first, remember what were the sub game not sub game perfect, but Nash equilibria of this game. The sub game perfect Nash equilibria will be subset of this set of Nash equilibria. So, just try to remember what are the Nash equilibria of this extensive game, there is more than one.

In fact, there are three, C H, F, C H and f; this is one; D G and E and D H, E. So, these are the three Nash equilibria profiles - strategy profiles, some of them will be sub game perfect, some of them will not be. We have to find out which are sub game perfect equilibrium. Now, if I have to use the method of backward induction, then what do I do? I start with sub games of length 1, this is the a sub game of length 1. In fact, this is the only sub game of length 1, this is basically following the history - non-terminal history C E. So, in this sub game, we have to look at the optimal action of the player who is going to move in the beginning of this game.

Now, this is this player, is basically 1, because P C E is 1, now what is optimal for 1? Optimal for 1 is obviously G, so G is optimal action for 1 after the history C E. So, this we are taking as given, then we are trying to go backwards, then basically we go to the sub game of length 2 and this is described as the following.

This sub game is following, the non-terminal history C, after C this sub game comes about, alright. Now, in this sub game where 2 is to move first, 2 basically has to compare between what and what, 2 has to compare between E and f, but given that player 1 is going to take this action G, because we know as I just mentioned that the players are dependent on the fact that the other players are rational.

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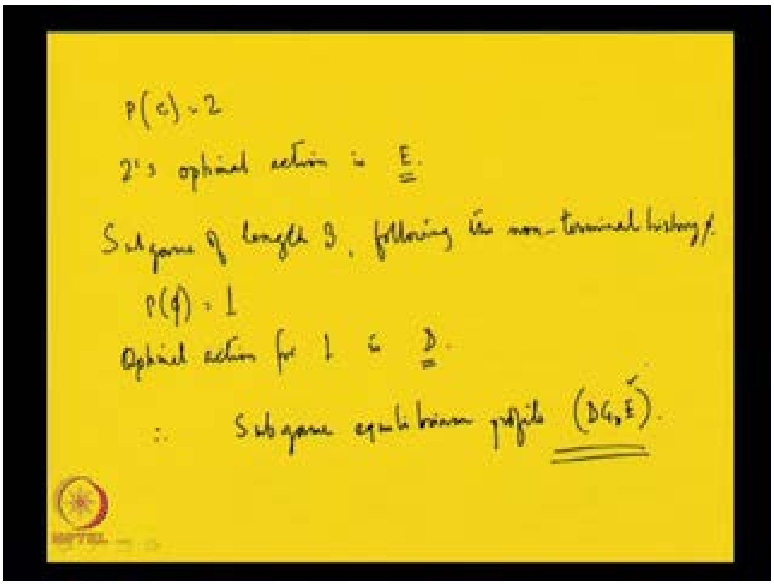

So, 2 knows that player 1 is rational, therefore 2 is believing or he is trusting the fact that 1 will take this action G. So, I can as well write this payoff here 1 2. So, the comparison is between 2 and 1, this 2 and this 1, obviously 2 is greater than 1 and action is E.

So, this will fall on the Nash equilibrium strategy, this will also fall on the sub game perfect of a Nash equilibrium strategy profile.

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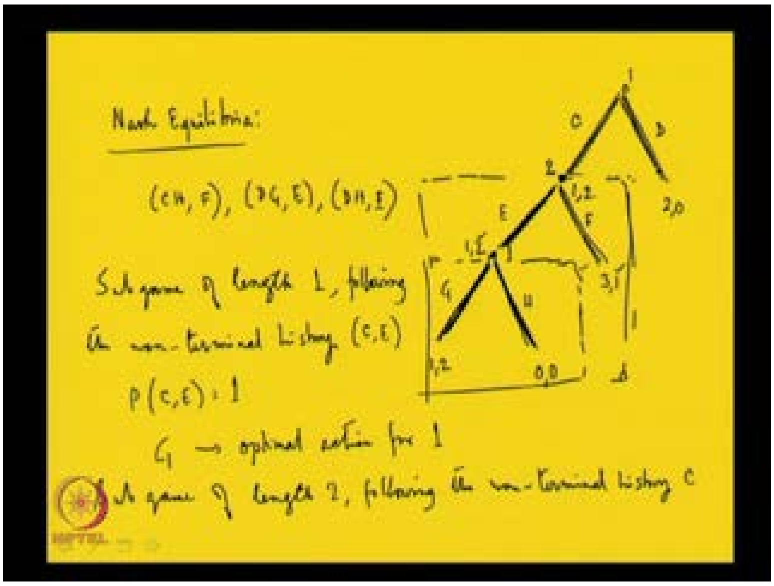

This is E and this is G, E and G they will fall on the Nash equilibrium strategy profile. Then, we again go backward and look at the sub game of length three, this is following the non-terminal history phi and who is the player? Who is going to move in this sub game? It is 1, 1 is depending on the fact that this action and this action are going to be taken by 2 and himself in the subsequent stages.

