Game Theory and Economics
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Module No. # 06
Illustrations of Extensive Games and Nash Equilibrium
Lecture No. # 01
Ultimatum Game

Welcome to the first lecture, of module 6, of this course called game theory and economics. So, this is the last module that we have in this course.

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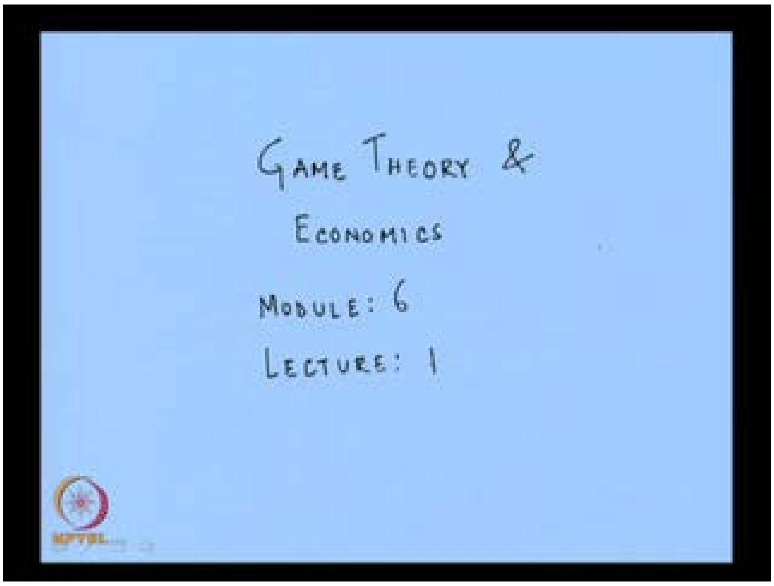

Before we start this module, let me just take you through what we have discussed in the previous module. What we have discussed in the previous module is, basically we have discussed the theory of the extensive games, how they are defined and what are the solution concepts. We have talked about two solution concepts, one is Nash equilibrium, in the case of extensive games and also sub game prefect equilibrium, because you have seen that the idea of Nash equilibrium may not be robust in case of extensive game with perfect information.

We have also seen one particular way of finding the sub game perfect equilibrium in extensive game of perfect information; this method is called the method of backward induction. What we propose to do in this last module, which consist of only two lectures is, to look at some of the well-known illustrations and applications of extensive game, and how we solve those games, and how our solution helps us to predict real life events.

Before we start this last module, what we propose to do is that just finish one last exercise that we have been doing in the previous lecture, which is related to extensive game with perfect information and the application of backward induction.

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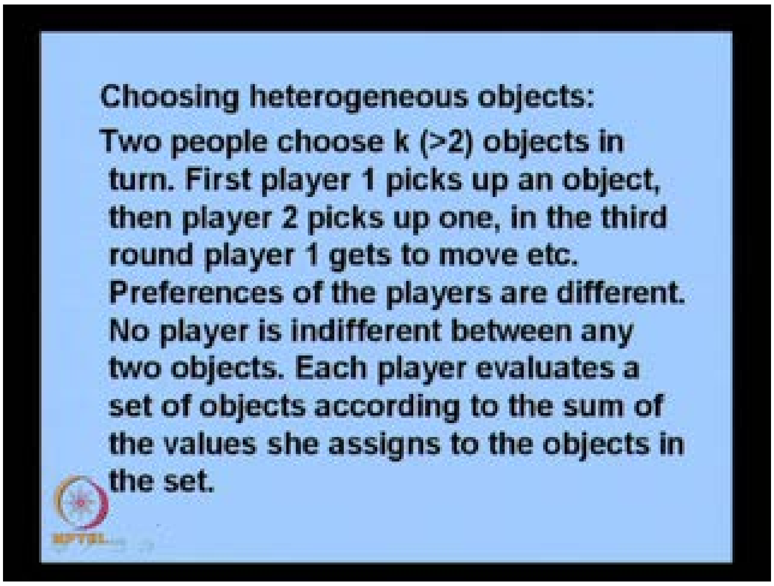

Let us finish that exercise first, and then, we shall start with this last topic of this last module. This was the exercise, choosing heterogeneous objects: two people choose k objects in turn, where k is greater than 2. First player 1 picks up an object, then player 2 picks up one, in the third round, player 1 gets to move etcetera.

These two players are alternatively choosing their objects from a collection of k objects. Preferences of the players are different, so they are not ranking the objects in the same manner. No player is indifferent between any two objects, so there is no indifference between the objects. If each player evaluates a set of objects according to the sum of the values she assigns to the objects in the set.

If a player has selected k number of objects or suppose, n number of objects then, the total payoff that player is getting from this n number of objects is the sum total of individual payoff from this n objects.

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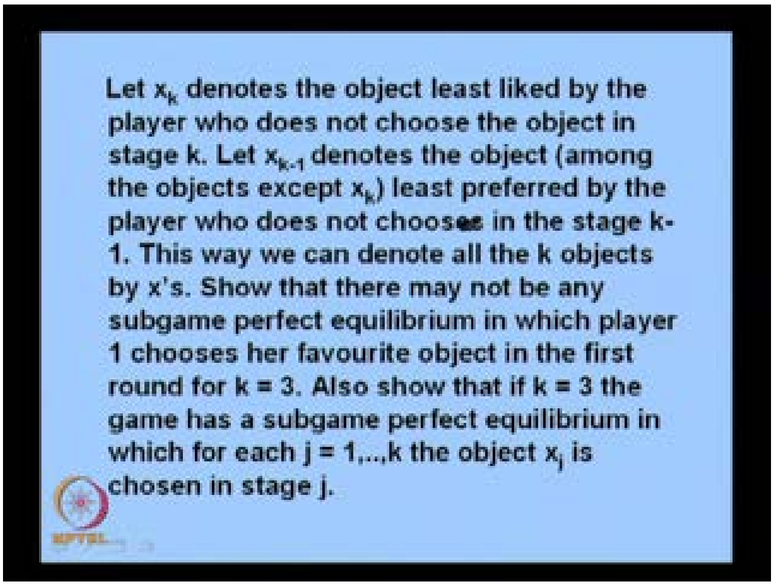

Let x k denotes the object least liked by the player who does not choose the object in stage k. Let x k minus 1 denotes the object, among the objects except x k, least preferred by the player who does not choose - this should be choose - in the stage k minus 1. This way we can denote all the k objects by x' s. Show that there may not be any sub game perfect equilibrium in which player 1 chooses her favorite object in the first round for k is equal to 3. Also show that if k is equal to 3, the game has a sub game perfect equilibrium in which for each j j going from 1 to k, the object j x j is chosen in stage j.

