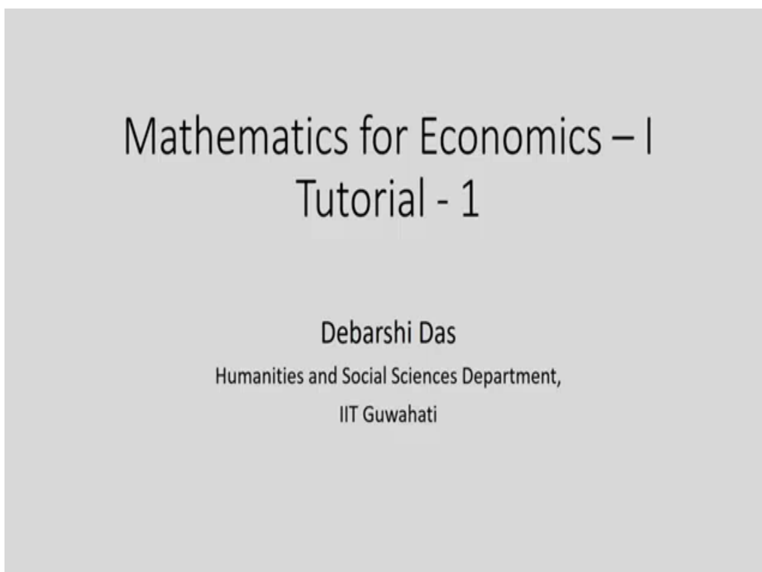
Mathematics for Economics - I
Professor Debarshi Das
Department of Humanities & Social Sciences
Indian Institute of Technology, Guwahati
Lecture 18
Tutorials – 1b

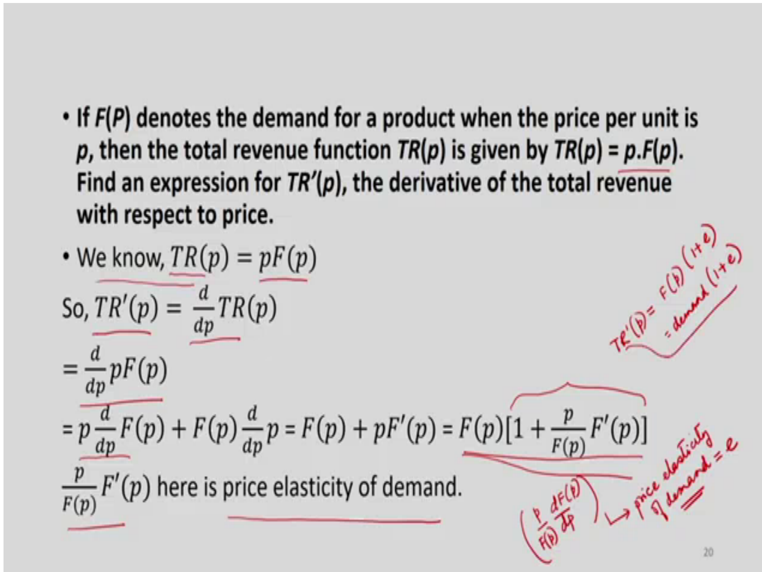
Hello, and welcome to another lecture of this course Mathematics for Economics Part I. Right now what we are doing is that we are doing some tutorials. These tutorials hopefully will help you to revise some of the topics that we have covered in this course and also to develop some skills of problem solving, because theory is fine. What we did mostly in this course is discussing the theory. But as we know problem solving skill is also important. So, in this series of tutorials, we are going to solve certain problems from the topics that we have covered.

(Refer Slide Time: 01:23)



So, as you can see on your screen, so we are talking about first set of tutorials, tutorial 1. And the topics that we have covered in this tutorial so far are, we talked about real numbers and mathematical logic. We also talked about sets and set operations. The last topic that we started with in the last lecture was on differentiation that is what we just started with. So, let us go to that problem that we discussed in the last lecture.

(Refer Slide Time: 02:10)



So, this was the problem that we were discussing. Let me just revise this a little bit and so that we can go forward. $F(p)$ is the demand function of a product, where p is the price per unit of the good and given this demand function the total revenue function is denoted by TR, which is again a function of p, and $TR(p) = p.\ F(p)$ that is price multiplied by the demand. Find an expression for$TR^{'}(p)$ that is the derivative of the TR with respect to the price that is $TR^{'}(p)$.

And that is what we did. We have started with $TR(p)$. We have used the definition. Total revenue is equal to price multiplied by demand. Demand is given by $F(p)$. So, it is p multiplied by $F(p)$. Then we took the first derivative of total revenue with respect to price and this is what we get $\frac{d}{dp}p.\ F(p)$.

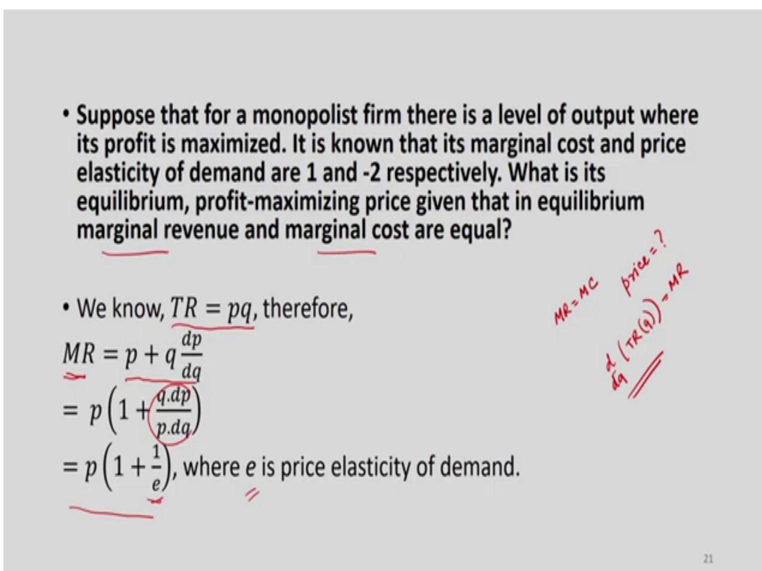
And then we use the product rule of differentiation in the next step. And then we get so $p\ \frac{d}{dp}F(p)\ +\ F(p)\frac{d}{dp}p$. And that with some manipulation gives us this expression. If we take $F(p)$common then within the brackets we have this expression $[1\ +\frac{p}{F(p)}F^{'}(p)]$.

Here remember $F(p)$ is the demand function. Demand function is a function of the price. And as we have noted here, $\frac{p}{F(p)}F^{'}(p)$ is called the price elasticity of demand. This we shall deal with later on also in more detail. But let us try to see what it decomposes into, $F^{'}(p)$ is what it is $\frac{dF(p)}{p}$ This is $F^{'}(p)$ and that is getting multiplied by $\frac{p}{F(p)}$. This is what the elasticity of demand is, price elasticity of demand.

And if we denote this by let us suppose simply as small e then what we are getting is total revenue the derivative of that with respect to price, $TR^{'}(p) = F(p)[1 + e]$ and $F(p)$ is nothing but the demand. So, demand at that particular price multiplied by 1 plus price elasticity of demand. So, this is what the $TR^{'}(p)$ is. So, this is the answer.

I have just added these comments so that the listeners or the students who are undergoing this course are aware of the implications of the result. Result is fine. So, the result is just this much. Here is the result. But the result also has an implication. So, this is the implication of that that if you take the derivative of total revenue with respect to price then this is what we get. It is actually a function of the price elasticity of demand.

(Refer Slide Time: 06:53)



This is a related problem. Suppose that for a monopolist firm there is a level of output where its profit is maximized it is known that its marginal cost and price elasticity of demand are 1 and -2, respectively. What is its equilibrium, profit maximizing price given that in equilibrium marginal revenue and marginal cost are equal?

So, there are certain information that we are given here that there is this monopolist firm. We have talked about what is a monopoly before. So, I am not going to spend time on that. And this monopolist firm is maximizing its profit. So, there is one output where its profit is maximized. It is also known that the marginal cost and the price elasticity of demand of this firm is given by 1 and - 2.

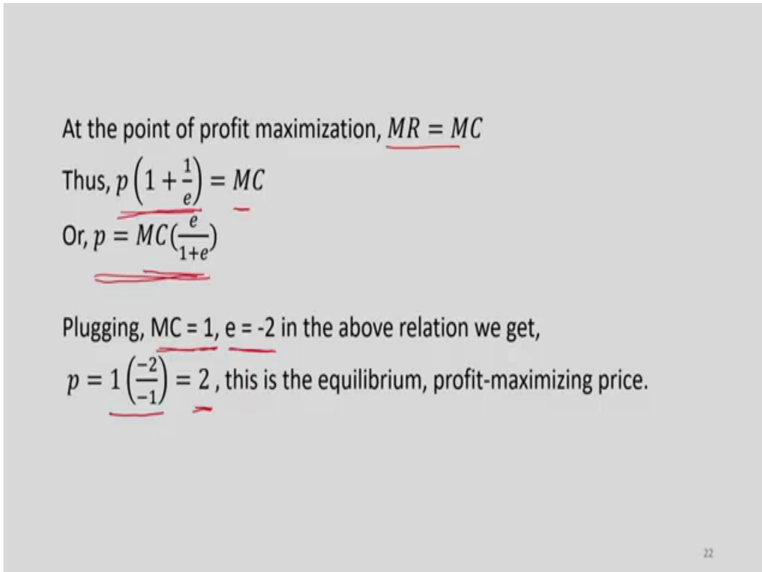
What is further given is that in this equilibrium where the profit is getting maximized MR, marginal revenue and MC they are equal. So, $MR = MC$ that is what determines the profit maximizing output. We have to find out what is the price at that equilibrium. So, what is the price that will be charged in the market, because the firm is maximizing its profit? So, that is the question. How do we solve this?