So, the choice is between this and this, C and D. If he plays D, he is getting 2, if he plays C, then 2 is going to play E and in the third stage, 1 is again going to play G, therefore 1 will choose D. So, we have got a series of optimal actions, 1 is G, the second is E and the first is D.

Sub game perfect equilibrium profile is D G and E. Look how I have arranged these actions before this coma in this parenthesis, before this coma I have D G. D G is basically the strategy of player 1, after the coma I have the strategy of a player 2 and the interpretation is the following. Player 1 is saying that after the history phi I am going to move D, I am going to play D. If C E is the history, then I am going to play G. Player 2 is saying the following; the player 2 is saying that if you play C, I will play E. This is the only sub game perfect equilibrium in this game.

So, this was a much easier game to solve, but it might be more complicated as I just told. That if I have indifference between two actions for any player or more than one player, then I have to consider many optimal action profiles or many optimal strategies of a player at a particular stage, then the task becomes more complicated.

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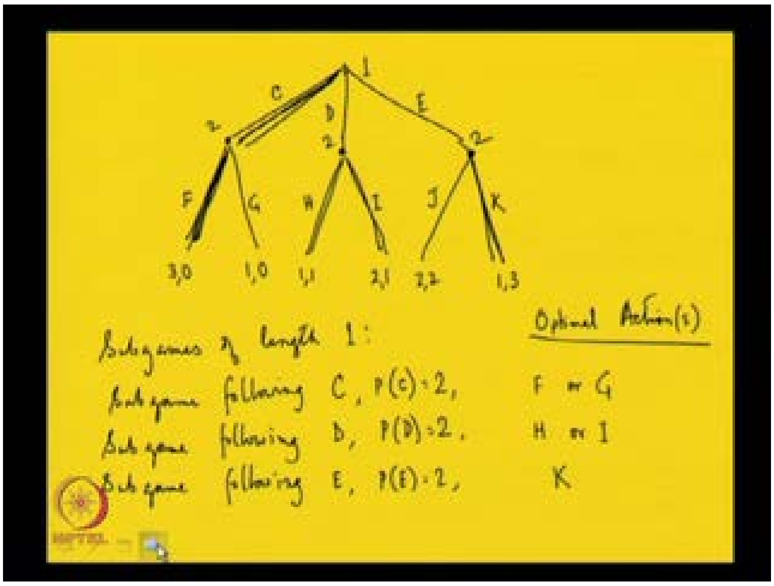

Let us take one example. So, player one has two three actions in stage 1 that is after the history phi C D and E. If he chooses C, then player 2 gets to move, he will have two options. If he chooses D, again player 2 has two options, H and I. If he plays E, again player 2 has two options, J and K.

So, there the game ends, there is no other action by any other player; these are the payoffs. One needs to find out what is the sub game perfect equilibrium or equilibria through backward induction. We follow the rule, we start with sub games of length 1, unlike the case before here I have three such sub games not one. So, sub game following C, if the history is C, then I have sub game, the player function is telling me that this is player 2 who is going to move. What is the optimal action? This is optimal action or actions. In this case, player 2 is indifferent; player 2 is indifferent between f and G. So, f or G both are optimal, then I have sub game following D, P D is equal to 2. What are the optimal actions? Again, I have 2 optimal actions H or I. Lastly, here however there is unique optimal, which is K, now what is the task? I have to figure out what are the optimal strategies of player 2, given that optimal strategies I have to know. What is optimal for player 1 in the beginning of the game? So, when you go back to sub game of length 2, there is just 1 sub game of length 2.

