Game Theory and Economics

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Module No. # 02
Strategic Games and Nash Equilibrium
Lecture No. # 07
Best Response Functions

Welcome to this lecture 7 of module 2 of the course called game theory and economics. Before we start, let me recapitulate what we have done in the previous lectures. What we have done is, we have defined what is known as Nash equilibrium - an equilibrium concept used in game theory. We have tried to solve different exercises just to give you an idea of how Nash equilibrium is actually used.

We have already defined what is known as a strict Nash equilibrium, where the action profile is such that if someone deviates and tries to take some other action other than the Nash equilibrium action, that person is strictly worse off and this applies for every player, so these are the preliminary things that we have done.

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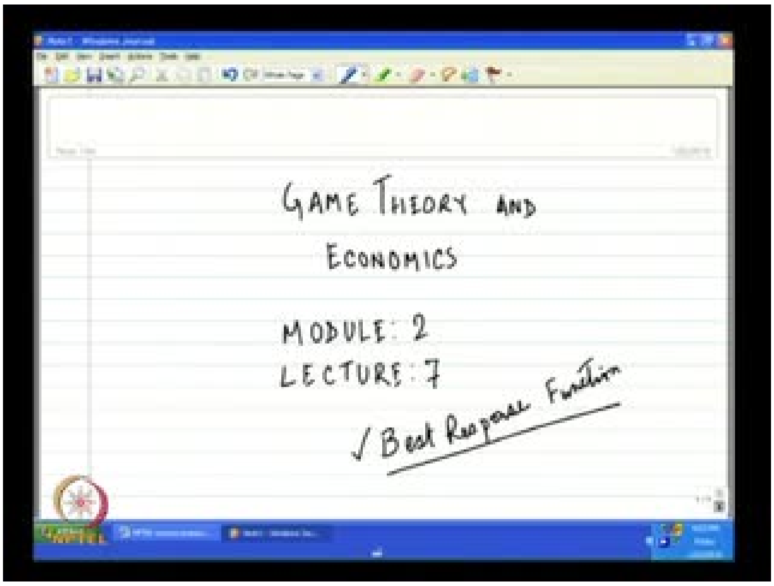

What we shall do today is, to introduce another concept of game theory which is extremely useful in finding what are the Nash equilibrium and this concept is called Best Response Function. The reason for introducing this idea of best response function is that in many cases, it may happen that the number of actions that each player can take is quite substantial, it is a large number of actions, it may even happen that the number of actions is infinite. Suppose, I can take any action or I can choose any number between the interval 1 and 2, in that case the number of actions that I can take is infinity.

Now, if that is the case then it is difficult to use the method that we have used so far to find out the Nash equilibrium. Each and every action profile has to be checked in terms of possible deviation and checking for whether that deviation is profitable or not. If I have infinite number of actions there will be infinite number of profiles to check.

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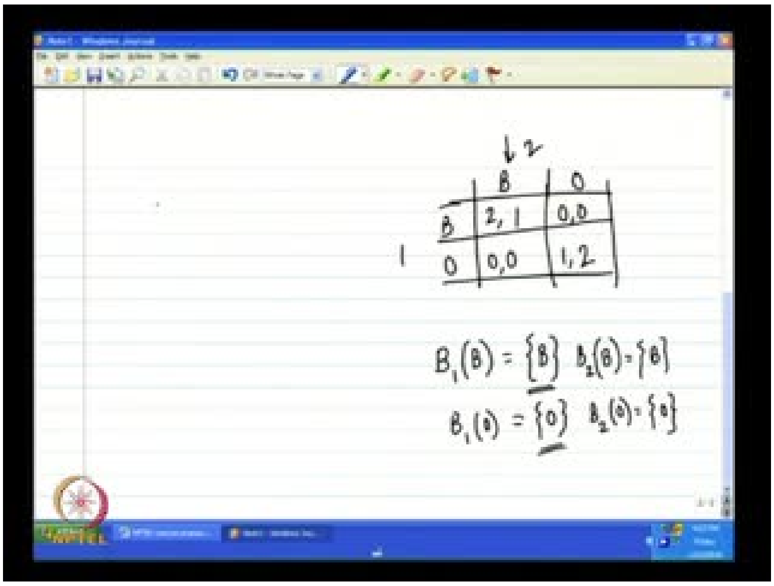

What do we do? Basically, in those cases use the concept of best response function. Let me try to define what the best response function is. Let me try to give an example so that it becomes clear and this example is from the case of discrete actions, then you can generalize this and give a general definition. Take the case of battle of sexes for example, player 1 has two actions, going to boxing match and going to the opera. Player 2 also has two actions, going to the boxing match and going to the opera and payoffs are 2, 1, 0, 0, 0, 0, 1,2.

Now, in this case, if player 2 is playing action B, player 1 has two actions to choose from B or O. Now, action B for player 1 is giving him 2, were action O is giving him 0. We say that with respect to action B taken by player 2, action B by player 1 is the best because, by action B is getting 2.

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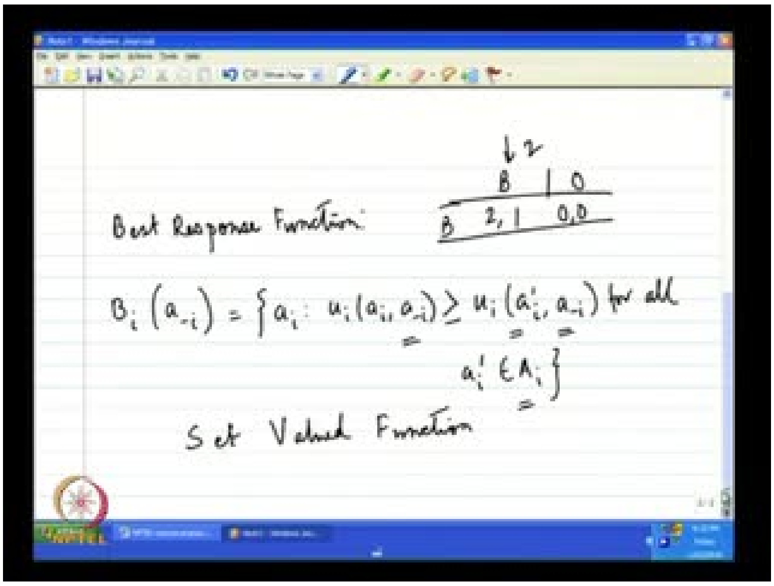

So, it is written like this B - B for best response - and we add the subscript 1 that is, the best response of player 1 with respect to the action of player 2, which suppose B is B; there are so many B's, so it might be look a little confusing well that is how it is. If player 2 plays action B then the best response for player 1 is playing B.