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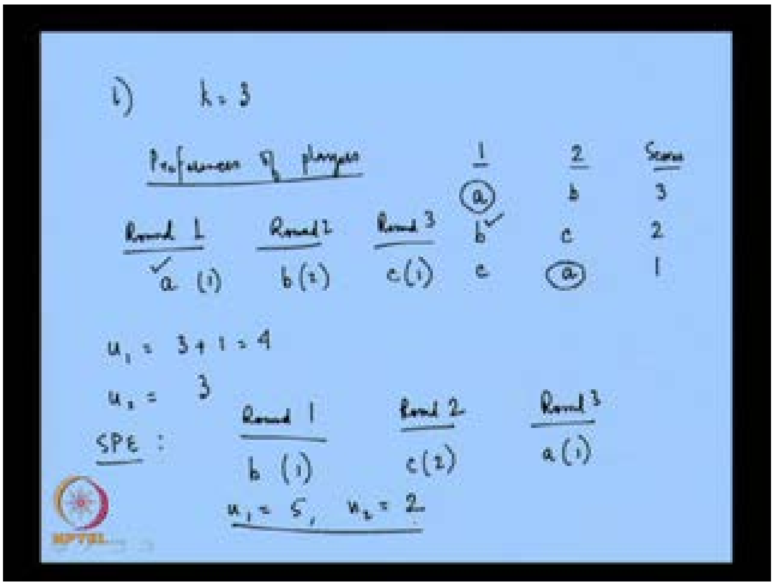

There are two parts to this exercise, the first part is saying the following that, suppose, k is equal to 3, so there are 3 objects. Then, there may not be any sub game perfect equilibrium, where player 1 chooses her favorite object in the first round itself. I might like a particular object of the 3 objects; particular object might be my favorite, but I might not pick the top in the first round; first round in the sub game perfect equilibrium how that can happen? This is what we need to show.

Let us take the following preference pattern, I just have to construct an example. There are only two players, player 1 and player 2. Suppose, this is a preference of player 1 and this is the preference of player 2 and these are the scores. Scores are nothing but the payoffs from the objects that this players are deriving. So, if player 1 gets object a, he gets 3; if he gets b, he gets 2; and if he get c, he gets 1, like that for player 2.

Here, in this case of preferences, suppose, player 1 does indeed pick up a in the first round - round 1 - remember in round 1, player 1 gets to move suppose, he picks up a, then we have round 2. Now, a has gone, it is no longer there in the choice set then what should do in round 2, because it is player 2's turn to move player 2 here, obviously will choose between b and c and b is preferred by him, so he will pick up b.

Therefore, in round 3 all the c is remained which we go to player 1 and so in this case what is the total payoff of player 1? It is payoff from a, which is 3; payoff from c, which is 1, so 4. What is the payoff of player 2? It is 2, so this is one outcome but the point is that this is not sub game perfect equilibrium here.

Why is that so? Remember, if we want to find out the sub game perfect equilibrium one fruitful way to do that is to start from the end of the game, here the end of the game is round 3 and whatever object remains in round 3 player 1 picks that up. Player 1 knows the preference pattern of player 2 and player 1 knows that this object a is less like by player 2. So, player 1 can figure out that in stage 2, when player 2 moves and picks up an object, he will never pick up a because a is less, this like by him.

So, player 1 is guaranteed of a in round 3, even if player 1 does not pick up a in round 1, he is sure that this a will be for him in round 3. In that case, player 1 need not pick up a in the first round which is doing right now, he can do better, so what could be the better a will not be picked up by player 1.

So, a has gone out of this question, between b and c for player 1 b is better, so the alternative which is the sub game perfect equilibrium is the following. In round 1, player 1 is picking up b, so b has gone out of the question. In round 2, only a and c are remaining and for player 2, c is better; so c is picked up by player 2 and in round 3 player 1 is left with a.

This is the sub game perfect equilibrium of this particular preference pattern and this game. In this case, player 1 is getting how much? Is getting a and b which means 3 and 2 which is 5 and player 2 is getting c which is 2, sorry, in the previous case player 2 was getting b, which is 3.

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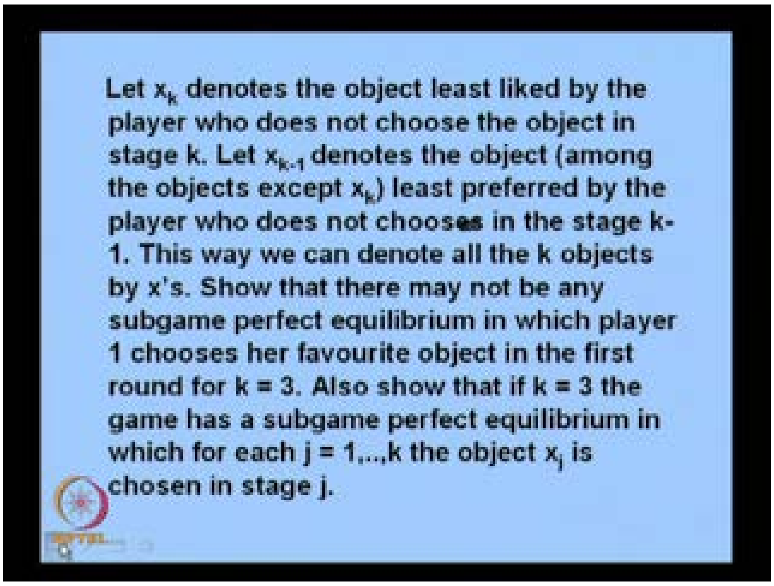

This is sub game perfect equilibrium, where player 1 is basically calculating from the end of the game. He is looking at the game and introspecting, and he is finding that in stage 3 is assured of a, so he need not pick up a in the first round. Therefore, he picks up b in the first round which means that he is not picking up his favorite object in the first round and which is what we needed to show. There is another part of this question, also show that if k is equal to 3, the game has a sub game perfect equilibrium in which for each j, j going from 1 to k the object x j is chosen in stage j.

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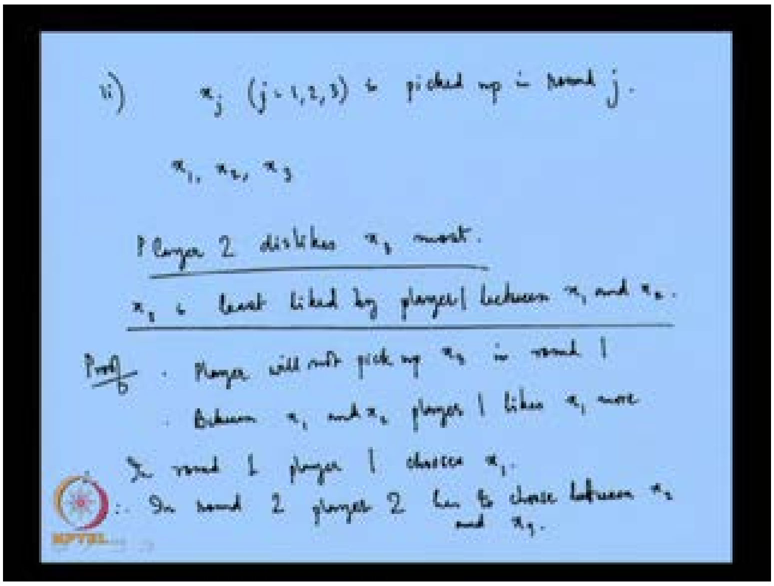

So, x j, j could be 1, 2, 3, remember here there are 3 k's, k is equal to 3, so x j is picked up in round j, so that is what we need to show in sub game profit equilibrium and there are 3 x's x 1, x 2 and x 3 and how this x 1, x 2, x 3 are defined x 3. The last x k 3 k is equal to 3, x 3 is the object least like by the person who does not get to pick up the object in stage 3 here, who is the player who does not get to pick up the object in round 3? It is player 2.