We start with the total revenue function. As we know total revenue is equal to price multiplied by quantity, $TR = pq$. And from here we try to get what is the expression for marginal revenue. As we know, if we take the derivative of the total revenue with respect to quantity, then we get the marginal revenue, so $\frac{d}{dq}(TR(q)) = MR$. Total revenue can be thought of as a function of quantity. So, this is what we are trying to get. This is MR.

So, we have basically used the fact that $TR = pq$and then we have taken the derivative of pq with respect to q. So, once again, product rule. But mind you compared to the previous problem here, the variable with respect to which the differentiation is being done is not price. It is quantity. So, this is what we will get. We will take p common. So, if I take p common it becomes $MR = p(1 + \frac{qdp}{pdq})$ and this can be written in terms of the elasticity.

In the previous problem itself we have seen that. Small e can be denoted as the price elasticity of demand and small e is given by this expression. This is nothing but $\frac{pdq}{qdp}$ and here we are actually talking about the reciprocal of that e. So, that is why I have written this as 1 divided by e. So, the upshot of this small exercise is that $MR = p(1 + \frac{1}{e})$.

(Refer Slide Time: 11:03)



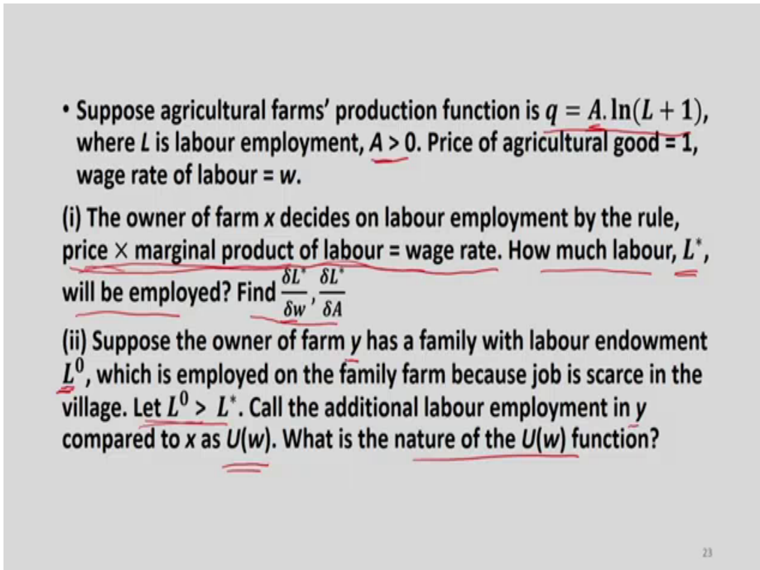
Now, that is just one step. Now, as we know further it is given in the question that at the point of profit maximization $MR = MC$ that is given. Now, we use that fact and the fact that marginal revenue is this, the left hand side, $MR = p(1 + \frac{1}{e}) = MC$ in the equilibrium. And then we express the thing in terms of p. On the left hand side I only have the price and on the right hand side if I take those terms there, I get $p = MC(\frac{e}{1+e})$. So, this is an important expression.

Price in a monopoly market is $p = MC(\frac{e}{1+e})$. So, as long as the elasticity is, suppose constant, then price has to be a fraction, in particular, $(\frac{e}{1+e}) > 1$. So, price will be some quantity, constant quantity multiplied by the marginal cost. It is something sort of markup over the marginal cost. This is called markup. But here our problem is somewhat different. We have to find out the price.

We are given the fact that the $MC = 1, e =- 2$. So, we put these values here $MC = 1, e =- 2$, if we put them in this expression, then this is what we get $p = 1(\frac{-2}{-1})$ and that is simply 2. So, this is the equilibrium profit maximizing price. Price is equal to 2. So, this is our solution.

Now, again, as a commentary, notice that in the equilibrium the marginal cost is 1 and the price is more than that marginal cost. It is 2. So, the marginal cost is added with something, with some markup and then we are getting the price. So, this is the answer.

(Refer Slide Time: 13:45)



Here is another problem and this is related to the economy of poor countries, the agricultural economy of poor countries. So, it could be very relevant for countries like India, where the agricultural sector is a large sector. It employs a lot of people. Actually, it is the largest employer in the Indian economy. So, what goes on in the agricultural sector that is an important question. And this problem is trying to answer some of the things that are important in the agricultural sector of a poor country.

Suppose agriculture farms' production function is given by this, $q = A.ln(L + 1)$, where L is labor employment, L is labor employment, and A, which is multiplied with this log function is greater than 0, $A > 0$. Price of agricultural goods is given to be 1, wage rate of labor is w, small w. So, this is the setting. Then we are having two questions here.

The first one is, the owner of farm x decides on labor employment by the rule, this rule, price multiplied the marginal product of labor = wage rate. So, this rule decides how much labor the owner of farm x will employ. So, we have to find out how much labor, let us call that L^*, will be employed, and furthermore, we have to find out $\frac{\delta L^*}{\delta w}$, $\frac{\delta L^*}{\delta A}$. So, these are partial derivatives of L^* with respect to w and A. Those things we have to estimate. This is the first part.

Second part, suppose the owner of farm y has a family with labor endowment L^o which is employed on the family farm because job is scarce in the village, $L^o > L^*$. Call the additional labor employment in y compared to x as $U(w)$ and U is a functional of small w, $U(w)$ as we shall see. What we need to answer is that what is the nature of $U(w)$ function. So, this is the second part. We have to find out the nature of this U function.

Just to think about it how to approach the problem and what this problem is trying to say. So, there are two farms in this agrarian economy x and y. x is a farm where the owner is deciding the employment, labor employment on the farm by this rule price multiplied by marginal product of labor is equal to wage rate and from this rule the L that is labor employment can be found out that we shall see and that employment is given by L^*. So, this is one kind of farm.

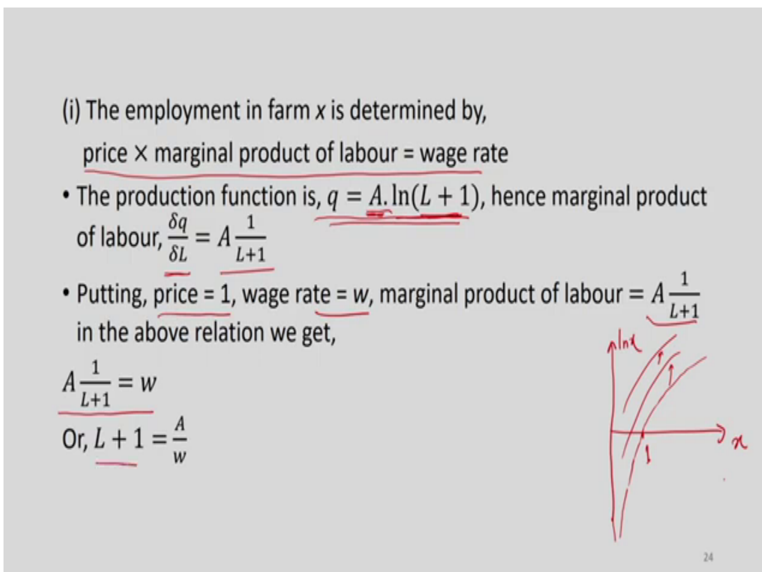
The other kind of farm which is there in part two is the y farm, where the employment decision is not decided by this price multiplied by marginal product of labor is equal to wage rate, but it is decided by how much labor is there in the family which owns that farm. So, the family is endowed with L^o amount of labor and that labor will be employed on the farm irrespective of what the wage rate is.

So, this rule is not applicable. And what we are assuming is $L^o > L^*$. There will be condition under which L^o will be greater than L^*. We shall see that. And then we are trying to find out what is the additional employment that will be done in farm y as compared to farm x.

So, the first farm is like a capitalist farm. It is also called a capitalist farm, where the owner does not have family labor to take care of. He is like a profit maximizing capitalist producer. He tries to maximize profit. And that profit maximization as we shall see later on gives us this rule, price multiplied by marginal product of labor = wage rate. That is one principle of how much labor to employ.

On the other hand, a farm might be owned by a family and the family has plenty of labor to employ and since there is unemployment in the economy that family endowment of labor will be put on the farm, because jobs are difficult to come by so the people who are there in the family work on their own farm. And so L^{o} will be employed. L^{o} amount of labor will be employed. So, we are trying to see the relationship between L^{o} and L^{*}. So, that is the background of this problem. It is a very important and quite familiar problem in the literature of development economics.

(Refer Slide Time: 20:15)



Let me start with the first part. This is the farm x and employment in farm x is decided by this rule, as we have seen in the problem, price multiplied by the marginal product of labor = the wage rate. Now, out of these three things, price and wage rate are known to us. Marginal product of labor is not known to us. But that can be found out from the production function. Production function is given by this $q = A.ln(L + 1)$.

Here are just few words about the nature of the function. Here L is the employment of labor. Notice, what we are getting here is ln that is $ln(L + 1)$. It is not simply L. The reason is as we know labor employment cannot be negative, it can take the value of 0 or positive amount of labor. Now, the trouble with log function is that at $L = 0$, suppose you are taking this kind of x and here you are taking log x and here is suppose 1, then here it is 0, but if it is less than 1, x is less than 1, then it becomes negative.

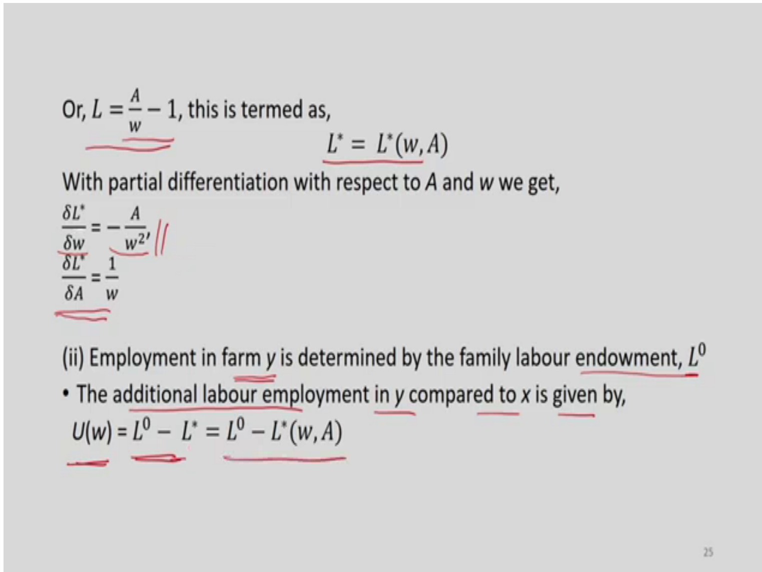
Now, if I had taken log of L then it would have been very difficult to interpret this function because that would have meant that if labor is less than 1 greater than 0 then output is negative which is difficult to justify or it is difficult to justify why if you have labor employment 0 the output is minus infinity. So, those things can be avoided if we take this form $ln(L + 1)$. So, here even if you take L, the minimum value is 0 it gives you log of 1. Log of 1 is 0. So, that is fine.