Similarly, if player 2 plays O best response is no longer B then the best response is O because from O he is going to get 1; if he plays B he is going to get 0. So, with respect to action taken by player 2 which is O, his best response is O. For player 2 it will be just similar, because the game is more or less a symmetric game, for player 2 if player 1 takes the action B his best response is B and if O.

So, that is how it is defined - I mean - it is an example but, if I try to give a general definition then what should it be? It will be that given the actions taken by other player which action is best for me. That action will be called my best response, with respect to those actions taken by other players, so it is a functional relationship.

The value that I am getting here is a set value, it is like the following, giving me an action which has to be taken. This is defined as B i - i is for any player - i can be 1, 2, 3, etcetera and it is defined over a naught i. So, a naught i is an action profile which gives me those actions taken by other players than i. So, this is the definition of best response function of any player i.

It is telling me that best response function of player i with respect to it is defined over the actions of other players is an action or it might be more than one action also, such that given what the other players are playing, which is a naught i. This action a i should give player i at least as much as any other action. This any other action is generically given by a i dashed; a i dash can be any action from his action set capital A i, so this is how it is defined that given the action set I have, i shall pick that action as my best response, which is giving me the best payoff.

Just a few comments; first it is a set valued function, which means that it is giving me some action which is to be taken by player i. It is not giving me any number or any other thing; it is giving me what action the player should take which gives him the best payoff, given what the other players are doing.

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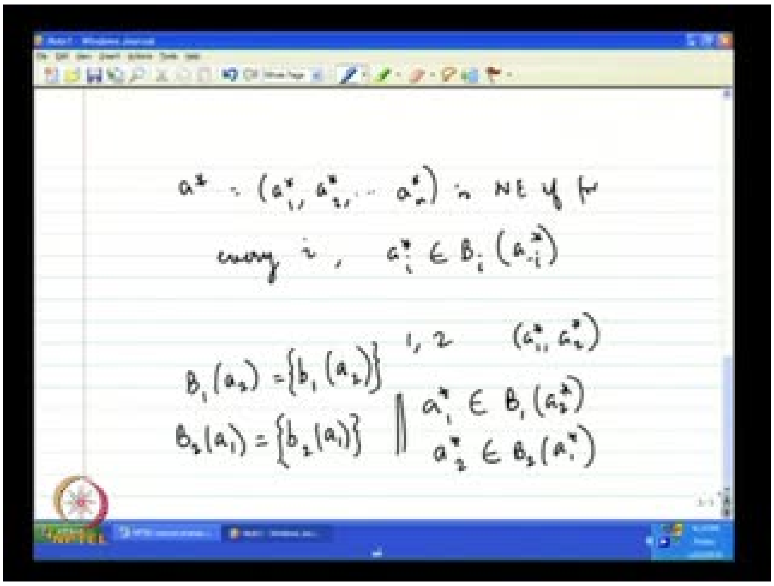

Now, what was the purpose of defining this best response function? The purpose of defining this best response function is that we want to find out that, if we can use this idea to find out the Nash equilibrium over the game, how can that be done? Well, it is the following. I have an important result; think about this, a star - a star is an action profile - it is like a 1 star, a 2 star, etcetera a n star. This action profile a star will be Nash equilibrium, if for every player i, a i star belongs to B i a naught i star.

So, which means that given other players are playing a naught i star, a i star the action that is, the Nash equilibrium action must belong to the best response function of player i. This is obvious because, if you remember what is the definition of Nash equilibrium; in the Nash equilibrium given what the actions of the other player, I should be doing my best and that means, that action must belong to my best response function, it is as simple as that and the crucial thing is that this should happen for everyone.

If I have two players, 1 and 2 and I am considering whether a 1 star, a 2 star is a Nash equilibrium action profile then, it must happen that given a 2 star taken by the other player then the best response for me should be such that it should include a 1 star, there might be other actions also but, it should include a 1 star.

Similarly, this should also happen that given the action taken by player 1 which is a 1 star, the best response function of player 2 should include a 2 star. Then, we have a sort of mutually reinforcing of actions, given what you are doing, I am doing my best and playing my action and given, this action taken by me, your action is that action with respect to which I took my best response action.

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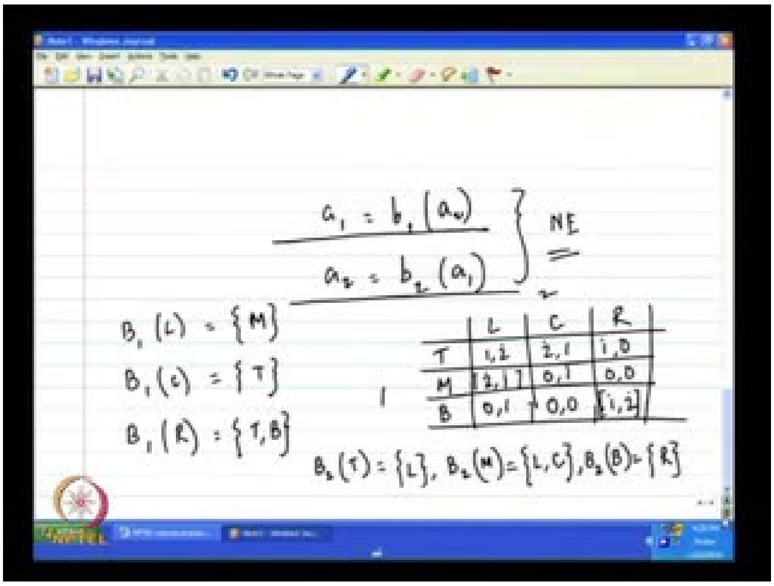

This best response function therefore, is closely related to the idea of Nash equilibrium. If it is the case that this best response function includes only a single element that is, it is a unique valued function, it is a single turn set and in that case the thing becomes easier. It is the following suppose B 1 2 is b 1, so it is a single value. Similarly, in that case what I need to solve - it does not have more than one element that is what I am trying to say, in that case what I need to solve - is these two equations one is, a 1is equal to b 1, a 2 and a 2 is equal to b 2, a 1.