Player 2 dislikes x 3 most or other player 2 likes x 3 least and how is x 2 defined? After we are excluding x 3, there is x 1 and x 2; x 2 is that object which is least like by player 1, because player 1 is not the player who is moving in round 2.

So, x 2 is least like by player 1 between x 1 and x 2 and then, we have x 1 is the rest the object that is remaining between x 1, x 2 and x 3. So, this is how x 1, x 2 and x 3 are defined in this game in terms of preference of this 2 players. What we need to show is that in the sub game perfect equilibrium, in round 1 player 1 will pick up x 1, in round 2 player 2 will pick up x 2, and in round 3 player 3 will pick up x 3. What is the proof? The strategy of the proof is just what we have done just before in the part 1 of the question. Player 1 when he introspects the game, he can immediately see that he need not pick up x 3 in round 1.

Because x 3 is least liked by player 2 of the 3 objects, so if player 2 is given a choice he will not pick up x 3 in round 2. When player 2 gets to move, it is round 2, player 2 will never pick up x 3, so x 3 is something player 1 is guaranteed to get in round 3. First conclusion that we can draw is, player 1 will not pick up x 3 in round 1, because he is assured of this x 3 in round 3. So, he is to choose between x 1 and x 2 and between x 1 and x 2 player 1 likes x 1 more, this is because of this x 2 is least liked by player 1 between x 1 and x 2.

So, between x 1 and x 2 player 1 likes x 1 more, therefore if player 1 has to choose any object in round 1, he will chose x 1. Therefore in round 2, now it is player 2, who is going to move player 2 has to choose between x 2 and x 3 and we know that player 2 dislikes x 3 most, therefore, he picks up x 2 in round 2.

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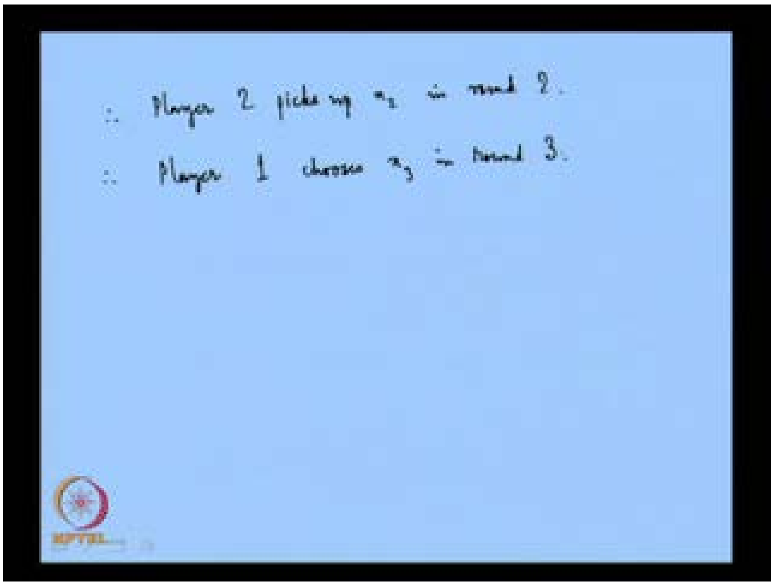

Therefore, player 1 chooses x 3 in round 3 or other he is left with x 3 which he picks up, so this is the proof that in sub game perfect equilibrium that player 1 chooses x 1 in period 1, player 2 chooses x 2 in period 2 and player 1 again chooses x 3 in period 3. This result holds for any k which is greater than 2 not only for a k is equal to 3 for k greater than 3 also this will hold.

Now, let me start with the module that we were suppose to start which is module 6. What we propose to do in this module is to take some well-known illustrations of sub game perfect equilibrium and extensive game with perfect information.

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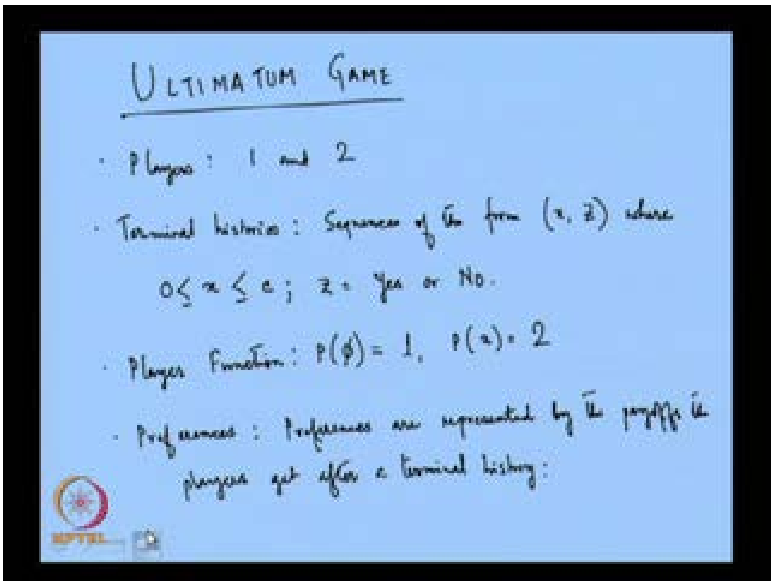

The first well known game is known as Ultimatum game, so what is the story here? Let me tell first you the story and then, we shall write down the story in terms of game theoretic language. Here, there are 2 players were involved and they want to divide a fixed sum of money or constant some of money. Suppose, the value c between them, so you can think of this has a pi which they want to divide between themselves. In stage 1, player 1 proposes a deviation of this pi or this amount of money c.

What player 1 does is to say is to specify the value of x where this value of x can take any value between 0 and c and this x is supposed to go to player 2, so what player 1 is essentially doing is, he is proposing a deviation of c between x and c minus x. This x is offer to player 2 in stage 1 after that player 2 decides whether to accept the offer or not. If he accepts the offer then, the offer basically stands, player 1 will get c minus x and player 2 will get x, but if player 2 rejects the offer, he does not accept the offer then the players will get 0 each basically both the players will get zero, so this is the game.

Before we will write down the game in terms of game theoretic language, some observations can be made; first is that here, the number of actions that player 1 can take can be infinite, we can take any x between 0 and c and this x could be a continuous variable in fact x is a continuous variable we shall assume that.