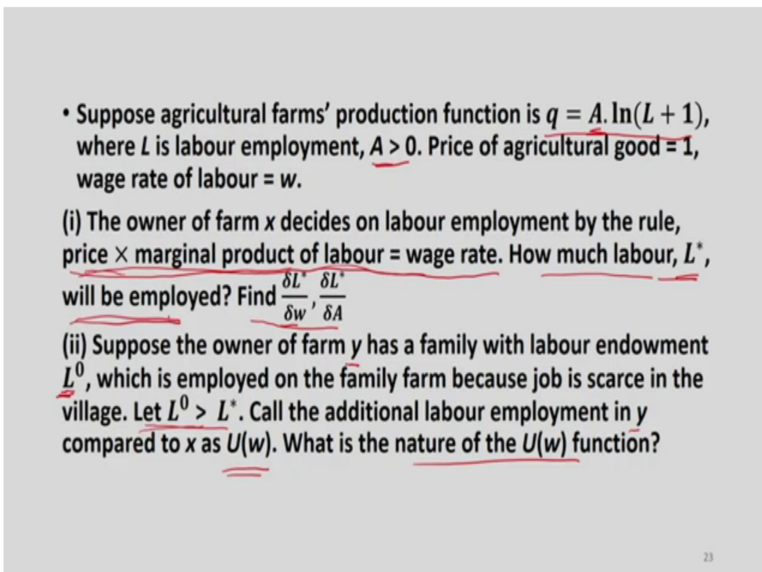
And then there is this constant term which is A. now, what is the significance of this A? Capital A is a kind of productivity parameter, which can be thought of as the amount by which this $ln(L + 1)$ is getting multiplied to give us the output. So, you can think about high A and a small A. If you have a high A, large A, then the same L will give you a lot of output, but if you have small A that same L will give you less output. So, this A is actually shifting the production function. It will be like this. As you have high A, it is shifting upwards. So, that is why I am saying that it is kind of the technology, kind of an indication of how developed the technology is for agricultural production.

Now, the main point is that we have to find out the marginal product of labor from this function, from this production function. Now, as we know the marginal product of labor is found out by taking the derivative. Here it will be partial derivative, because A might also vary. So, I take the partial derivative of q with respect to L and that gives me $\frac{A}{L+1}$, simple derivative of a logarithmic function.

And then we apply this rule: price multiplied by marginal product of labor = the wage rate. So, there I put price = 1, wage rate = w and marginal product of labor is equal to $\frac{A}{L+1}$ and we get simply this, very simple looking function. Then we actually found, try to find out what is L.

(Refer Slide Time: 24:57)





So, in the next step, we get this, $L = \frac{A}{w} - 1$. And this as we have seen in the question is termed as L^*. There you can see how much labor L^* will be employed that we have just found out. Which is given by $\frac{A}{w} - 1$. Now, the next step is to find out the partial derivatives. So, for that I have taken the partial derivative of L^* with respect to w and if we take the derivative with respect to A it is simply $\frac{1}{w}$.

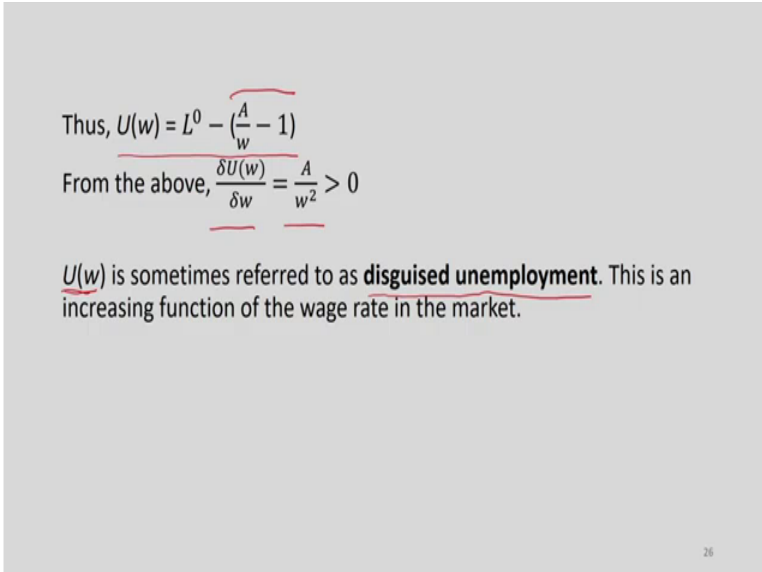
What is the implication once again? Let us try to understand what is the implication? L^* is the equilibrium employment or the optimal employment to be done by the owner of farm x. The owner of farm x is a profit maximizing farmer and he is employing L^*. But that L^* can vary with respect to two things. One is the wage rate. As wage rate rises, we are getting L^* to be declining, because it is a negative thing, $\frac{\delta L^*}{\delta w}$ is negative.

In simple terms, as the wage rate in the market rises, the owner of farm x is going to reduce employment. On the other hand, as A, that is A rises, L^* will rise, because $\frac{\delta L^*}{\delta A} = \frac{1}{w}$, which is positive. So, as the technology becomes more productive, the owner of farm x will employ more labor on his farm. So, that is what the meaning of these two terms is.

The second part, we are talking now about the other farm which is y. Farm y employment is determined by the family labor endowment. So, again to recall, y is not owned by a profit maximizing farmer. It is owned by someone who has a family to look after and that family has L^0 amount of labor endowment. So, that L^0 is employed on that farm as we are told in the question.

Therefore, the additional labor employment in y compared to x is given by this $U(w)$ which is given in the question. $U(w) = L^0 - L^* = L^0 - L^*(w, A)$. How much is the difference between the employment in the family farm that is L^0 and the employment in the other farm, capitalist farm so that is L^*. So, this is what we are getting. $U(w)$ is equal to a function of w.

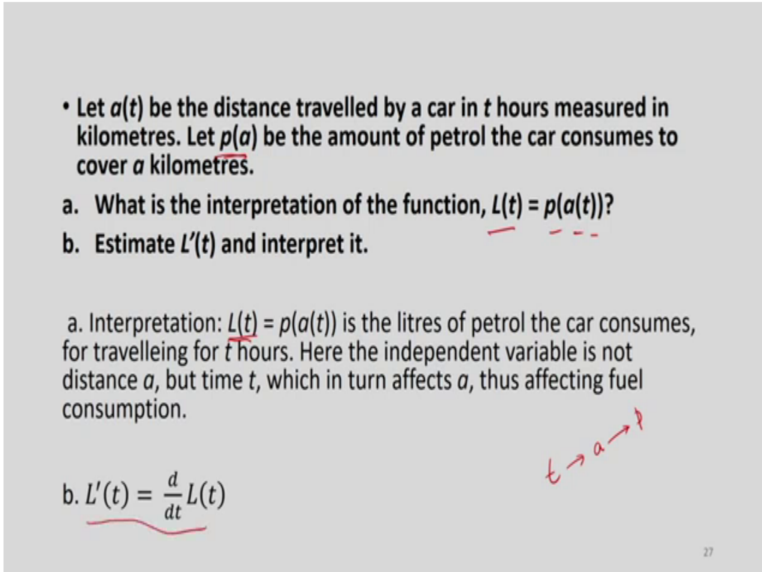
(Refer Slide Time: 28:34)



And I write the expression for L^* which is this $\frac{A}{w} - 1$. And thereafter I take the partial derivative of $U(w)$ with respect to w and I get simply $\frac{A}{w^2}$ and which is positive. So, this is the answer we were asked to find out the nature of the $U(w)$ function. And I have taken the partial derivative of $U(w)$ with respect to w and found it to be positive. So, $U(w)$ is an increasing function of the wage rate, which basically means that as the wage rate in the market rises, the additional employment in the family farm compared to the x farm that is the capitalist farm, that additional employment is actually rising.

Now, $U(w)$ has a name actually it is sometimes called the disguised unemployment. This additional labor employment in the family farm is compared to the x farm is sometimes known as disguised unemployment. Those of you who will be doing courses on development economics will be quite familiar with these expressions.

(Refer Slide Time: 30:15)



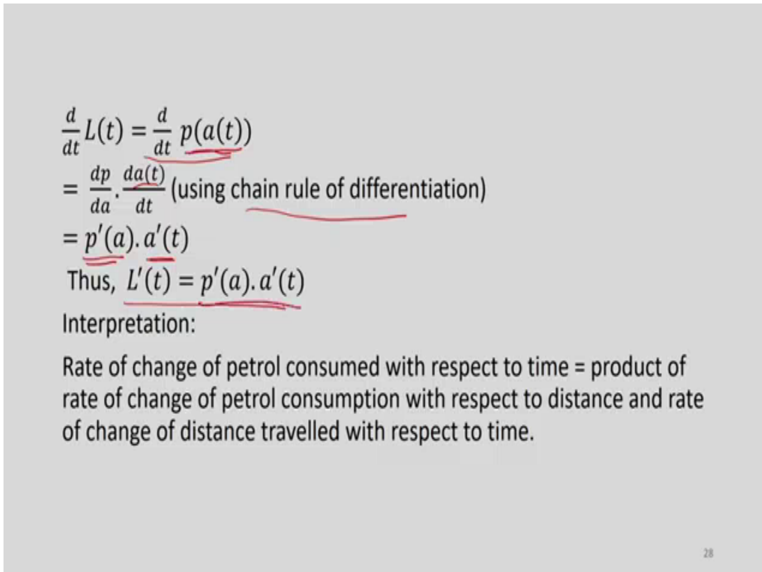
Here we take up another problem related to differentiation. Let $a(t)$ be the distance traveled by a car in t hours measured in kilometers. Let $p(a)$ be the amount of petrol the car consumes to cover a kilometers. What is the interpretation of the function $L(t)$ which is $p(a(t))$? Estimate $L'(t)$ and interpret it. So, this is the question.