If I can solve these two, I will find the Nash equilibrium, because b 1 will be a function of a 2, so I have one equation in two variables and here also b 2 is a function of a 1, so another equation in two variables - two equations two variables - if I solve them I get a Nash equilibrium, that is how it works. Before I tell you how to solve these problems in case of continuous variables, let us look at how this idea of best response functions can be used, if we have discrete actions like we have been having so far.

Take this game and we shall try to see, whether we can use the idea of best response functions to find out the Nash equilibrium. This is the payoff matrix suppose, we have to find out the Nash equilibrium or Nash equilibria by using the best response function then, what done is that firstly, let us try to find out what are the best responses of player one.

Now, B 1 that means, B 1 given L is what? B1 L given L means, if player 2 takes the action L then what is the best action for player ? Well, it is a single action which is M, so we have a single turn set. If the action of player 2 is C then his best response is T. If the action of the second player is R, player 1 has two best responses, one is T and the other is B. So, these are the best responses of player 1, what about player 2? If player 1 takes the action T his best response is L. If player 1 takes the action M, he has two best responses L and C. If player 1 takes the action B, there is single element R at which he is getting the maximum payoff 2 which is R.

So, this is how best response functions can be used in this exercise but, what at the Nash equilibrium? Well, to find that out we shall use a very simple technique. If player 1, player 2 is playing l, player 1's best response is M then, we put a dot over 2, why over 2? Because we are considering the best response of 1 that is why I am putting a dot over the payoff of 1 which is 2.

If player 2s action is C his best response is t, so a dot here; if his action is R there are two best responses here one and here another. So, these takes care of the best responses of player 1. Now, let us talk about the best response of player 2. If player 1 is playing T his best response is L, so I have a dot over 2. If M, L and C; if B on the R, once I have put all the dots, what is the Nash equilibrium? Well, the Nash equilibrium action profile will be that action profile while I have two dots, not the single dot but two dots and this is happening here and here (Refer Slide Time: 19:35).

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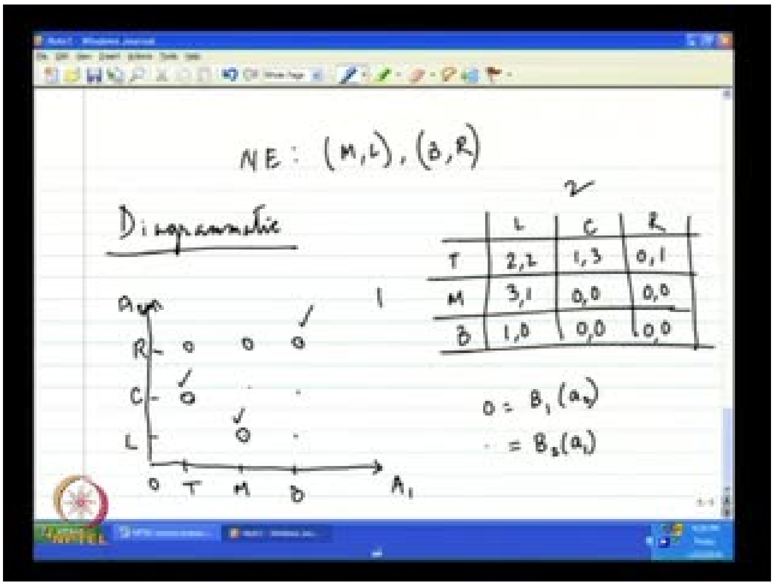

There are two Nash equilibrium in this case, one is M, L; the other is B, R - two Nash equilibriums are there. Now, this was about Nash equilibrium in case we have a payoff matrix and all that but, the same idea of finding the best responses and then try to match the best responses of two player, because what happening here is that we are trying to match the best responses.

If the two best responses are intersecting, they are matching with each other which is happening at M, L and B, R then, those action profiles is defined as the Nash equilibrium. The same idea can be done diagrammatically also, how it can be done? Let us take another exercise, this is a diagrammatic technique, here the game is following. Now, instead of putting dots here, what we are going to do is to put some dots but not in this matrix itself. What we are going to do is to draw to a axis like x-axis and y-axis and we are going to put the actions of player 1 here A1 and actions of player 2 here A 2.

So, player 1 has three actions T M B and player 2 has three actions as well, they are L, C and R. Now, given player 2 is taking action L, what is the best response of player 1? It is M, so here is L M, I put a circle here; circle is representing the best response for player 1, so circle is like B 1 a 2. If player 2 plays C then 1 has one single best response which is T, so C T I put another circle here. If player 2 is playing R, player 1 has all the three are best responses, so T R, M R and B R. Now, let us think about player 2 and player 2's best responses will be given by dots in this diagram.

Now, if player 1 is taking the action T player 2’s best response is C - so T C, here I put a dot and this dot, since there is a circle already there I put the dot inside the circle. If player 1’s action is M, player 2’s best response is L, so M L is another. Finally, if player 1’s action is B, all three L, C, R are best responses for player 2, so B L, B C, B R. Now, one can guess from the exercise that we have done just now through the matrix that is this one. One can guess the Nash equilibria will be those action profiles, where I have the dot as well as the circle. So, there will be three Nash equilibria here, one is here, the other is here and the third one is here.