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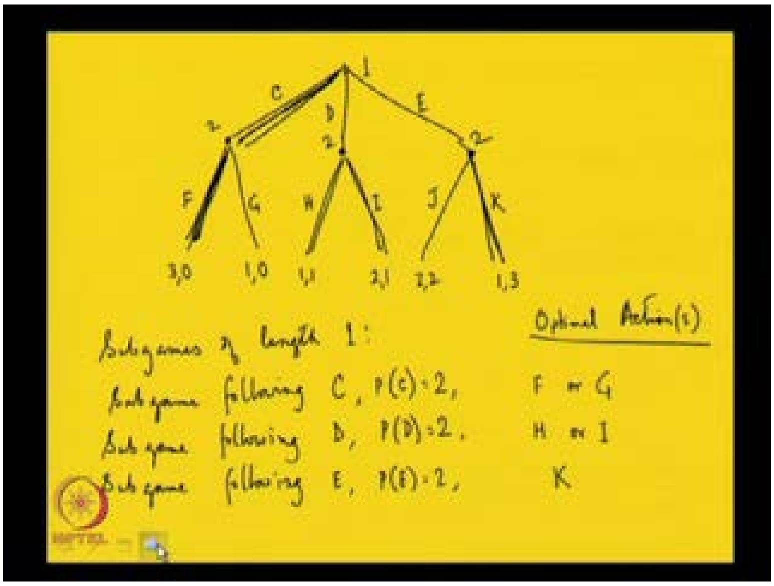

Now, the point is since there are indifference, since there is more than 1 action which is optimal, I have to consider all combinations of this actions. So, the strategy profiles, optimal strategy profiles, in fact there are more than one, it could be the following; F H K his saying F H and K, so this is optimal.

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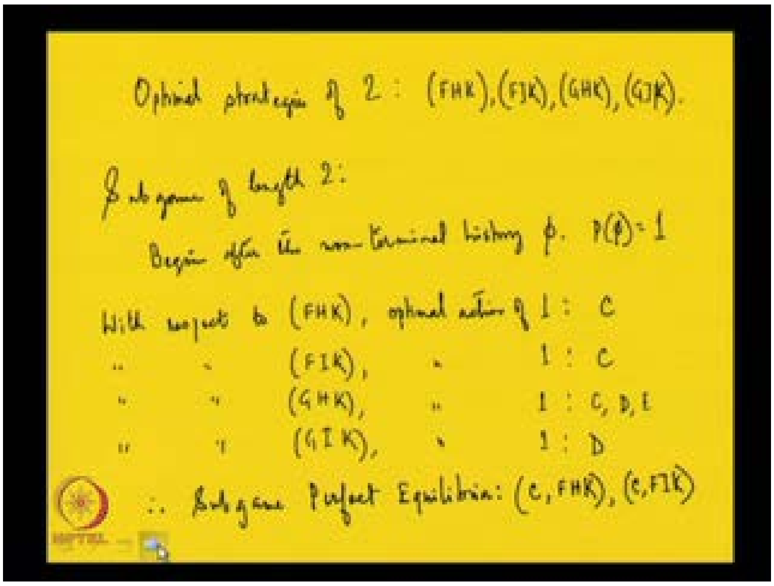

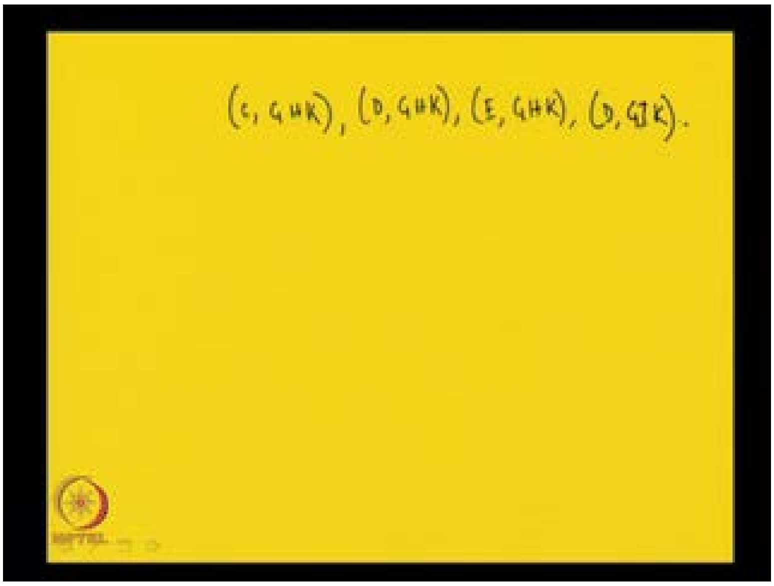

But, equally optimal is for example GHK or for example FIK or GIK. So, there are basically four optimal strategies of player 2; one is FHK, then I have FIJ, GHK, GIK. So, these are the four optimal strategies of player 2 with respect to each of them, I have to know what is the optimal action of player 1 if I go back and look at sub game of length 2.