Let us try to understand the question. Two functions are given a is the distance. It is in terms of kilometers and it is a function of t. How much time is spent in traveling and that affects the amount of distance covered and that distance is given by a. On the other hand, if you are covering more distance, the petrol consumed also rises. So, p is the petrol consumed. Obviously, p will be a function of a, $p(a)$. Both of these will be increasing functions, both a and p. We have to find the interpretation of the function L. So, the L function is defined as $p(a(t))$.

So, interpretation is the following that $L(t) = p(a(t))$ is the litres of petrol the car consumes for traveling t hours. Here petrol is a function or litres of petrol is a function of the time. It is not like $p(a)$. In $p(a)$litres of petrol was a function of distance. Here it is a function of time.

Here the independent variable is not distance per time t which in turn affects a, thus, affecting fuel consumption. So, this is easy to understand. So, t is their time. It affects distance a. And distance affects petrol that is how much petrol is being consumed. So, ultimately p is a function of t, $p(a)$ and that is being written here as $L(t)$. Secondly, what is $L'(t)$. $L'(t)$, the derivative of the L function with respect to time. So, that is $\frac{d}{dt}L(t)$.

(Refer Slide Time: 33:10)

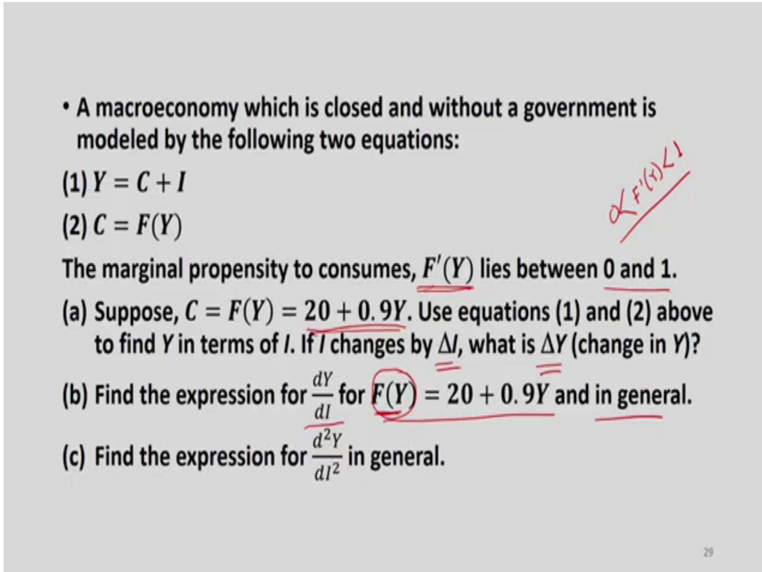


And that we write it as following because I know $L(t) = p(a(t))$. So I write that explicitly and then I take the derivative of this using the chain rule of differentiation and that will give us $\frac{dp}{da}.\frac{da(t)}{dt}$ where a is a function of time, $a(t)$.

Now, I also know $p(a)$. So, this becomes $p'(a)$ and the second term is nothing but $a'(t)$. So, in short, $L'(t) = p'(a).a'(t)$. So, this is what we are getting. $L'(t)$ that is the derivative of the litres of petrol consumption with respect to time = $p'(a).a'(t)$.

The rate of change of petrol consumed with respect to time = the product of rate of change of petrol consumption with respect to distance and rate of change of distance traveled with respect to time. So, that is the interpretation in terms of words.

(Refer Slide Time: 34:35)

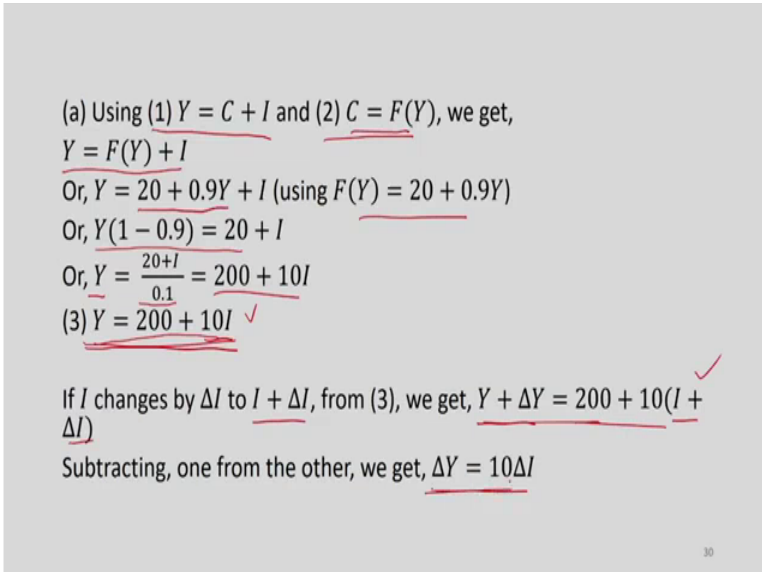


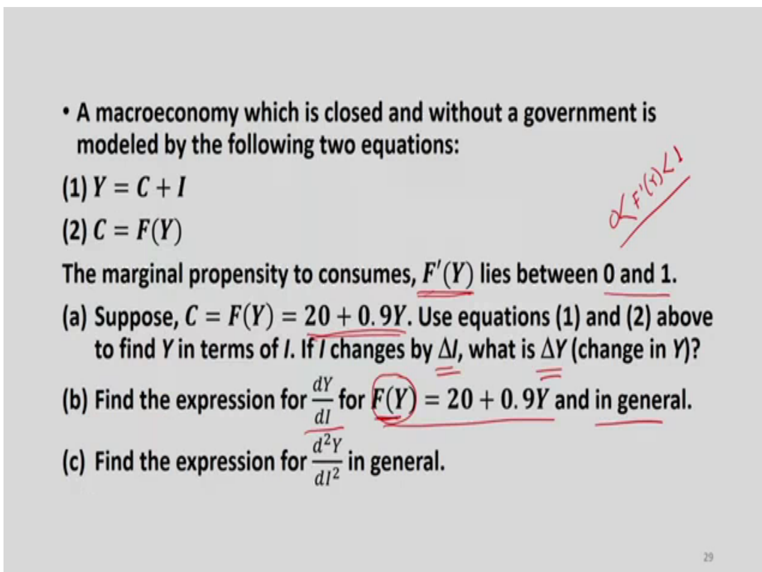
Here is a sort of different problem. Here the problem is macroeconomic. A macroeconomy which is closed and without a government is modeled by the following two equations; $Y = C + I$ and $C = F(Y)$. The marginal propensity to consume that is $F^{'}(Y)$ lies between 0 and 1. So, I can write it simply as this $0 < F^{'}(Y) < 1$ it lies between 0 and 1.

Now, there are three parts to this problem given the setting. Suppose $C = F(Y)$ is this. So, it is taking this particular form $C = F(Y) = 20 + 0.9Y$. Use equations 1 and 2 above to find Y in terms of I. If I change this by ΔI how much or what is ΔY, ΔY is the change of Y. This is the first part.

Second part, find the expression for $\frac{dY}{dI}$ if $F(Y) = 20 + 0.9Y$and in general, in general means $F(Y)$ is having this form, $F(Y) = 20 + 0.9Y$. $F(Y)$ is just $F(Y)$. It is not taking any particular form. And finally, find the expression for $\frac{d^2Y}{dI^2}$ in general. In general means again suppose $F(Y)$ is just $F(Y)$. So, these are the three parts of the problem. Let us start with these two equations that we are given.

(Refer Slide Time: 36:36)





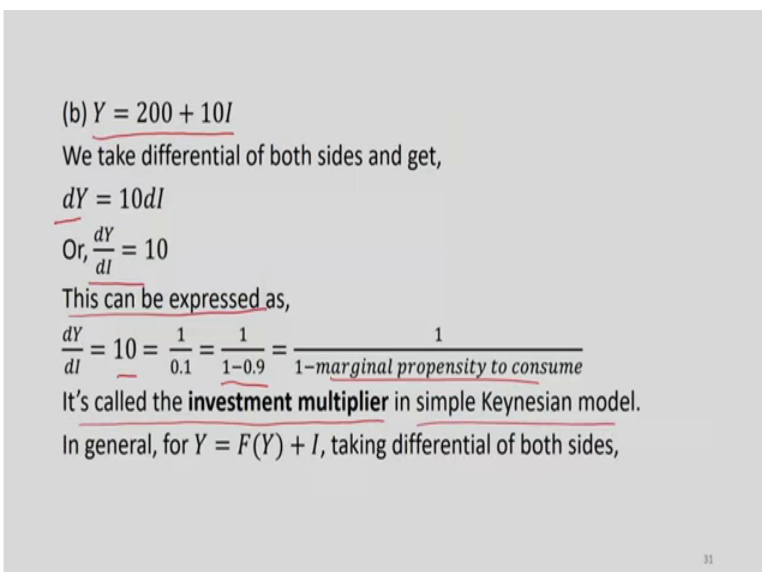
This is one equation. Number one, $Y = C + I$. And number two is $C = F(Y)$. If we combine them, so I substitute $C = F(Y)$ here in the first equation, I am getting $Y = F(Y) + I$. And I have been given the form of $F(Y)$, it is this $F(Y) = 20 + 0.9Y$. I am using that. And now I can express Y as a function of I. So, that is what I am doing. I am taking all the Y terms to the left hand side.