So, that is one point and second point is that here player 2 can use some pressure tactics on player 1, because he can threatened to reject the offer in which case player 1 will get

0. Though player 1 has some advantage it seems that he gets to make the offer but player 2 also has some well pins with himself that he can reject the offer and there by hand over 0 to player 1.

Let me first write the game in terms of the 4 elements, so there are 2 players 1 and 2, terminal histories where x can take any value between 0 and c, it is a continuous variable, so it is not indivisible, units are divisible and z could be yes or no. Yes means that player 2 is accepting the offer, and no means he is rejecting the offer. Player function after the history 5, that is, after null history it is player 1 stand to move, that is, player 1 is the first mover and after any x specified by player 1, player 2 gets to move preferences are represented by the payoff is that the players are getting at the end of the game.

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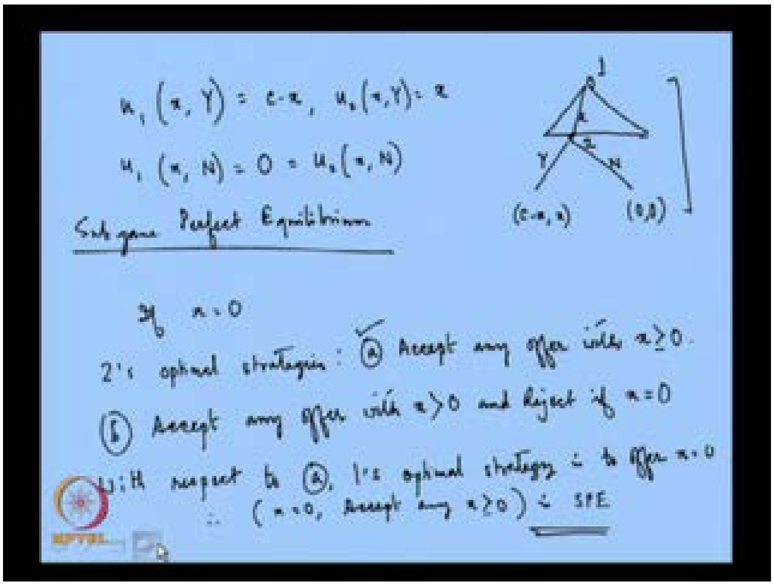

Here, what are the preferences that payoff I need to mention, first let me write down the payoff of player 1, this is the terminal history x and y which means, c is offering x and player 2 is accepting that. Then, player 1 will get c minus x and in this case, what is the payoff of player 2 it is x. If the response is no, then player 1 get 0 and same is the case for player 2; if he says no, then he himself is getting 0, this is how the preferences are defined.

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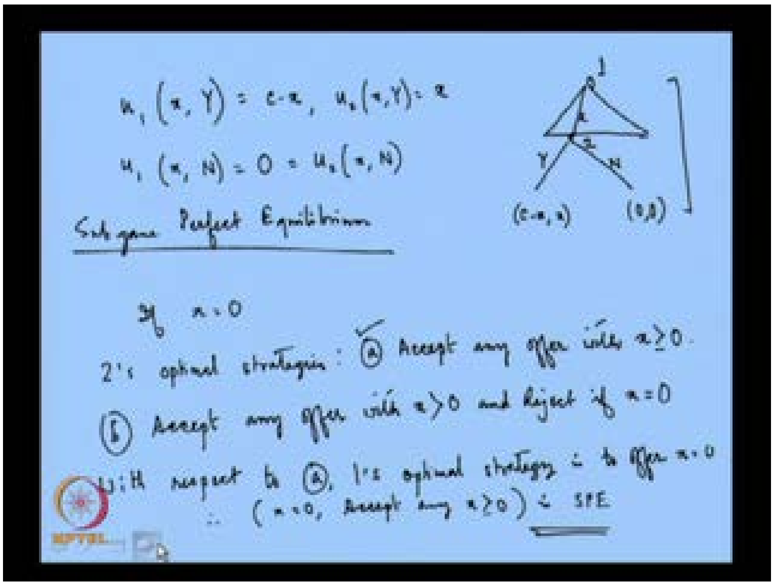

We can draw this game in terms of the familiar game tree, this is the player of the terminal history. He can make a continuous or continuum of offers x's. Let us take any particular x, suppose, this is x and after this x is offered then player 2 gets to move. He can either accept the offer or reject, if he is accepting the offer the payoffs are c minus x x 0 0.

This is a game of perfect information player 2 is aware of player 1 is action in the first round and it is a sequential extensive game you can see that. One needs to find out what is the sub game perfect equilibrium or what are the sub game perfect equilibrium in this game. We again apply the idea of backward induction, we start with the sub game of length 1 which is there, just 1 sub game of length 1, and in the sub game look at the optimal action of player 2; player 2 is to choose between x and 0. If he says yes, he is getting x; if he saying no, he is getting 0.

Ideally, he should choose y that is yes, if x is positive. If x is 0 then player 2 is indifferent, this is the strategy that player 1 will keep in mind when he decides his optimal action at the start of the game, where he is moving first that is, at the start of this sub game applying through which is the entire game.

He ideally will like to minimize the x which is going to player 2, because if he minimizes the amount of money that goes to player 2 then, his share is going to raise, it is a constant sum game, if you reduce the payoff of the other player, your payoff of raises.

Ideally, he will like to give player 2 - 0, so if x is equal to 0 his own payoff his maximum. However, if x is equal to 0 then there is a complication, the complication is something which I have mention just now, that if x is equal to 0 player 2 is indifferent between choosing y or n.

He can chose either of these 2 strategies and both strategies are optimal, these 2 strategies are the following. One is that accept with x greater than or equal to 0, so it does not matter whether x is 0 or not, he is going to accept any offer. In this case, we are basically saying that if he is given the offer 0 he is going to accept that, because by rejecting that also he is getting 0, so is indifferent, why not accept? This could be 1 strategy of his of player 2.

The other optimal strategy of player 2 could be, reject if x is equal to 0. So, he is going to accept any offer if x is positive, but he is going to reject if f x is equal to 0. Now, both of them are optimal for player 2 as we have just seen. As far as player 1 is concerned, if the strategy is this one accept any offer with x is positive or equal to 0, then player 1's optimal action with respect to this strategy is to offer 0.

So, with respect to a 1's optimal strategy is to offer x equal to 0, so player 1 offer 0 which by a is accepted by player 2 and that is the end of it. In this case, where indeed having a sub game perfect equilibrium. This is a sub game perfect equilibrium but what happens if player 2 adopts the second strategy which is also optimal for him that he is going to accept any offer if x is positive and reject if x is equal to 0.