Here, it begins after the non-terminal history phi; P phi is equal to 1 that is player 1 has to move in this sub game of length 2. I have to know what is the optimal action or optimal actions of player 1 with respect to each of these strategies of player 2, with respect to FHK - F H and k, the player 1 will basically compare between C D and E, with C he is getting 3, with D he is getting 1 and with E he is getting again 1, C is best, so C.

With respect to FIK, optimal action of 1 – FIK – now the choice between 3, 2 and 1, again 3 is best, so again I have got C. With respect to GHK, now there is a little complication, because here the choice is between 1 and 1 and 1 and all of them are equal. So, what is the optimal action for player 1 in this sub game of length 2? All of them are optimal, C and D and E all of them are optimal, so C D E.

Finally, if I have got GIK optimal action of 1, G I and K the optimal action is I, mean D, because if he plays D, player 2 will play I and he will get 2, in other cases he will get 1. Therefore, what is the sub game perfect equilibrium or equilibria in this case C, FHK, C, FIK and then you have C, GHK, D, GHK and E, GHK and finally, D, GIK.

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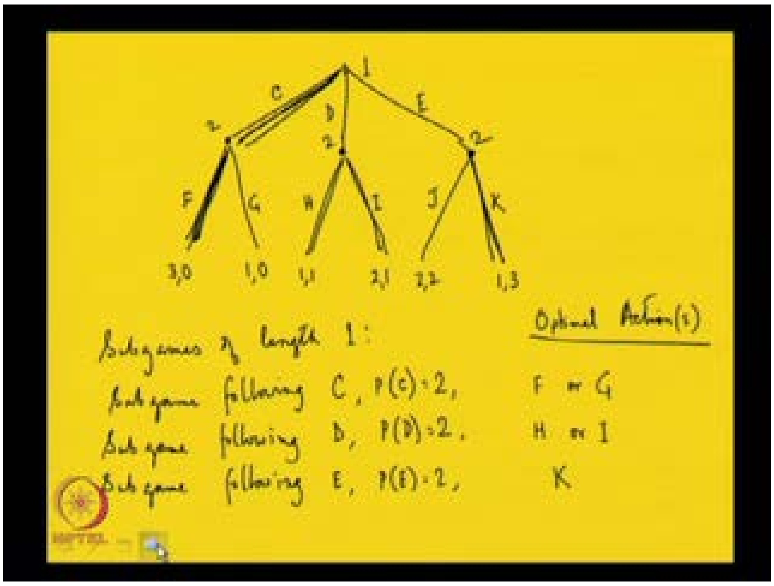

So, essentially I have a six sub game perfect equilibrium, what could be interpretation of them, let me just take you through one of them, one sub game prefect equilibria. This is C F I K, let us look at the diagram; what this equilibrium is saying if the followings C F I K? Player 1 is saying that at the beginning of the game, when the history is 5, I am going to take this action C. Player 2 is giving as the following strategy, he is saying that if the history C, I will play F, if the history is D, I will play I, if the history is E, I will play K. The equilibrium strategy - this equilibrium strategy profile is going to give me this terminal history, where player 1 is going to get 3 and player 2 is going to get 0.

So, likewise we can interpret the other equilibrium strategy profiles, so this is more or less - what we have done is that we have looked at how to find the sub game prefect equilibrium through backward induction, what I proposed to do now is to take another example, where the action set of a player is not finite, which we have seen here the action set of the players where finite. You could take only discrete number of actions and it is a finite number of discrete actions.