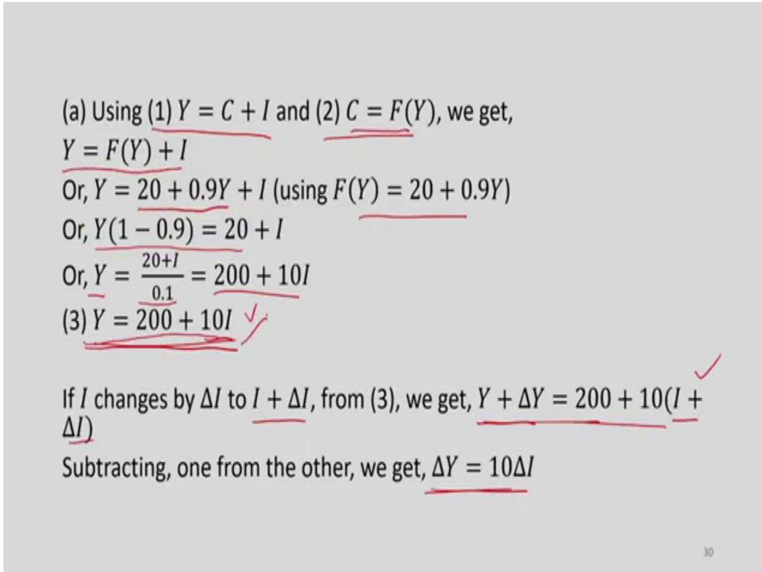
And after that, I get this $Y(1 - 0.9) = 20 + I$. So, I can divide both sides by $(1 - 0.9)$. $(1 - 0.9) = 0.1$. So, I am dividing both sides by 0.1. So, that is what I am getting $\frac{20+I}{0.1}$ that turns out to be $200 + 10I$. So, $Y = 200 + 10I$ that I think was one question that was asked to find Y in terms of I. That is what we have done. Here Y is the dependent variable and I is the independent variable. I have expressed Y in terms of I.

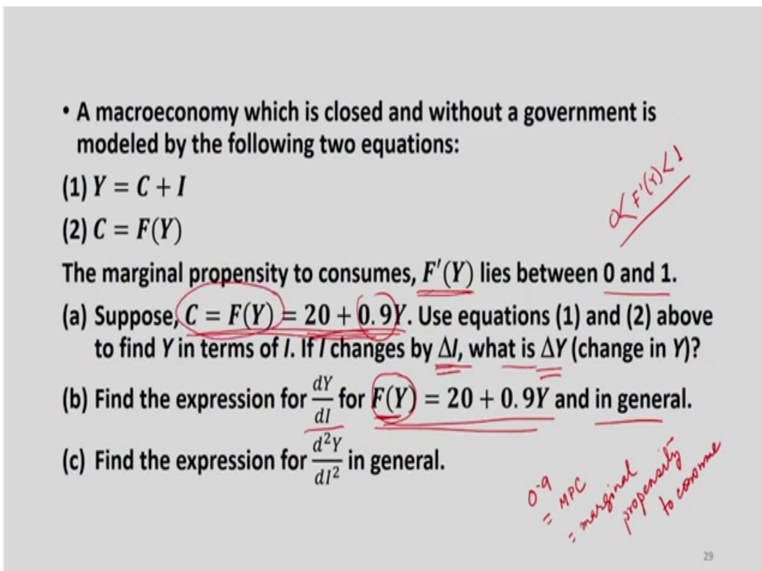
The second part here in the first question itself was if I changes by ΔI then how much does Y change. So, to do that, let us try to see if I changes by ΔI, the new I is $I + \Delta I$. So, that I substitute here. In that case Y will change to a new Y. So, suppose that new Y is given by $Y + \Delta Y$.

So, $Y + \Delta Y$ will be $200 + 10(I + \Delta I)$. What is the new I. New I is $I + \Delta I$. And we take this equation and we take this equation, we subtract this one from this one. Then if we do so, then we shall get $\Delta Y = 10\Delta I$. I think that was the second part. Second part was if I changes by ΔI what is ΔY. Then $\Delta Y = 10\Delta I$.

(Refer Slide Time: 39:33)







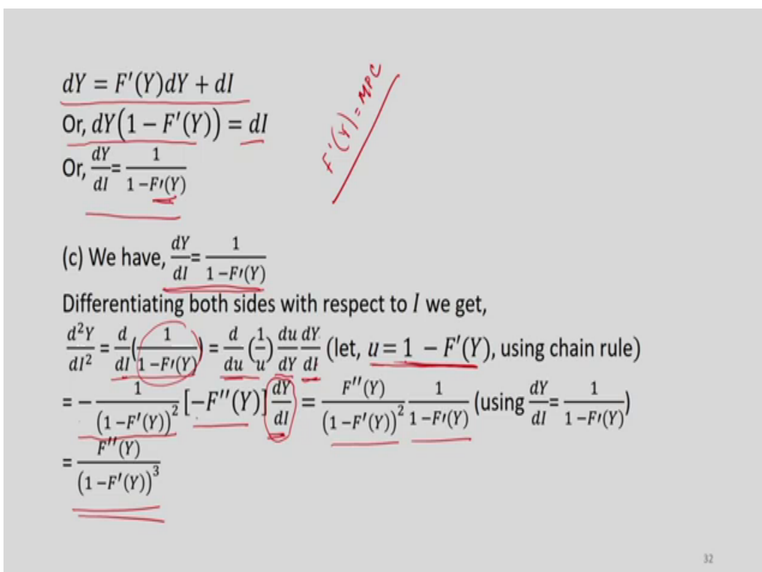
Now, we come to (b). (b) is what. Find the expression for $\frac{dY}{dI}$ for this particular form of $F(Y)$ and in general. We again start with the equation that we have just derived this one. We start with that. And we take the differential of both sides and we are going to get $dY = 10dI$. The reason is 200 is a constant. So, it will drop out. Y and I add the variables. So, we take the differentials. And so, we get $dY = 10dI$.

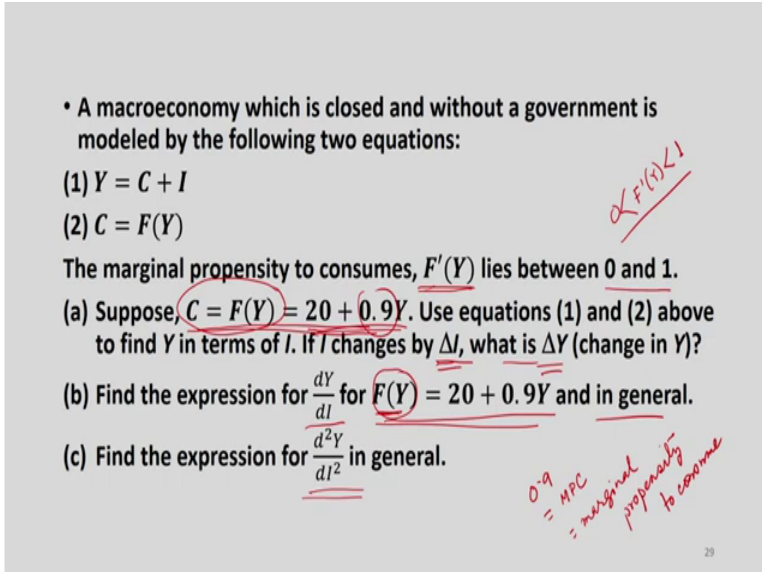
And in the next step I simply get $\frac{dY}{dI}$. This actually can be expressed as the following, $\frac{dY}{dI}$ and that can be written as $\frac{1}{0.1}$ turns out to be $\frac{1}{1-marginal\ propensity\ to\ consume}$. 0.9 was the marginal propensity to consume, remember. If you have this consumption function then 0.9 is equal to MPC or marginal propensity to consume. So, that is what we have written here that $1 - 0.9 = 1 - marginal\,propensity\,to\,consume$.

So, $\frac{dY}{dI} = \frac{1}{1 - marginal\,propensity\,to\,consume}$. This is also known as the investment multiplier in the simple Keynesian model. So, this sort of model is called a simple Keynesian model. And in this kind of model $\frac{dY}{dI}$ is called the marginal propensity to consume and in this model it is given by $\frac{1}{1 - marginal\,propensity\,to\,consume}$.

But this was the case where the consumption function was having this particular form. The form was this form $C(Y) = 20 + 0.9Y$. But suppose I take a general consumption function of $F(Y)$ instead of this linear form, then what do we get?

(Refer Slide Time: 42:26)





This is what I get by taking the differential of both sides. $dY = F^{'}(Y)dY + dI$. Again, I take all the dY terms to one side, so $dY(1 - F^{'}(Y)) = dI$. And in the next step, I am going to get $\frac{dY}{dI} = \frac{1}{1-F^{'}(Y)}$. Again, if you remember $F^{'}(Y)$ is MPC.

So, previously when the form was given, linear form, the MPC was 0.9 so that is what was appearing here. But if I take $F(Y)$ to be the general consumption function, then $F^{'}(Y)$ is the MPC. That is why here instead of 0.9 I am getting $F^{'}(Y)$. So, the basic form of the investment multiplier remains the same. It is $\frac{1}{1-MPC}$.