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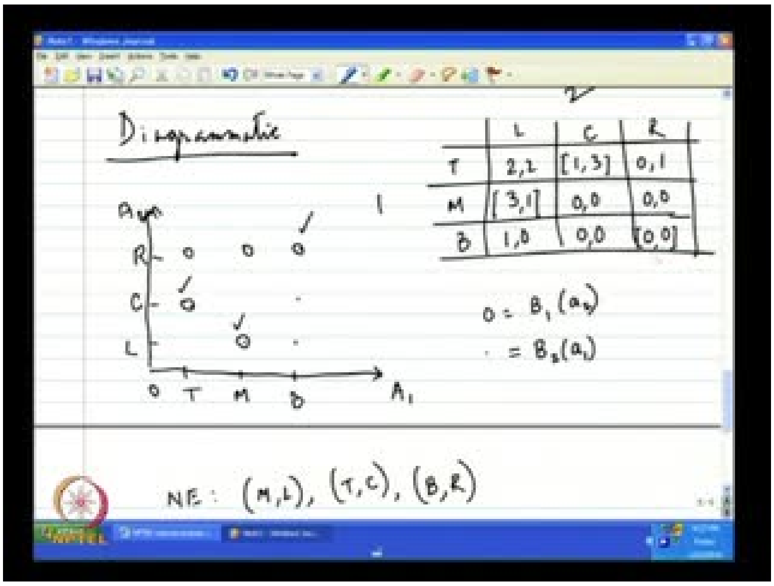

The Nash equilibrium will be for the Nash equilibria three Nash equilibria are there one is M L then we have T C and we have B R, this can be checked from the matrices suppose M L, I am considering this one. Now, obviously player 1 deviates he cannot be better off, he can either get 2 or get 1 which are less than 3. What about player 2? If he deviates, he gets 0 and 0, again they are worse than 0, they are worse than 1, then we have T C from this if player 1 deviates he gets 0 which is less than 1.

If player 2 deviates he gets either 2 or 1 which is less than 3 and B R, similar to the logic provided before if player 1 deviates he gets 0 which is not better than 0 which player one is getting here. Similarly for player 2, if player 2 deviates he gets 0 which is not more than what he is getting in the Nash equilibrium, so these three are the Nash equilibrium.

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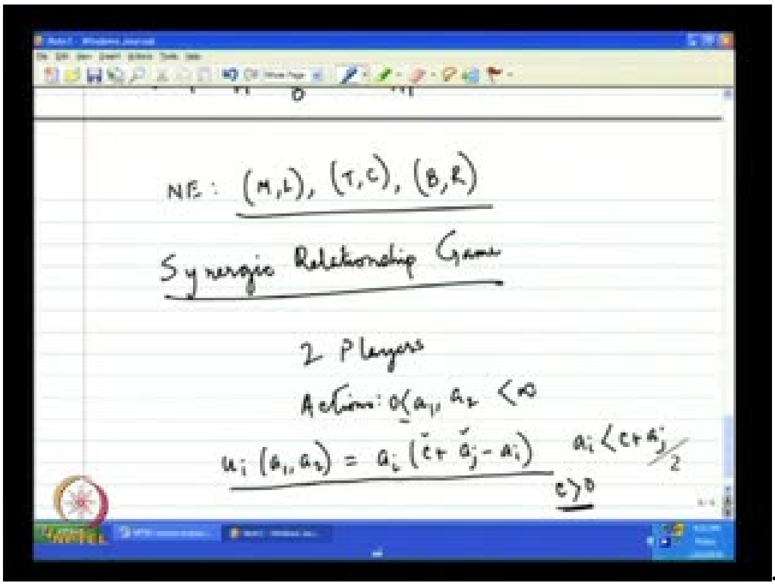

This were the cases where we had discrete actions of each player and we have seen that we can find out the Nash equilibrium in this games by using either of this two techniques, either directly using the payoff matrix or by using a diagram which is a representation of the payoff matrix itself. Now, it so happens that a number of actions they are not discreet but they are continuous variables then, what happens then? Can this idea of best response functions be used profitable and the answer is yes.

Let us take a case, this game is called Synergic Relationship Game and the idea is the following that I have two players, so players and the actions that they can take, so let us call a 1 is the action of first player and a 2 is the action of second player, they can be any number between infinity and 0. So, any positive or a non-negative number is allowed as an action.

What about the payoffs? It is represented by the following for player i the payoff that player i, it is a function of two variables a 1 and a 2 and it is given by a i c plus a j minus a i, where a i is his effort level. Here, a i and a 2 a 1 and a 2 are the actions which represent the effort level put in by each player, effort level for what? Well, this is a relationship with a stories that though two players are involved a relationship.

If I put more effort given the effort level of the other player I should get more benefit out of the relationship, it helps me also, it gives me more satisfaction, but if the effort level put in by the other player is constant it is not changing it is not increasing. Then, if I go on putting more and more effort after a point of time, the benefit that I get goes on declining because, if the other person is not contributing, I do not feel very happy about it, so that is represented by this payoff function. Initially, given c and given a j, if a i rises then this u i also rises because suppose a i is less than c plus a j divided by 2, I forgot to mention c is positive constant, it is a constant given from outside.

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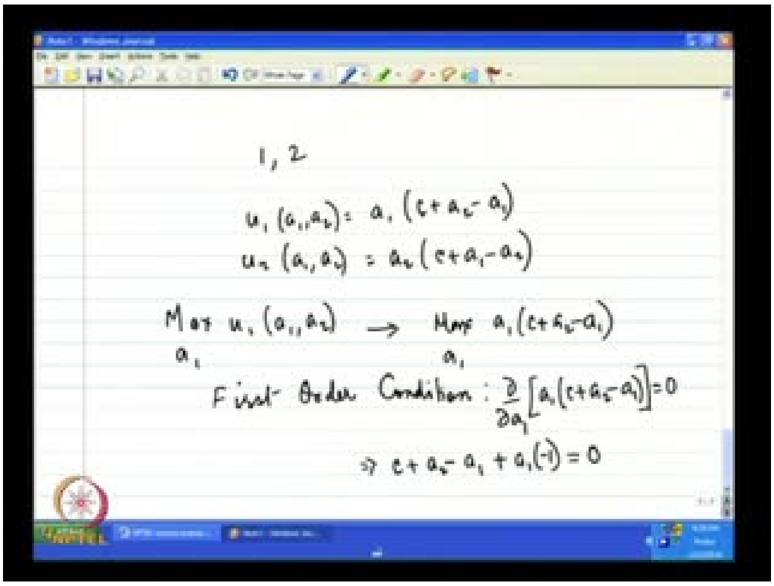

Now, if a i is less obviously, if a i rises the whole u i rises. Here, I have a game theoretic situation where the actions taken by each player could be continuous variable a i and a j. In particular I am going to consider two players only 1and 2, so the payoff functions will look like the following. This is how the payoff functions look; question is, what is the Nash equilibrium here?