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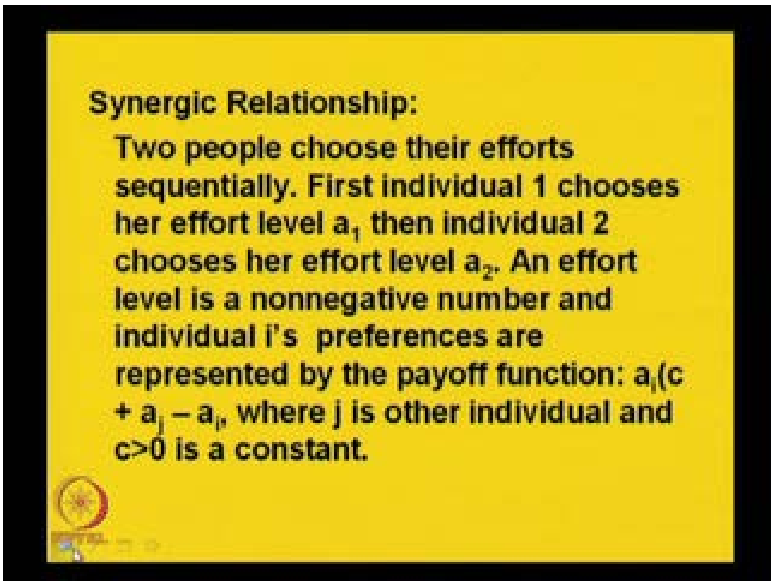

So, this is the example. Suppose this is the case of synergic relationship; this synergic relationship game we have seen before also, in case of games where the players were taking their action simultaneously, those where strategic games. They could be applied in the case of sequential game that is extensive game also. This is the description of the game; two people choose their efforts sequentially, first individual one chooses her effort level a 1, then individual 2 chooses her effort level a 2. An effort level is a non-negative number, individual i's preferences are represented by the payoff function: a i multiplied by C plus a j minus a i, where j is the other individual and c is greater than 0 is a constant.

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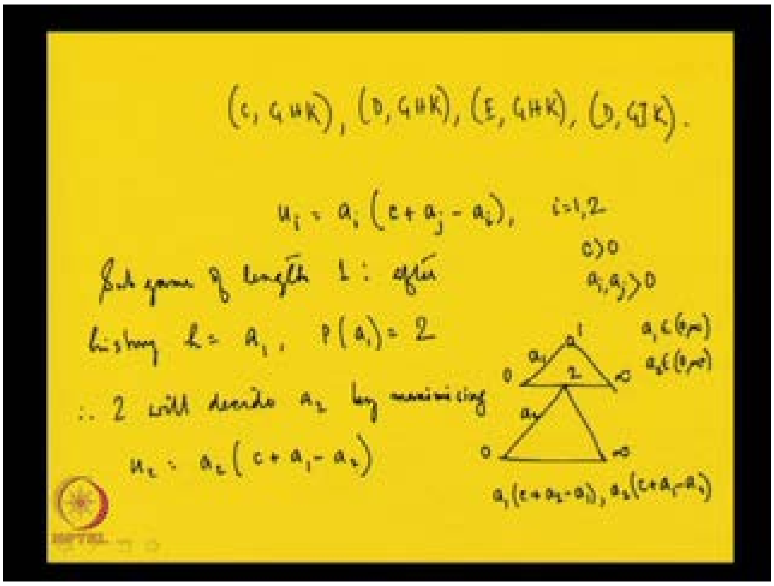

So, if this is the game, then I have to find out for example what is the sub game perfect equilibrium in this game. Remember, in this game player 1 is moving first after player 1 has chosen a 1 his effort level, then player 2 moves choosing a 2. Let us write down the payoff functions, a i multiplied by c is positive, a i, a j are also positive.

Now, how to go about solving this game and finding the sub game perfect equilibrium. So, we basically follow the philosophy of backward induction. Here, it is player 1 who is moving first, so 1 is here. Now, one's actions it is a positive in number, it could be anything. So, suppose I take this, it stretches from 0, excluding 0, and infinity.

So, a 1 belongs to 0 infinity, similarly a 2 also belongs to this same set 0 and infinity. After a 1 has been decided, a 2 moves, now there is no particular position for this point; this point could be any body, it does not matter much, it is just a 1. Here also a 2 is decided, which is stretching all the way from 0 to infinity. After that the game ends, the payoffs are a 1, c a 1; c plus a 2 minus a 1 and a 2 c plus a 1 minus a 2.

So, as I was telling I have to apply this idea of backward induction. So, to do that I have to find out in the sub game of length 1, there is just 1 sub game here, after history h is equal to a 1 this sub game occurs. This player function is giving me that the player 2 is moving in the sub game.