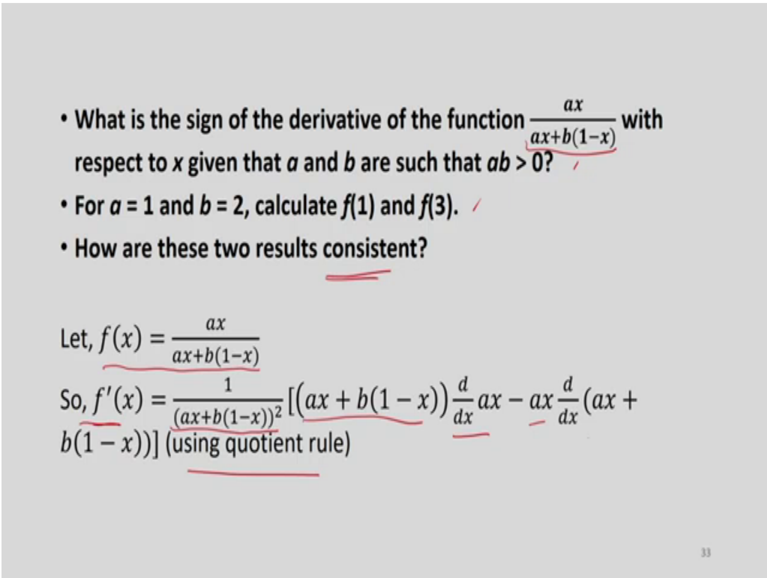
Now, the third part. Third part is, we have to find out this, the second derivative, $\frac{d^2Y}{dI^2}$. $\frac{dY}{dI}$ was found out. It is called the investment multiplier. But if I take the second derivative of Y with respect to I, then what do we get? The first derivative is this. We are talking about the general form. I am not taking that linear consumption function. Now, differentiating both sides with respect to I, that is what we get to get the second derivative. So, that is what we are doing here.

Now, here, to simplify this, I am using the chain rule. I am assuming that $u = 1 - F^{'}(Y)$. So, if I do so, then this expression becomes $\frac{1}{u}$. And if I have I here, I cannot differentiate the function if it is a function of u. So, I write du here and using the chain rule, I write $\frac{du}{dY}$ and $\frac{dY}{dI}$. In the next step $\frac{d}{du}(\frac{1}{u})$.

So, what is that? It is $\frac{-1}{u^2}$. So, $\frac{-1}{u^2}$ is coming here, because this was u. So, $\frac{1}{u^2}$means$\frac{1}{(1-F'(Y))^2}$. $\frac{du}{dY}$ so I have to take the derivative of this with respect to Y and that will be $-F''(Y)$ that is coming here multiplied by $\frac{dY}{dI}$ and this is repeated here and then I simplify it a little bit.

I use the fact that $\frac{dY}{dI}$ that is nothing but the investment multiplier is equal to $\frac{1}{1-F'(Y)}$. So, this is what it is turning out to be. Remember, here is a minus sign and here is a minus sign they are getting multiplied. So, they are cancelling each other. So, finally, I am going to get $\frac{F''(Y)}{(1-F'(Y))^3}$. So, this is the expression that we were supposed to find out $\frac{d^2Y}{dI^2}$ is found to be $\frac{F''(Y)}{(1-F'(Y))^3}$.

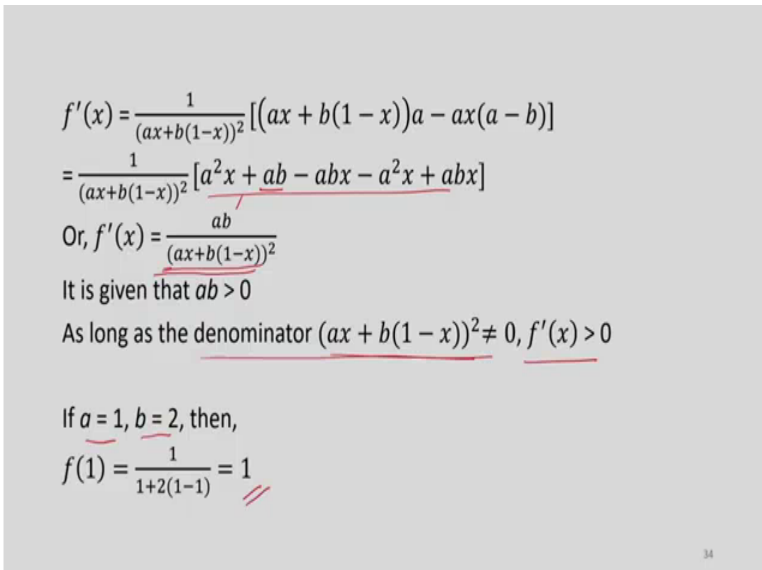
(Refer Slide Time: 46:43)

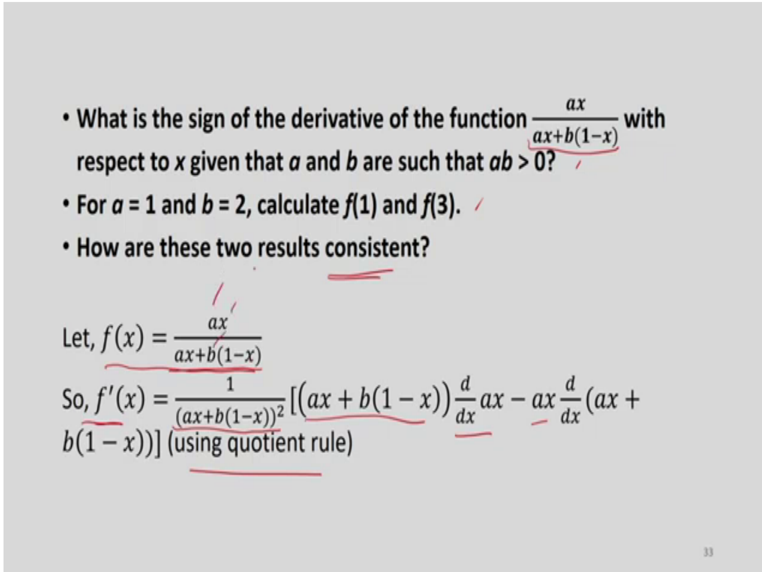


Here is another problem from differentiation. What is the sign of the derivative of the function, this function, $\frac{ax}{ax+b(1-x)}$with respect to x given that a and b are such that a multiplied by b is greater than 0. So, this is the first part. We have to find out the derivative of this function. Second part for a = 1 and b = 2 calculate $f(1)$ *and* $f(3)$. And thirdly, how are these two results consistent? So, two results means, this result and this result, are they consistent?.

So, let us suppose that this function is given by $f(x) = \frac{ax}{ax+b(1-x)}$ we have to find out $f^{'}(x)$. That we find out by using the quotient rule. What was the quotient rule? $\frac{1}{(ax+b(1-x))^2}[(ax + b(1 - x))\frac{d}{dx}ax - ax\frac{d}{dx}(ax + b(1 - x))]$. So, this messy expression has to be simplified.

(Refer Slide Time: 48:34)





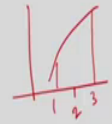
And this is just simplification. Actually, many terms will get cancelled out at this stage. And if sometimes getting cancelled out, but only one term is remaining which is ab in the numerator. So, $f^{'}(x) = \frac{ab}{(ax+b(1-x))^2}$. Now, it is given that ab is greater than 0. Now, $ab > 0$, that means numerator is positive. The denominator is positive always as long as the denominator is not 0 it can be defined also.

So, I have written that, the denominator cannot afford to be 0 for $f^{'}(x)$ to be meaningful. So, as long as the denominator is not 0, that is $(ax + b(1 - x))$ is not 0, the first derivative $f^{'}(x)$ x is positive. So, that is the answer to the first question I think. First question was what is the sign of the derivative of the function as long as $ab > 0$? As long as $ab > 0$ the sign is positive, provided the denominator is not 0.

The second part is what is the value of the function that is $f(x)$ at $x = 1$ and $x = 3$ if $a = 1\ and\ b = 2$. If $a = 1\ and\ b = 2$, what is the value of $f(1)$. That is very simple. I have to use this expression. I have to put $a = 1$ here, $b = 2$ here and $x = 1$ here. And that turns out to be simply 1, $f(1) = 1$.

(Refer Slide Time: 50:40)

$f(3) = \frac{3}{3+2(1-3)} = \frac{3}{3-4} = -3$



We found that $f'(x) > 0$, which seems to contradict the result that $f(1) > f(3)$.
However, we need to remember that $f'(x) > 0$ is valid if

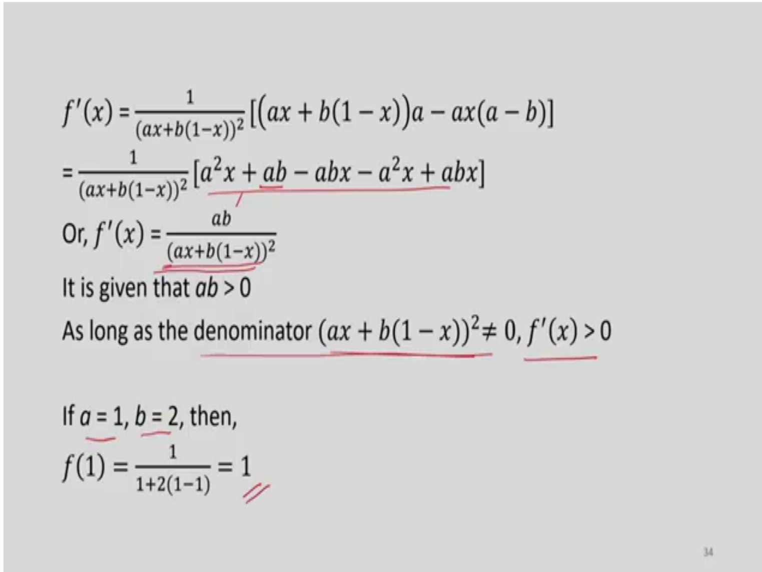
$$ax + b(1-x) \neq 0$$

At $a = 1$, $b = 2$, $ax + b(1-x) = x + 2 - 2x$
At $x = 2$, $ax + b(1-x) = 0$,
Thus, at $x = 2$, the function $f(x) = \frac{ax}{ax+b(1-x)}$ is not defined, it is discontinuous.