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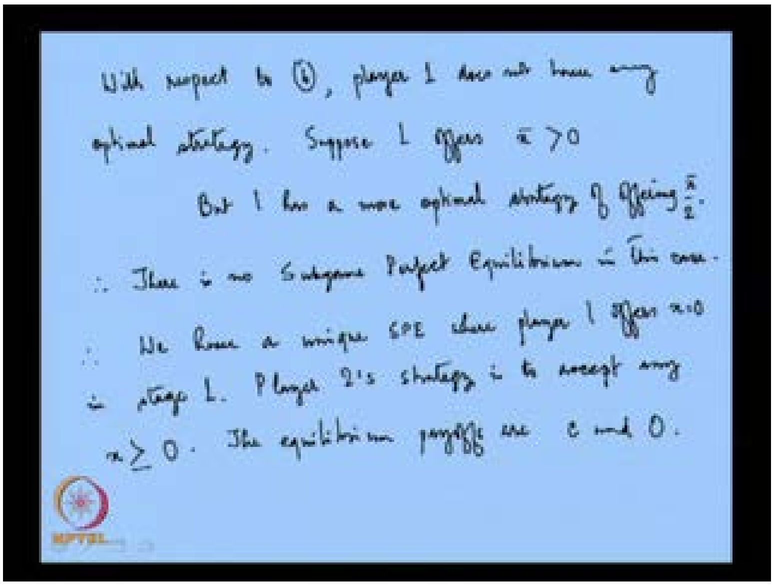

If this is the strategy of player 2, then player 1 does not have any optimal strategy. This is because of the familiar reason that here the variable x is a continuous variable. What player 2 is saying that if you offer me 0, I am going to reject that. So, one will like to give a positive x to player 2, but if one tries to give any positive x 2 player 2, there could be another x which is better for 1 for example, suppose 1 offers x bar which is positive.

So, I know that this is going to be accepted by player 2, but 1 has a more optimal strategy of offering x bar divided by 2 for example. In that case player 2 is getting x bar divided by 2 and 1 is getting c minus x bar divided by 2 which is higher than c minus x bar. So, because of the familiar reason that we have continuous variable, player 1 does not have an optimal strategy, when player 2 is taking the strategy that he is going to accept only positive x's. Therefore, there is no sub game perfect equilibrium in this case.

We have the unique sub game perfect equilibrium, where player 1 offers x is equal to 0 in stage 1. Player 2's strategy is to accept any x which is non-negative. Here, the equilibrium payoffs are c and 0. There 1 is getting the entire pi and player 2 is getting nothing. So, this might sound and this might see in a little unusual, because if you remember that we started out by making the observation that player 2 though he is moving second has some tactical advantage, because he can reject any offer and there by punishing player 1. If player 1 plans to make any unequal offer which is not to the advantage of player 2, but what is driving this result, what is giving us this result of c 0 payoff is that here, the players are concerned only about their own payoffs. So, it does not matter what the other player is getting if my payoff is higher then that is better for me.

So, that is the assumption which is driving the result that even if player 1 gives the other player 0, the other player might accept that offer, because if he rejects the offer, he is again getting 0, so by deviating the other player is not better off. If the other player is not better off, then we have this Nash equilibrium in this sub game and therefore, Nash equilibrium in every sub game and therefore, sub game perfect Nash equilibrium. So this is the reason why we are getting this strange result.

One more interesting observation is that this might not seem like a bargaining process, but it is indeed a bargaining which is going on here, there is a fixed value which is to be assigned to two players 1 and 2. These two players are making just a 1 short bargaining in the sense that player 1 is making an offer, player 2 is accepting or rejecting and there the game is ending. We can generalize this process a little bit more by incorporating the following procedure that after player 2 is rejecting an offer, it is not the case that they are getting 0 each, but then player 2 gets to make his own offer to player 1, so that is stage 3.

In stage 3, player 2 is going to make an offer after saying no and after that may be player 1 and 2 again says, accept or reject and then, makes his own offer, if he says no. That way it can be seen, this process of ultimate of game can be extended and can be seen as a process of bargaining. We have some very interesting models, if we have a bargaining model - interesting results in that model - but right now our focus is on this simple ultimatum game.

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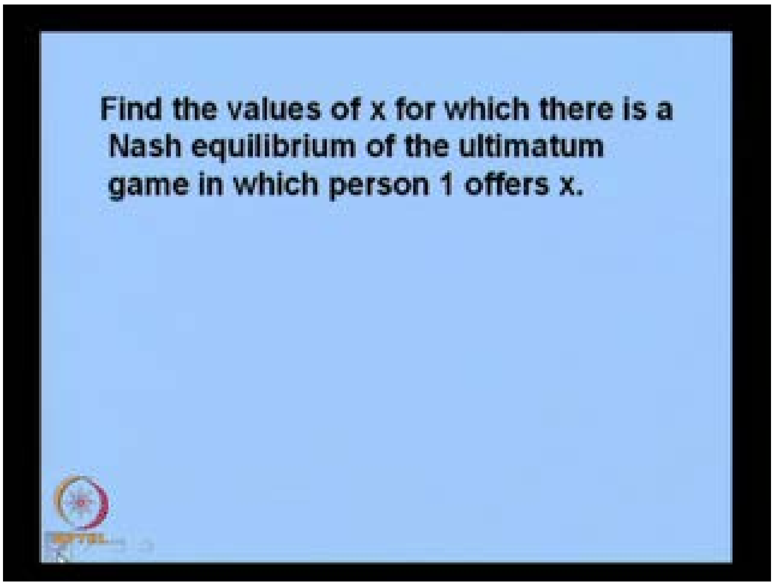

What we purpose to do now is to look at some of the exercises related to this ultimatum game. Let us take some exercises, so this is one exercise. Find the values of x for which there is a Nash equilibrium of the ultimatum game in which person 1 offers x?

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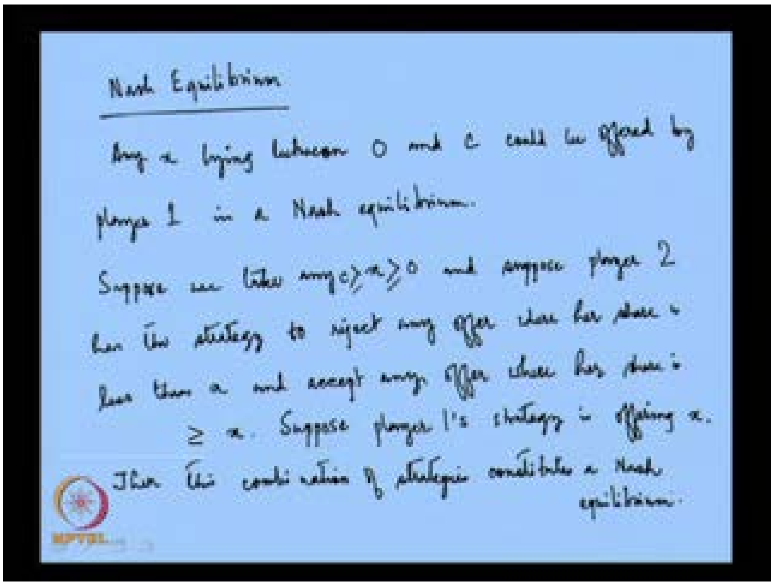

We are taking about Nash equilibrium; the question is the following that, what are the values of x for which there is Nash equilibrium, where x is offered by player 1 to player 2. Our claim is the following that there could be infinite number of x's. Any x lying between 0 and c could be offered by player 1 in Nash equilibrium and how do I prove that? Basically, when you claim that any particular x for example - not all x’s - any particular x a Nash equilibrium I have to specify what are the strategies of the players which are supporting this Nash equilibrium. Because, if you remember in an extensive game for any Nash equilibrium, I have to specify what are the strategies and in particular what is the strategy profile which is basically justifying this Nash equilibrium.