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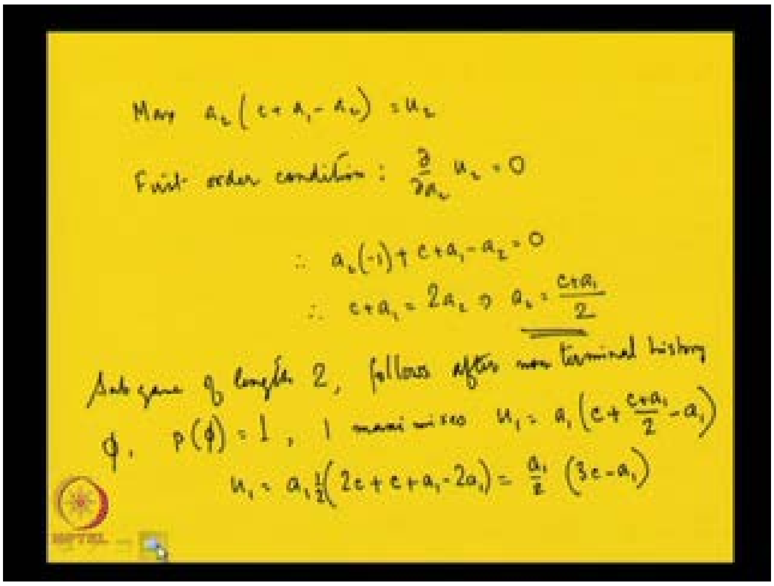

Now, player 2 will choose that action which is optimal for him, he will try to maximize 2 will decide a 2, here by maximizing his payoff function which is u 2. Now, here a 1 is just anything, it is a variable. So, given a 1, a player 2 is maximizing this function. We know how to do that it is just a quadratic function in a 2. So, first order condition is that I differentiate this with respect to a 2 and set this equal to 0, which gives me the following; a 2 multiplied by minus 1 plus C plus a 1 minus a 2 is equal to 0.

So, c plus a 1 is equal to 2 a 2, which means a 2 is equal to c plus a 1 divided by 2. So, this is the optimal action of player 2, it is nothing but the best response function if you remember. Player 2 is going to take this action which is a function of a 1. Whatever be the value of a 1, this is going to be the value of a 2.

Now, what we are going to do is to now go back to sub game of length 2. So, sub game of length 2, this follows after non-terminal history phi. First player is the player who moves here, so one has to choose is optimal action given this. So, 1 maximizes u 1, which is equal to a 1 multiplied by c plus a 2. Now, a 2 is something which we know, now minus a 1. This is what if I take half out 2 c plus c plus a 1 minus 2 a 1 and this is a 1 by 2 3 c minus a 1.

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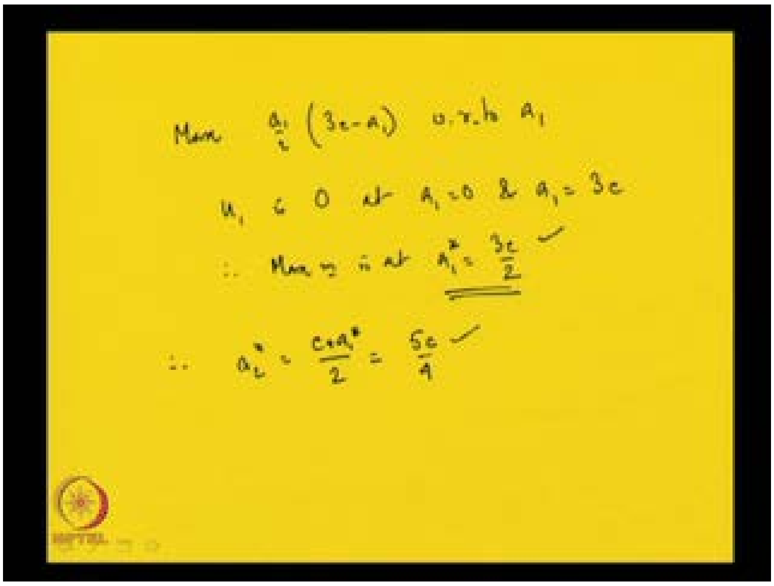

So, this is what player 1's optimal payoff function is, he will like to maximize this payoff function. So, maximize with respect to a 1, this has to be set the derivative of this with respect to a 1, will be set equal to 0 that will give me the solution. Otherwise look at it this, this function achieves 0.