Now, we are going to use the technique of best response functions because, as you can see you cannot draw payoff matrix for this game, because the number of actions is infinite. Now, what was the definition of best response function? Given the action of the other player I should pick up that action or those actions which is maximizing my payoff.

In other words, what I should do to find out the best response function is that I should maximize u 1 with respect to u 1. I should find out that value of a 1 which maximizes u 1, given the value of a 2 and that will be my best response to a 2. So, this translates into maximizing a 1 c plus a 2 minus a 1, this I have to maximize with respect to a 1.

Now, this is a very common problem in differential calculus, if I want to maximize a function with respect to variable what I do is, I have to satisfy what is known as the first order condition which is the derivative of this function with respect to a 1 has to be equal to 0, this is the first order condition. Now, let us see what I get out of this condition here. It will be c plus a 2 minus a 1 plus a 1 multiplied by minus 1 is equal to 0, it is a product of two functions a 1 and c plus a 2 minus a 1.

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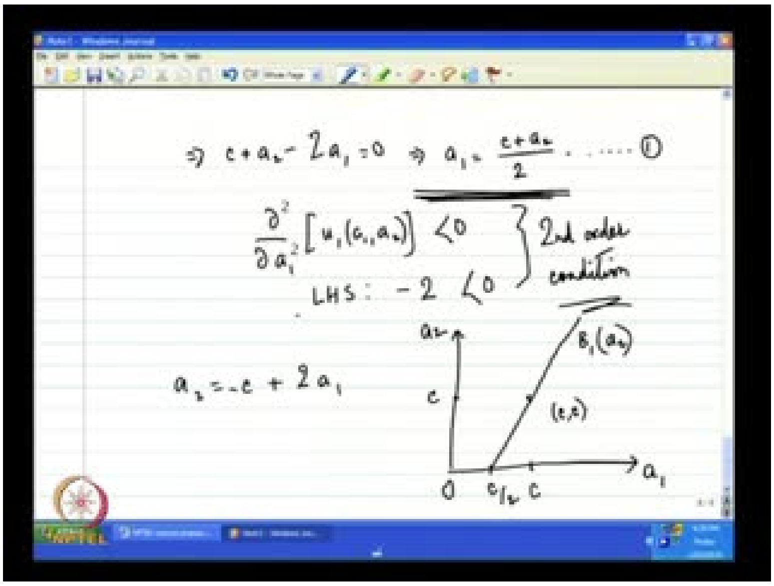

So, by the product rule it will be c plus a 2 minus a 1 multiplied by derivative of a 1with respect to a 1 which is 1 plus a 1 multiplied by the derivative of c plus a 2 minus a 1 with respect to a 1, which is minus 1. If I simplify this, what do I get? The c plus a 2 minus 2 a 1 is equal to 0 which means that a 1 is equal to c plus a 2 divided by 2 this is first equation.

Notice, if I maximizing a function with respect to a variable - this is a necessary condition - this a first order condition but, there is a second order condition which is the sufficient condition which says, that this should also be satisfied and which is satisfied here because, what is the left hand side? The left side hand side is minus 2, which is less than 0, so the second order condition is satisfied which means, that this is indeed the best response function for player 1. What does it say? It says that given the changing value of a 2, a 1 should vary according to relationship or according to this best response function one. If I want to draw this in a diagram, how does it look?

Now, this one can be simplified as a 2 is equal to minus c plus 2 a 1 which means, it has a slope of 2 and an intercept of minus c. In particular if a 1is 0, a 2 is minus c; if a 2 is 0, a 1 is c divided by 2. Suppose, this is c and this is c divided by 2, this is again c. So this line - this is not a curve - this is a straight line and this should look like this one, it has a steep slope of 2 and that is, horizontally intercept of c divided by 2.

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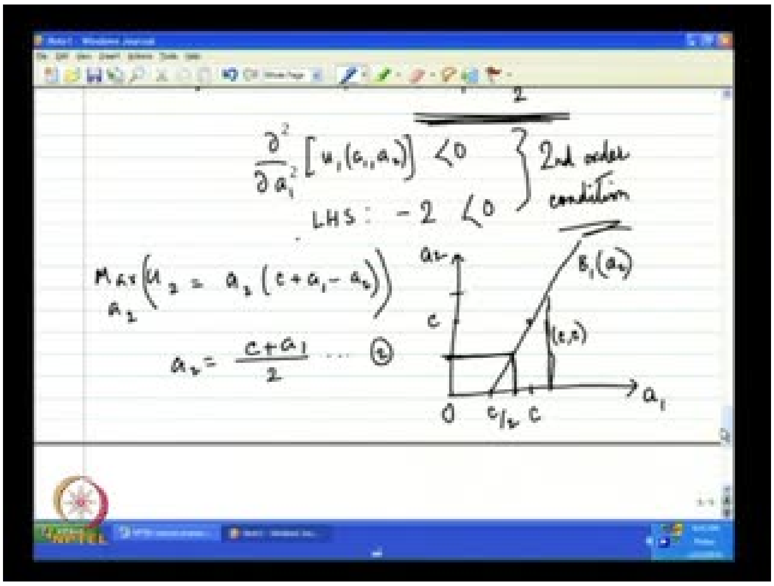

Now, let us concentrate on the other best response function. Now, before concentrating on the other best response function let us see, what it means? What it shows for this best response function of player 1? It shows for each value of a 2 if i take any value of a 2 suppose this arbitrary value I find that value of a 1 which is the best response for player 1. If I take this value I draw a perpendicular on this line, from the intersection point i draw another perpendicular and this point should give me the best response of player 1 with respect to this action of player 2. So, this is how this line is interpreted for.