Suppose, we take any x lying between 0 and c, suppose player 2 has the strategy to reject any offer where her share is less than x, and accept any offer where her share is this. Suppose, this is the strategy of player 2. Now, if this is the strategy of player 2 and player 1’s strategy is offering x, then, this is a Nash equilibrium this combination of strategy of player 1 and player 2. Our claim is that this is Nash equilibrium and this is not very difficult to see, what player 2 is saying that if you offer me less than x, I am going to reject that and if you offer me is equal to x or more than x as my share, I am going to accept all those offers.

With respect to this, player 1 is offering player 2 x, obviously in this case the outcome will be that player 2 will accept the offer. This is Nash equilibrium for the following reason that given player 2 strategy, obviously player 1 is not going to give him more than x because in that case is own payoff will go down, so player 1's strategy is optimal. Player 2' s strategy is optimal, because he is saying that if you offer me less than x, I am going to reject that offer and player 1 is indeed offering him x.

If player 1 had offered him less than x, then rejecting that offer would have been sub optimal for player 2, because by rejecting he is getting 0, by accepting any x which is positive, but less than this x which he is specifying any strategy, he is getting some positive payoff. So, that is sub optimal but that is outside the terminal history described by this strategy profile. Therefore, it does not matter what he is specifying outside the terminal history. On the terminal history, his strategy is optimal and that is what we need to know in case of Nash equilibrium, because we are not taking about sub game perfect Nash equilibrium here.

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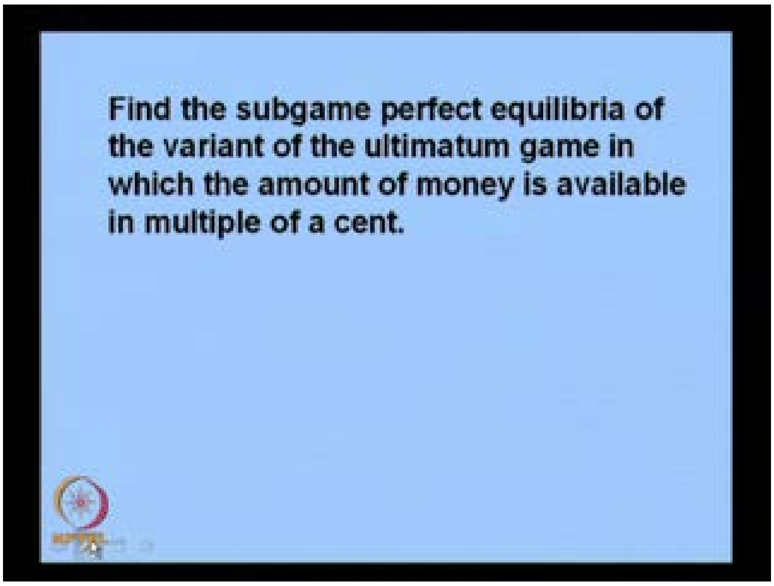

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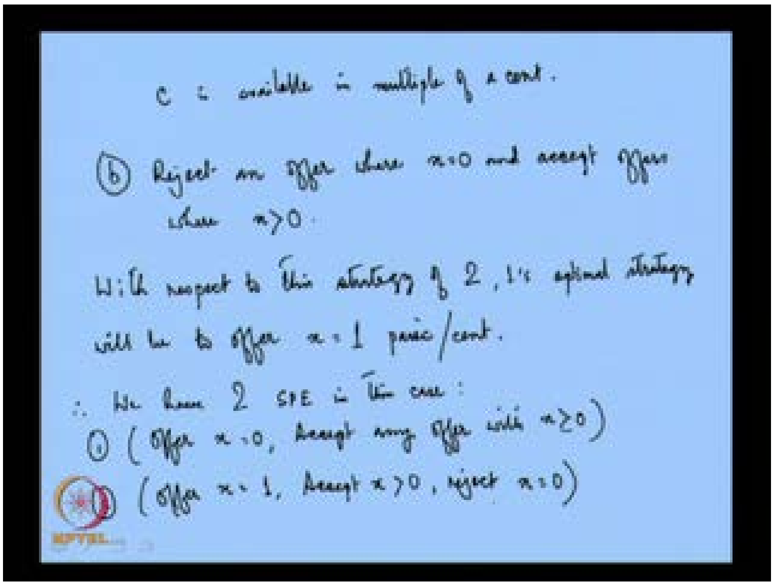

The result is that any x which lies between 0 and c could be the x that is offered by player 1 to player 2 in Nash equilibrium. Let us take another exercise, find the sub game perfect equilibria of the variant of the ultimatum game in which the amount of money is available in multiple of a cent, so c is available in multiple of a cent, cent is the smallest denominator, in India convention it could be paisa, so c is available in the multiple of paisa. In this case, does the result change in the ultimatum game that is the question? The answer is in fact it does, because if you remember in the sub game perfect equilibrium we say that player 2 has 2 optimal strategies. One is, he is saying that I am going to reject any offer; this was the strategy b, reject an offer where x is 0 and accept offers where x is positive, this could be an optimal strategy for player 2.

He is saying that if you give me positive share only then I am going to accept that because, if you give me 0 then I may as well reject that, if I reject that I will get 0, you have accept that also I will get 0. I am indifferent, so I might as well reject an offer where x is 0. With respect to this strategy of player 2, player 1 did not have any best response, if x was in continuous terms, but here what we are saying in this change circumstance we are saying that c has this nature that it is measured in terms of units which are not devisable, which are in devisable.

In that case, player 1 now will have optimal best response. With respect to this strategy of 2, 1's optimal because there is some minimum here, in the previous case when x was continuous there was no minimum value for x which is greater than 0, here 1 is the minimum value of x. We have 2 sub game perfect equilibria in this case: 1 is an offer 0, this is the strategy of player 1 and then after comma we have the strategy of player 2, accept any offer with x positive or 0 and there is other sub game perfect equilibrium is offer x is equal to 1, accept x greater than 0, reject x is equal to zero. This is the strategy of player 2, he is going to accept x's which are positive and he is going to reject any x which is 0. We have two sub game perfect equilibriums in this game.