So, u 1 is 0 at a 1 is equal 0 and at a 1 is equal to 3 c, this is a concave function, therefore the maximum is at a 1 is equal to 3 c divided by 2. So, this is the optimal action for player 1 in the beginning of the game, so a 2 - this is a 1 star, let us say a 2 star will be nothing but c plus a 1 star divided by 2. If we solve this, it turns out to be 5 c divided by 4.

So, this is how we can solve the games with infinite number of actions and continuous actions, which are sequential games through application of backward induction. So, before we end this lecture, what we have done in this lecture is to look at backward induction method of solving games of perfect information, but which are sequential, which are extensive games, this is something which we shall discuss in the next class also thank you.

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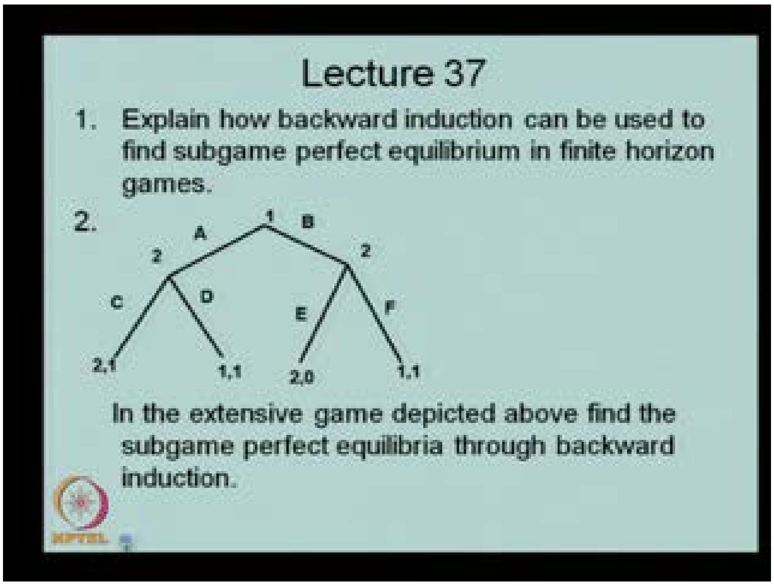

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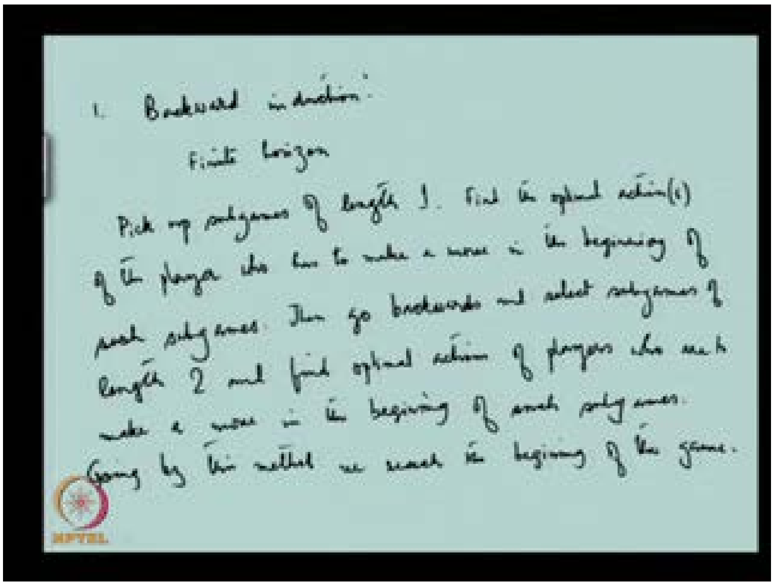

Explain how backward induction can be used to find sub game perfect equilibrium in finite horizon games. So, backward induction, if we have a finite horizon game then only we can use backward induction. Now, in a finite horizon game, what we have is that the length of the longest terminal history is finite.

So, if we have that then we can go to the last stage of the game, we can pick up sub games of length 1. What is meant by sub game of length 1? In the sub game, where the longest terminal history is 1 that sub game will be called sub game of length 1. Find the optimal action; it could be actions also of the player, who has to make a move in the beginning of such sub games.