For each value of a 2 this line gives me the best response of player 1. What about player 2? Remember, this is u 2 and so I have to maximize this with respect to a 2. If I do so again, I will have a first order condition, I will have a second order condition but, the exercise will be exactly similar that we have done before. If it is exactly similar then the function that I am going to get for the best response function for player 2, it will look like the following (Refer Slide Time: 40:00). This is how it should look, because for player 1 the best response function was a 1 is equal to c plus a 2 divided by 2 for player 2 it will be c plus a 1 divided by 2.

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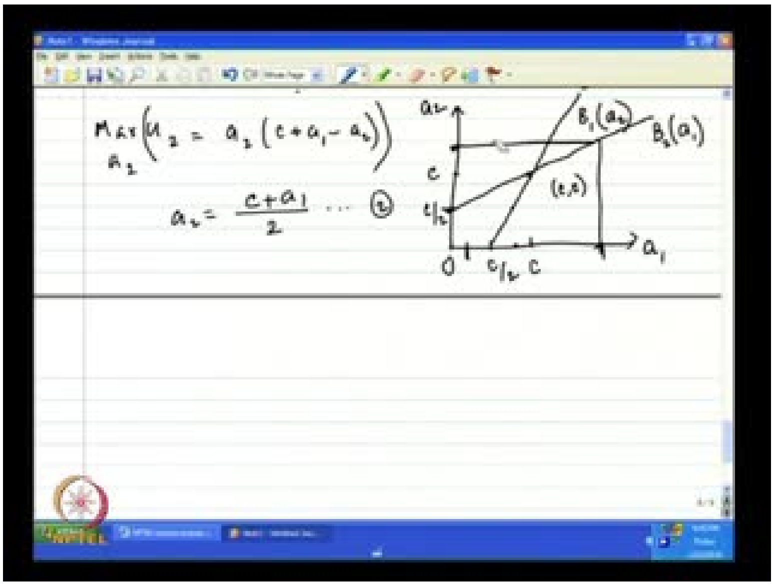

Now, if I want to plot this line how will it look? We just remove these lines so that it becomes more clear If a 1 is 0 a 2 is c divided by 2, so this is c divided by 2 and this line passes through this point. What is the slope? Slope is half, so it will be a flat line like this. This is the best response of player 2 and the interpretation is just like the interpretation before it is telling me, what is the best response of player 2 given any action of player 1. So, I have to read from here, given any action of player 1, suppose this is the action of player1 a 1, I am going up to the line of B 2 and from that I draw a perpendicular on the a 2 axis. So, this is the best response of player 2 with respect to this action of player 1, so this is how it is read.

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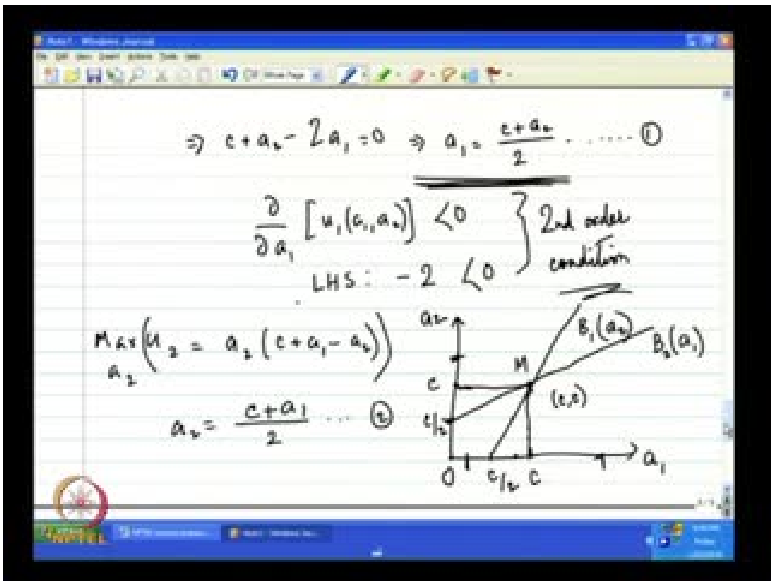

Question is, which is the Nash equilibrium? The answer is very simple; Nash equilibrium is that point where these two lines are intersecting with each other. Since, both these lines are straight lines there will be a unique point of intersection, because these lines are not parallel. It means, there will be a only one point in which they will intersect and that point is here.

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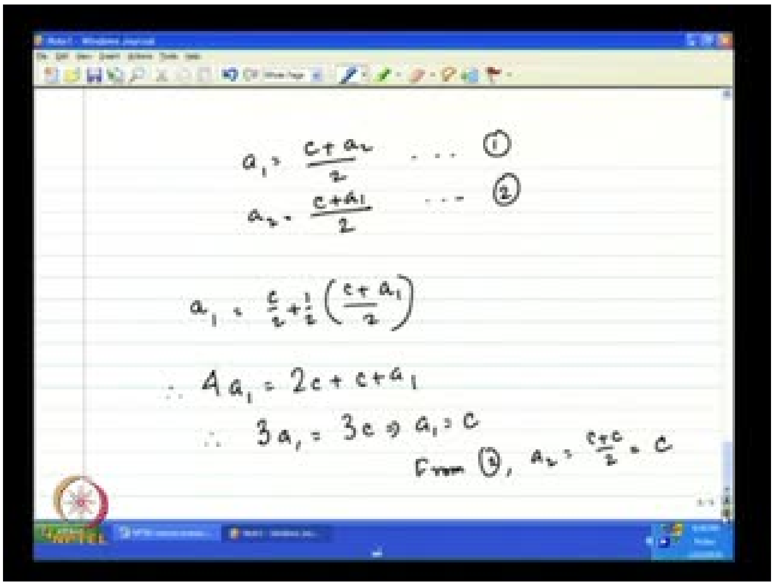

Let us call this point M and you can figure out intuitively at least from the diagram that this point has the coordinate c c. This can be verified by solving one and two also, if I solve these two equations a 1 is equal to c plus a 2 divided by 2 and a 2 is equal to c plus a 1 divided by 2. If I solve them together I should get the same solution because - let me just verify a 1 is equal to c divided by 2 plus half a 2 and in place of a 2, I can write this.