- **What is the sign of the derivative of the function $\frac{ax}{ax+b(1-x)}$ with respect to *x* given that *a* and *b* are such that *ab* > 0?**
- **For *a* = 1 and *b* = 2, calculate *f*(1) and *f*(3).**
- **How are these two results consistent?**

Let, $f(x) = \frac{ax}{ax+b(1-x)}$

So, $f'(x) = \frac{1}{(ax+b(1-x))^2}[(ax + b(1-x))\frac{d}{dx}ax - ax\frac{d}{dx}(ax + b(1-x))]$ (using quotient rule)

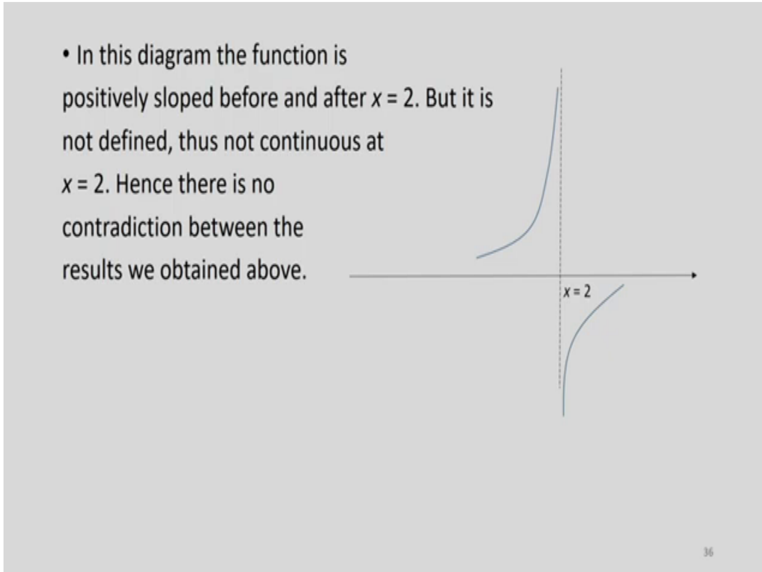


Second is, what is $f(3)$? Again, I use the functional form, this. I put $x = 3$ along with $a = 1$, $b = 2$ and I do the simplification I get to actually -3. Now, here is the sort of puzzle that we found out that $f'(x) > 0$. $f'(x) > 0$ means what that the function is something like this. I mean, this is just an illustration. It could have some other shape also, but it is a rising function if $f'(x) > 0$.

Now, this seems to contradict the result that $f(1) > f(3)$. So, here is 1 and here is 3, $f(3)$ should have been greater than $f(1)$ if the function is increasing, but here it is less than $f(1)$. $f(3)$ is - 3, whereas $f(1)$ is 1. So, how do we explain that? Well, the reason for this seeming puzzle is that $f'(x) > 0$ is valid if the denominator is not 0. Denominator is not 0 means $(ax + b(1 - x))$ cannot be equal to 0. If that is equal to 0 then we are in trouble.

Now, let us suppose if there is a problem somewhere, if $a = 1$ *and* $b = 2$ then this expression turns out to be this $x + 2 - 2x$. And if you put $x = 2$ here, where is 2, 2 is between 1 and 3. So, here is 2. Then actually this expression turns out to be 0, which means that the function that we are given $f(x) = \frac{ax}{(ax+b(1-x))}$, this function is not continuous at $x = 2$. It is not even defined at $x = 2$. Now, that is basically creating the problem.

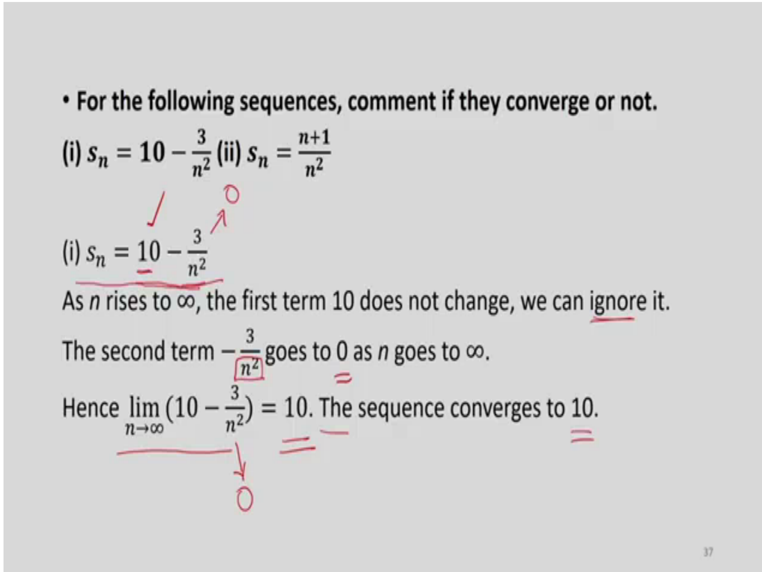
(Refer Slide Time: 53:16)



What we have is something like this. Again, this is just an illustration, not an exact representation of this function. This function is discontinuous at $x = 2$. At $x = 2$ it is not defined. So, before $x = 2$ it is, you can see it is increasing. After $x = 2$ again it is increasing. So, in general, the function is the rising function. That is what we found in the first part $f^{'}(x) > 0$.

However, at a particular point $x = 2$ the function is not even defined. So, there is a discontinuity there. And that is basically the reconciliation of these two results. The function is in general a rising function except for the fact that it is not defined at $x = 2$. And that is the reason why the value of the function before $x = 2$ could be higher than the value of the function after $x = 2$. So, there is no contradiction between the results that we obtain here. So, this is the reason. There is no contradiction. It is just that the function is not defined at a particular point.

(Refer Slide Time: 54:43)

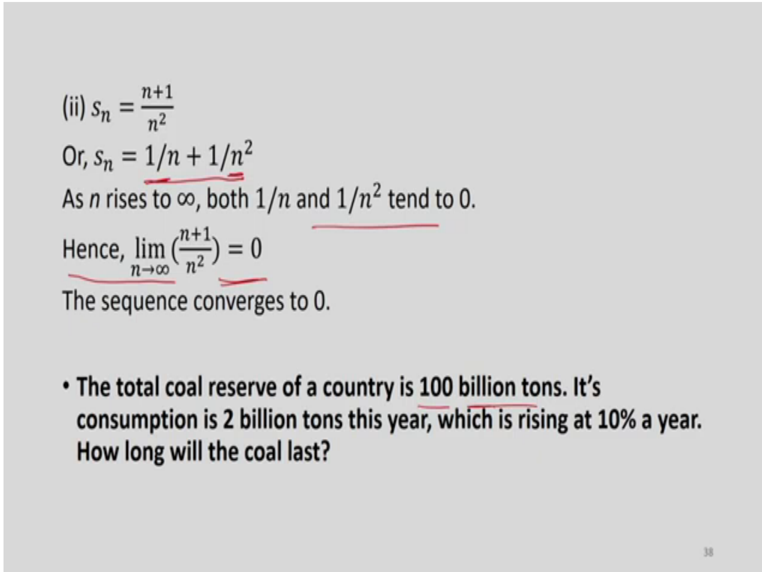


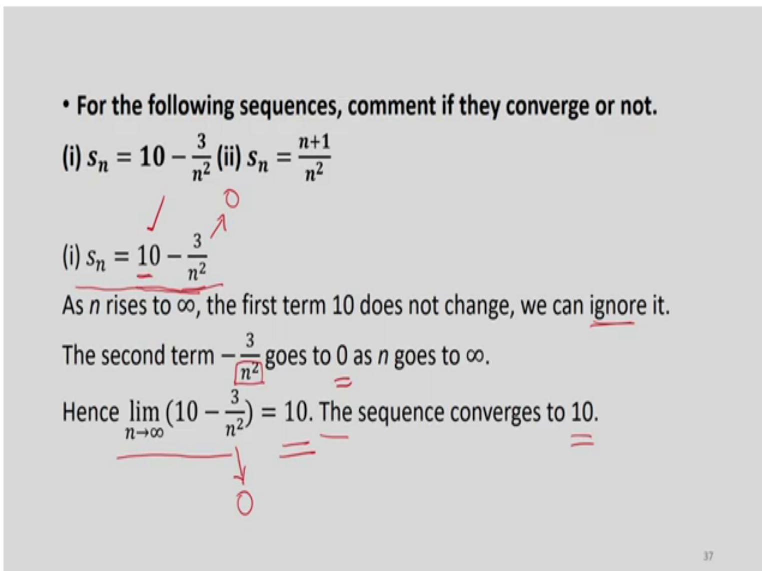
Here I am getting a little bit away from the differentiation theme. I am talking about sequences now. For the following sequences, comment if they converge or not. So, two sequences are given. As we know sequences are generally denoted by S_n. S_n, the first part is given by $S_n = 10 - \frac{3}{n^2}$. In a second part $S_n = \frac{n+1}{n^2}$. Let us take the first one. $S_n = 10 - \frac{3}{n^2}$. Now, we have to actually find out as n goes to infinity, what happens to S_n?

Now, here, as n goes to infinity, the first term is 10. It does not change. It is constant. So, we can ignore it. What about the second term? Second term is $-\frac{3}{n^2}$. Now, as n goes to infinity, the denominator goes to infinity. As the denominator goes to infinity, the numerator is constant. So, the term actually goes to 0 as n goes to infinity. So, this part remains constant at 10 and this part, the second part goes to 0. What happens to the entire thing? What happens toS_n?

So, that is what I have written here. $S_n = \lim_{n\to\infty} (10 - \frac{3}{n^2}) = 10$, because this part actually is going to 0. And as we know, that is the definition that if S_n goes to some constant, it tends to some constant as n goes to infinity, then the sequence converges. In this case, the sequence converges to 10. So, this is the answer. It is a sequence which converges.