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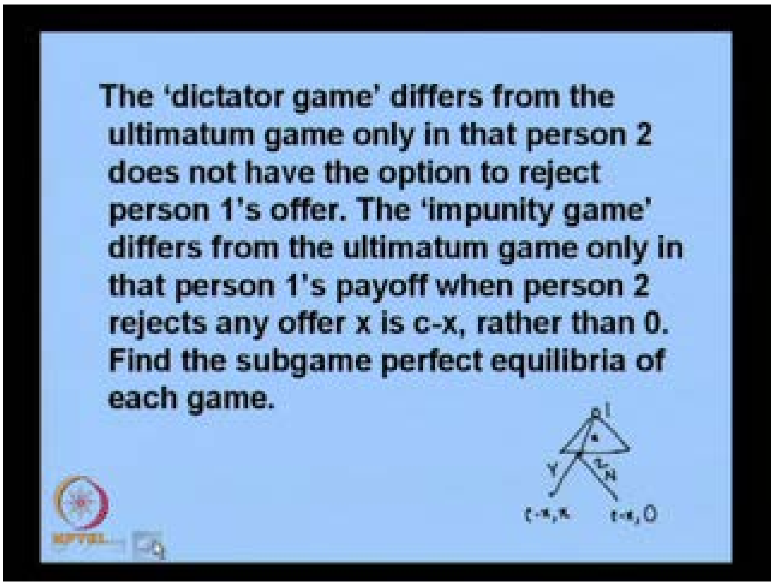

We can have some variations of ultimatum game and we can talk about two kinds of variations. One is, what is known as the dictator game, the dictator game differs from the ultimatum game only in that person 2 does not have the option to reject person 1's offer. Here, person 1, that is, player 1 is offering some x 2 player 2 in stage 1. In stage 2 player 2 cannot reject that offer, so that will be called a dictator game because here player 1 is acting as the dictator other player does not have any power. The other variation is called what is known as the impunity game, it is also a variation of the ultimatum game only in that person 1's payoff when person 2 rejects any offer is c minus x rather than 0.

If person 2 rejects any offer, it is not that person 1 is going to get 0, but he is going to get c minus x. It looks like the following 1 he is making some offer x and again, here player 2 can say, yes or no, if yes the payoffs are this; if no, player's 1 payoff remains c minus x but player 2's payoff becomes 0. It is called impunity game, because player 2 though he can say no, but in that no action, player 1 is not getting 0; he is having some impunity and that is why he is getting c minus x. By saying no player 2 is in fact harming himself, not harming the other player.

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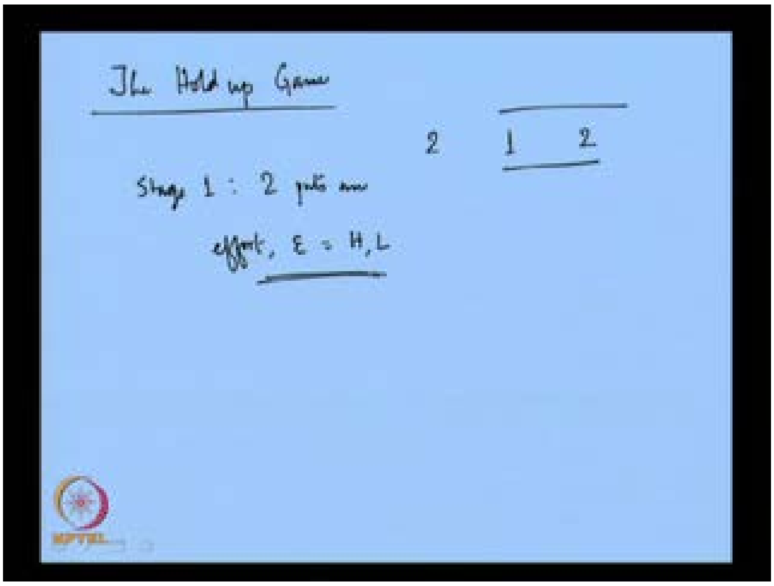

One interesting other variation of ultimatum game is what is known as the holdup game. I shall just introduce this game here and let then we shall call it ((A)) may be we shall take this game up in the next lecture - the last lecture. Here, the holdup game the story is the following, there are 3 stages here instead of 2 stages. So, first player 2 gets to move, then player 1 gets to move and then again player 2 gets to move.

Now, in this last 2 stages the game is just like the ultimatum game as we have seen before, but in stage 1 what does player 2 do? Player 2 puts an effort in stage 1 and this effort could be high or low. If he puts high effort then, the size of the pi that is c is high, it is a big pi; if he puts low effort then the size of the pi is also small, so the size decreases. After these two actions are taken then only player 1 makes an offer to player 2, it could be from the big pi or from the small pi and then only player 2 gets to accept or reject, so that is the game.

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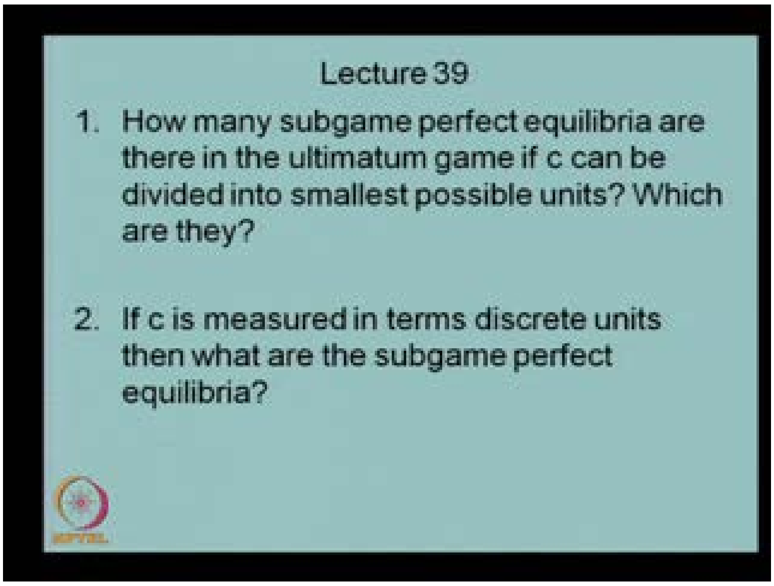

How many sub game perfect equilibria are there in the ultimatum game, if c can be divided into smallest possible units? Which are they? So, try to remember the ultimatum game.

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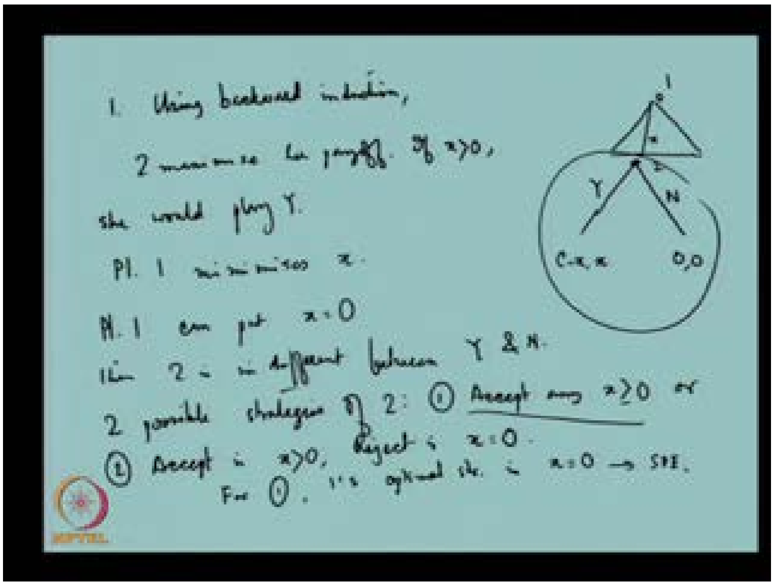

First, player 1 makes an offer, he basically is deciding x such that this x is something which will go to player 2 and if x goes to player 2, player 1 will be left with c minus x. This is the case where if player 2 says yes, that is, accepting the offer, if player 2 says no, both the players are getting 0, so this was structure of the ultimatum game.