So, we pick up all the sub games of length 1, pickup then find out what is the optimal action or actions of the players who are to make a move in the beginning of such sub games. Then, go backwards and select sub games of length 2 and find optimal actions of player, were to make a move in the beginning of such sub games.

So, going by this method we reach the beginning of the game, the resultant set of optimal actions will be the set of sub game perfect equilibrium. So, this is how we use the method of backward induction.

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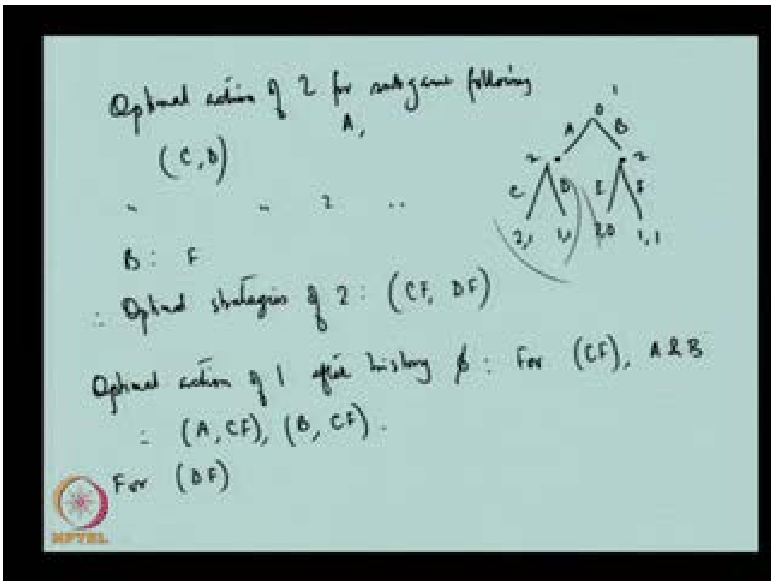

In the extensive game depicted above, find the sub game perfect equilibria through backward induction. So, let us look at this game and find out first the sub games of length 1. There are two sub games of length 1, player 2 makes a move in the beginning of both the sub games. If we look at this sub game what is the optimal action of player 2? The optimal action, they both of them are optimal actions C and D, for sub game following A C D, optimal action of player 2 for sub game following B, is going to be F.

So, optimal strategies of 2, it could be CF DF, alright both are optimal. Now, we go backward and find out the optimal action or actions of 1 after history phi, it depends on what player two’s optimal or player 2 strategies. For CF, player 2’s, it is both a and b.

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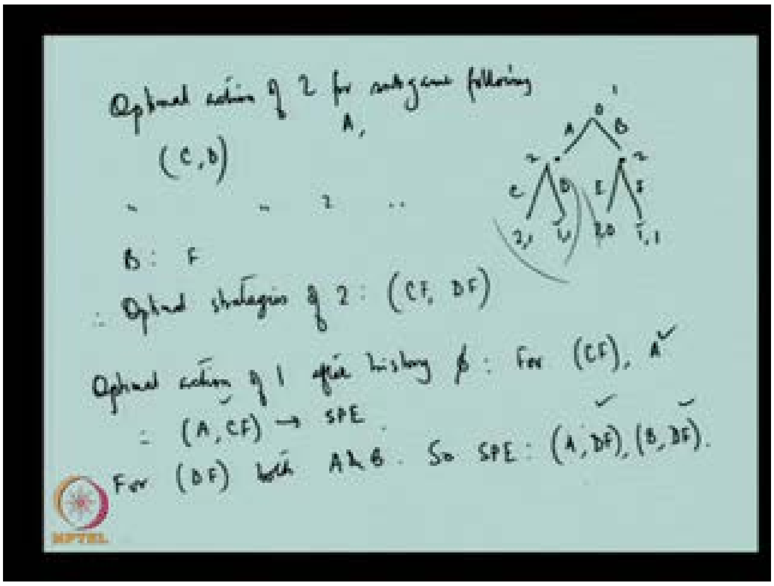

So, A CF B CF are both optimal and for DF, no for CF the optimal action is A in fact; yes this a. So, this is the sub game perfect equilibrium SPE, because from C, player 1 is going to get 2, from f, he is going to get 1, so he is going to be this 1 A. For DF, it is both cases player 1 getting 1, so both a and b, so sub game perfect equilibrium here is going to be A DF and B DF. So, we have three sub game prefect equilibrium; Thank you.