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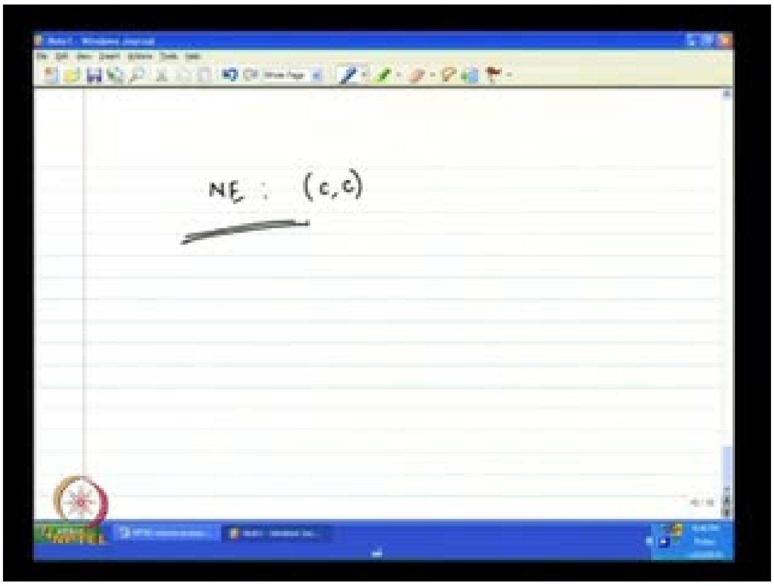

Now, if I multiply both sides by 4, so a 1 is equal to c and from 2 we get a 2 is equal to c plus c divided by 2 which is equal to c. In that from the diagram itself what we have seen the intersection point is c c and that also can be obtain by solving these two equations 1 and 2.

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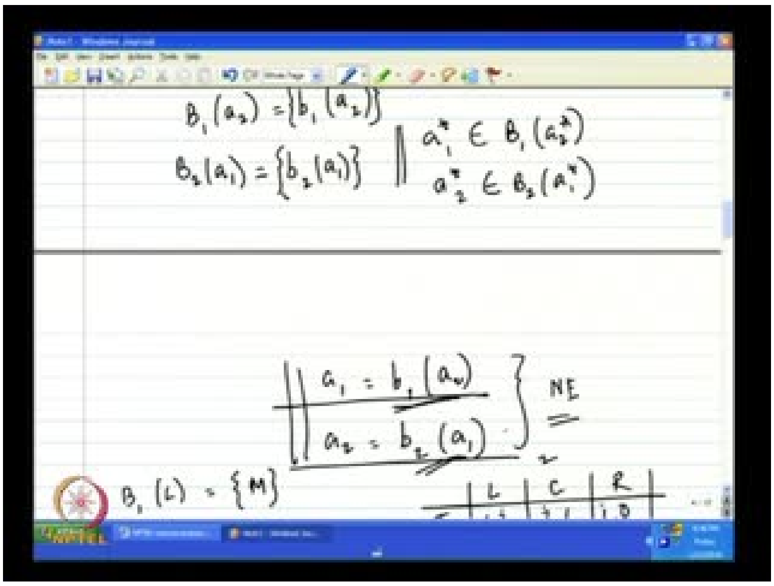

Now, why is this point being called Nash equilibrium? Because of the thing that we have seen said before remember, this is what we have said before. If I have two player and if it so happens that there is one point at which the best response function is giving me a unique point then, I have this unique point as b 1 and b 1 is a function of a 2; b 2 is a function of a 1, I solved them I get a unique Nash equilibrium.

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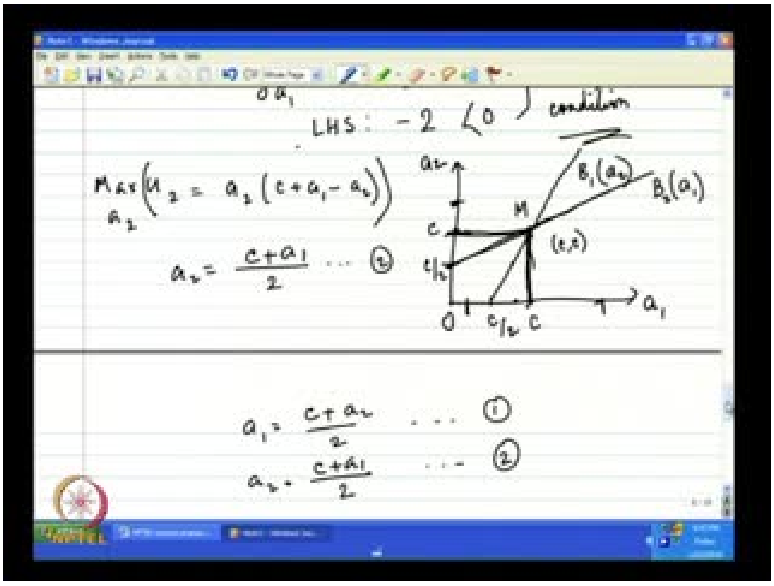

If I look at the diagram also the same thing is verified. Here from c, if c is the action taken by player 2, what is the best response for player 1? I go here and I reach here, this is the point c by the horizontal axis. With respect to the actions c by player 2 c by player 1 is the best response. Similarly, from here if i go up I reach B 2, this B 2 line and from B 2 I read out that this is again coming back to the same point c. So, going from the same point and we come back to the same point and that is why this is Nash equilibrium.

Now, this was the case where the best response functions were linear, there was straight lines. In general best response functions are not straight lines, it can be curves and if there curves then it is not necessary that the points of intersection is just there is a single point of intersection. There can be more than one point of intersection and if there is more than one point of intersections then obviously, the number of Nash equilibria will be more than one, so that is how it is solved.

Before we end this lecture let me just take you through what we have done in this a lecture. Basically, we have introduced the concept of best response functions and how to find out Nash equilibrium from best response functions. We have studied two exercises in the case of continuous variables and we have also shown how in case of discrete actions the idea of best response functions can be used to find out the Nash equilibrium, thank you.