We have to find out what are the sub game perfect equilibria here. Again using backward induction, we know player 2 is going to maximize his payoff in this sub game of length 1. Now, player 2 is going to maximize the sub game of his payoff as long as, x is positive player 2 is going to say yes, because if he says no, he is going to get 0, but remember player 1 minimizes x, because if he minimizes x his player 1's payoff raises.

Now, if player 1 minimizes x, then to what extent he can reduce x? Player 1 can put x equal to 0 that is possible, but however if player 1 puts x equal to 0 then 2 is indifferent between pi and n. He can take 2 sorts of strategies both are optimal, 1 accepts any x greater than or equal to 0, this is optimal because even if x is 0 the person is indifferent and he is accepting this is optimal. Obviously, if x is positive he is accepting there that is also optimal, but there is another strategy which accepts if x is positive and reject if x is 0, this is also optimal (Refer Slide Time: 58:20).

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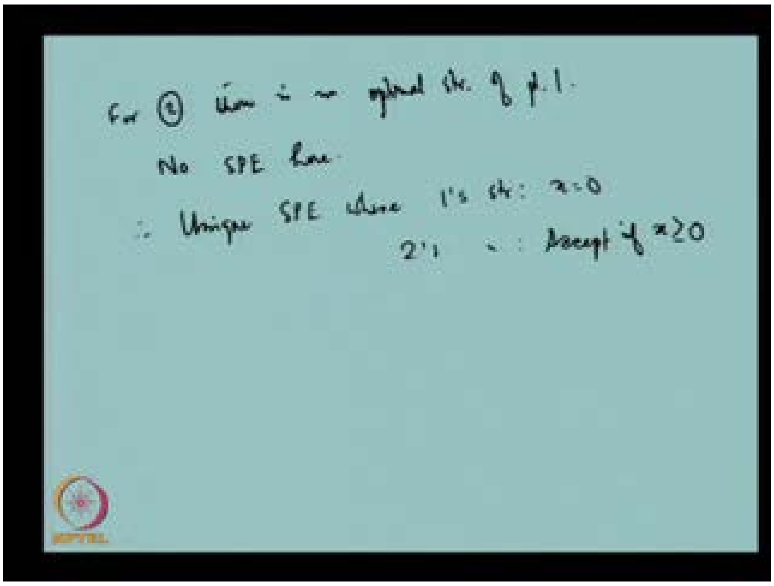

Now, here is the problem: for 1, if 1 is the strategy of player 2, 1's optimal strategy is x is equal to 0, so this sub game perfect equilibrium. Player 1 offers x is equal to 0 and player 2 accepts that given this strategy, but for 2 that is, if player 2 is saying I am going to accept only if x is positive, there is no optimal strategy of player 1. No sub game perfect equilibrium here, so there is unique sub game perfect equilibrium where 1's strategy is x equal to 0; 2's strategy is accept if x is positive or 0. So, there is unique sub game prefect equilibrium.

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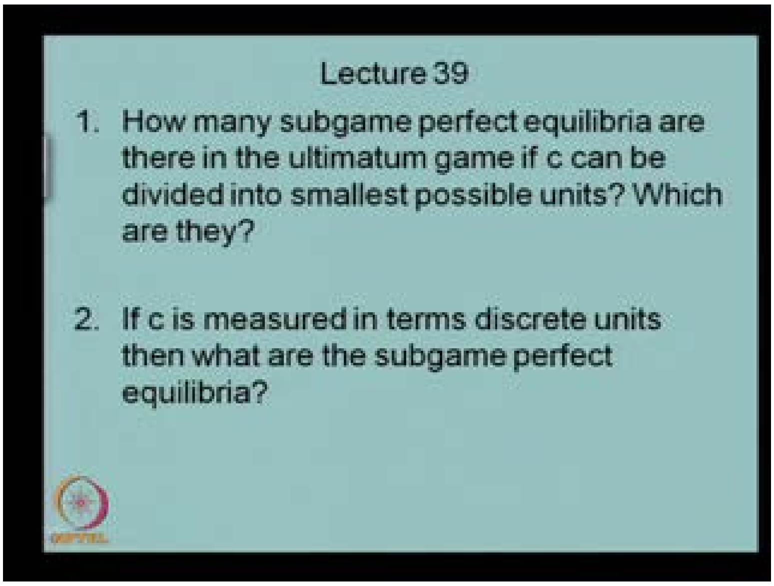

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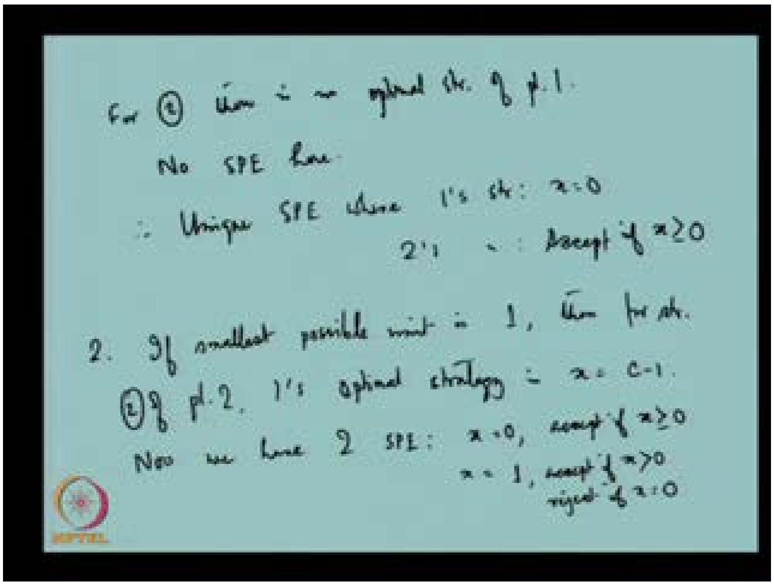

If c is measured in terms of discrete units then what are the sub game perfect equilibria? If smallest possible unit is 1, then for strategy 2 of player 2, 1's optimal strategy is x is equal to c minus 1. Now, we have two sub game perfect equilibrium, one is x equal to 0 and x equal to 1. For x is equal to 0 what is player 2 strategy is accept if x is greater than equal to 0, accept if x is positive reject if x is equal to 0, thank you.