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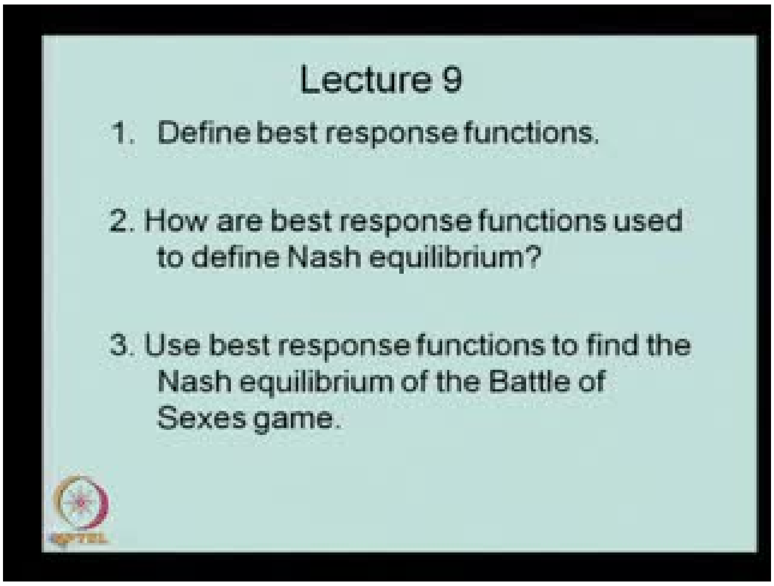

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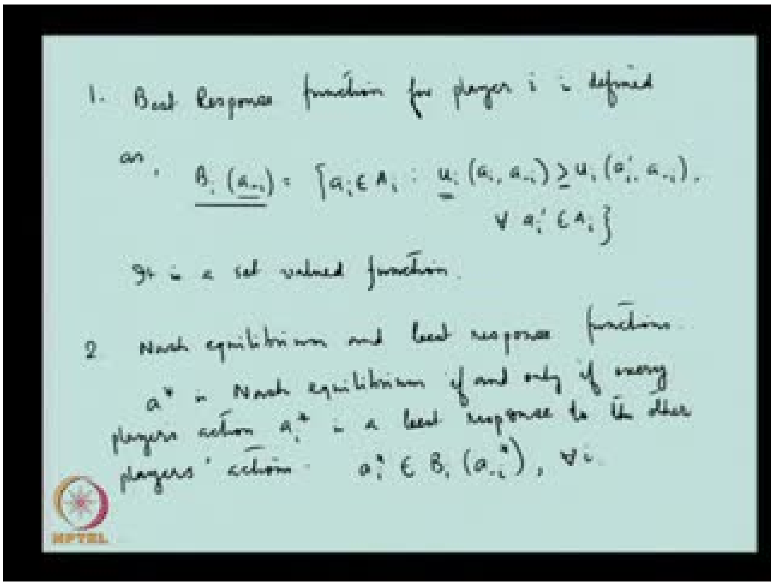

First, define best response functions. A best response function is defined for a particular player. It is defined as, this is the notational definition of best response function of player i written as B i. Now, this is a function of a naught i; a naught i is the vector of actions of other players, so best response function of a particular player is defined over the least of actions of other players a naught i and it is a set valued function.

It gives us a set of actions of player i which are best given player i's payoff function and given what actions the other players are taking, so that is how the best response function is defined. In particular the best response function could be null valued which means, there is no unique or even more than one best responses to other players action or it could be a function which specifies more than one action which are best given what the other actions are.

Let us go to the second question, how are best response functions used to define Nash equilibrium? This is the relationship we want to prove. We say that a star is Nash equilibrium remember, a star is a vector of actions, it is like a 1 star, a 2 star, etcetera, a n star. So, this vector of actions is Nash equilibrium if and only if every player's action that is generically, a i star is a best response to the other player's actions.

Symbolically, this can be written as a i star belongs to the best response function of player i and this is true for all i. So, this is how this best response function a is relating to the concept of Nash equilibrium. In the Nash equilibrium every player's action should be belonging to his or her best response functions and this should be true for every player.

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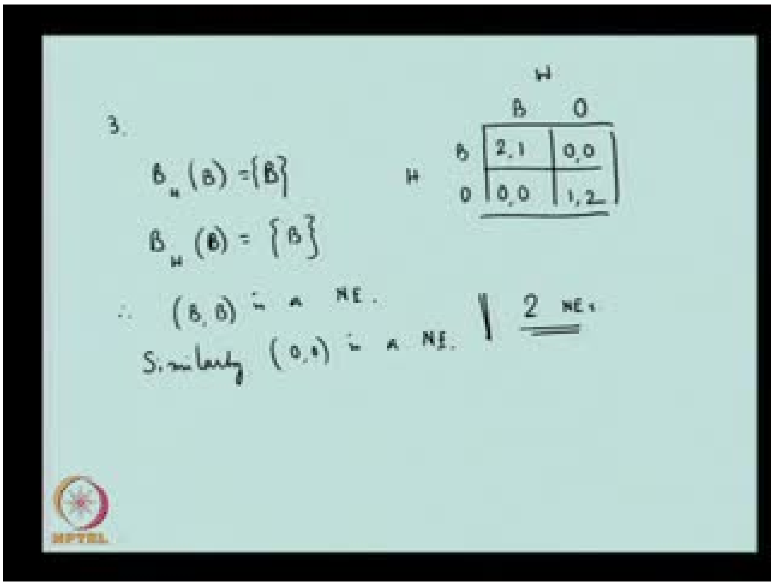

The third question is, use best response functions to find the Nash equilibrium of the battle of sexes game? If you remember the battle of sexes game it looks like the following, this is wife, this is husband and there are two actions here and these are the payoffs.

Now, we can easily see that the best response function of the husband given the wife is a going to the boxing match; the best response function of the wife given that the husband is going to the boxing match, is also going to the boxing match. By using the relationship that we just described B, B is Nash equilibrium. Similarly, 0 0 is Nash equilibrium, because given the wife is going to the opera, the husband will go to the opera that is his best response. Given the husband is going to the opera, wife will go to the opera, so these are the two Nash equilibrium.