(Refer Slide Time: 57:19)



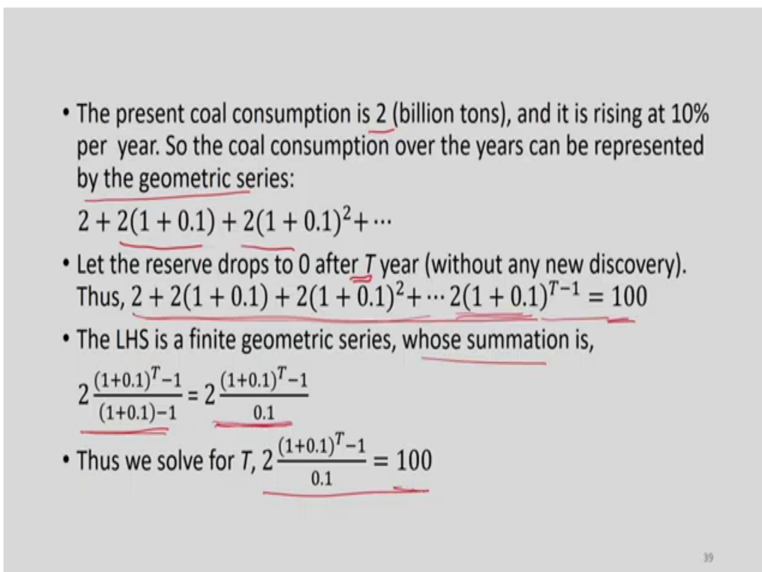


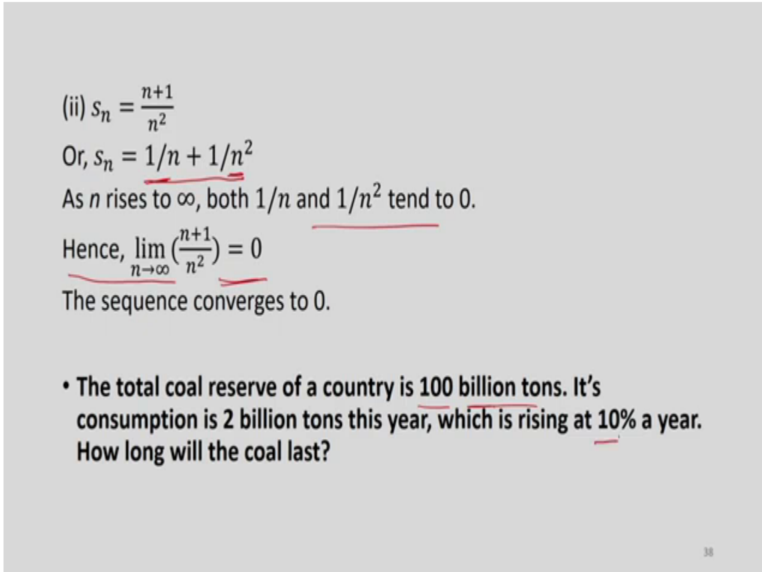
The second sequence is $S_n = \frac{n+1}{n^2}$. And actually, this can be simplified as $1/n$, n, n will get cancelled from the numerator and the denominator in the first term, plus $1/n^2$. Now, once again, as n goes to infinity, what happens to these two terms? As n goes to infinity, this first part, again, the denominator is going to infinity, which means that the fraction will go to 0. Similarly, here also the denominator goes to infinity. Therefore, the fraction again goes to 0. So, both of them tend to 0.

Therefore, $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n+1}{n^2} = 0$. Now 0 is a constant. Therefore, the sequence actually converges to 0. It is a case of convergence. I think that was the question. Comment if they converge or not? Yes, the second one also converges.

Here is another problem in the same slide. The total coal reserve of a country is 100 billion tons. Its consumption is 2 billion tons this year, which is rising at 10 percent a year. How long will the coal last? So, it is a question of how fast a country will run out of its natural resources, a natural resource, in this case, which is non-renewable. So there is a depletion, but there is no replenishment of that resource. So, if this resource is consumed and the consumption is rising over time then how fast will the resource be exhausted?

(Refer Slide Time: 59:43)





So, that is the question. Now, how to get around this problem? The present coal consumption is 2, let us suppose, I am just writing in terms of numbers and forgetting about the units, 2 billion tons and it is rising at 10 percent per year. So, the whole consumption over the years can be represented by a geometric series given by the following; $2 + 2(1 + 0.1) + 2(1 + 0.1)^2 + ...$ 2 in this year.

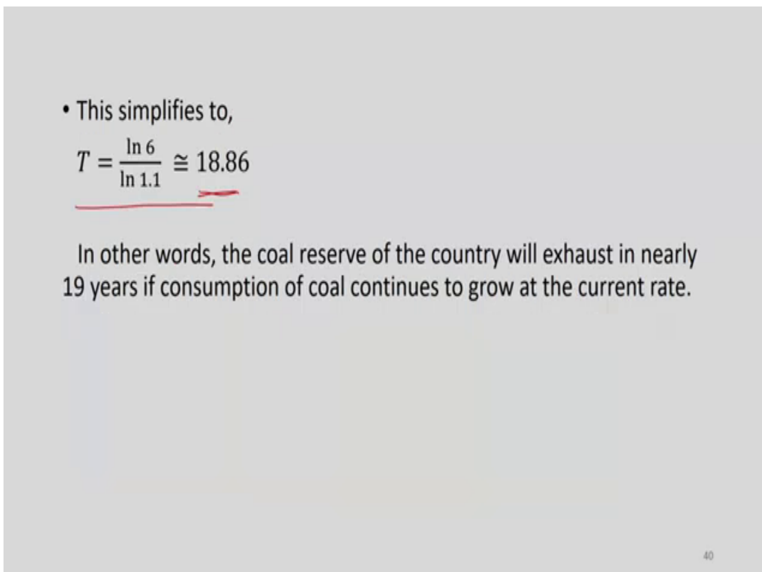
In the next year it will be $2 + 0.1 . 2$, because it is rising at 10 percent year. So, next year it will be $2(0.1)$, because 0.1 is equal to 10 percent. So, that is written as $2(1 + 0.1)$. This is the second year. In the third year, it will be $2(1 + 0.1)^2$ and like this it will go. In the fourth year it will be $2(1 + 0.1)^3$.

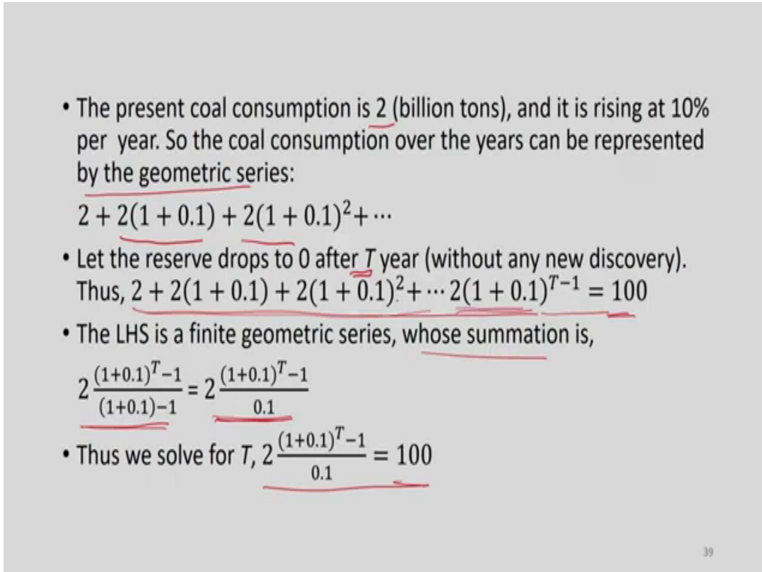
Now, let us suppose that the reserve drops to 0 after T years. T is some constant. Suppose T is the number of years after which the reserve actually becomes 0. And we are assuming that there is no new discovery of coal reserves. In that case, what will happen after T years? How much is the total consumption? Total consumption is given by this, $2 + 2(1 + 0.1) + 2(1 + 0.1)^2 + ... 2(1 + 0.1)^{T-1}$. So, this is the last term in the T-th year. T-th year's consumption will be this much. And this is the total consumption over the years and that is getting equal to 100. 100 is the present reserve.

So, since we are assuming that after T years the total reserve will be exhausted, so this relationship must hold. Now, the left hand side is a finite geometric series whose summation is given by this, $2\frac{(1+0.1)^T-1}{(1+0.1)-1}$.

This is the formula that we have discussed in the duration of the course, the summation of a finite geometric series. So, this is the formula. And so this simplifies to this expression. And we have to actually solve for capital T in this equation. The left hand side is the total consumption. The right hand side is the total reserve.

(Refer Slide Time: 62:44)

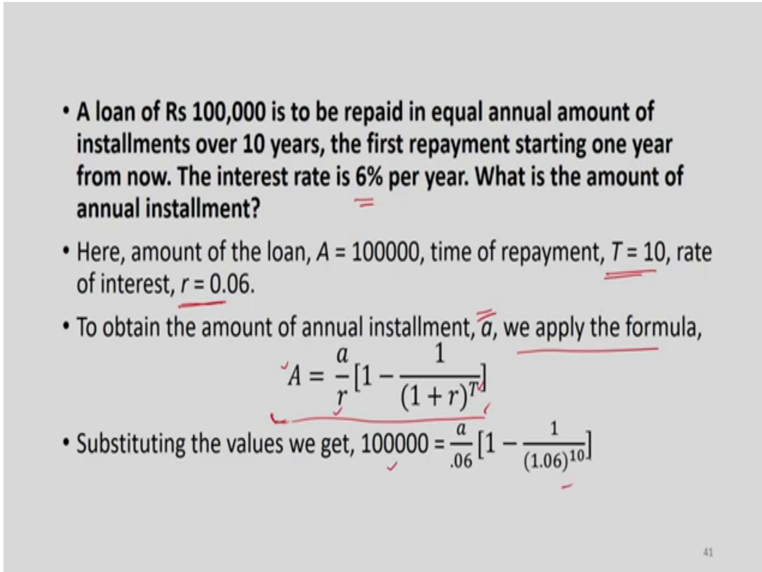




And if we do so, actually this turns out to be very simple, $T = \frac{ln\, 6}{ln\, 1.1}$ and if you have a calculator, it is easy to find out. $T = \frac{ln\, 6}{ln\, 1.1} \approx 18.86$. In other words, the coal reserves of the country will be exhausted in nearly 19 years. It is very close to 19, 18.86. If consumption of coal continues to grow at the current rate, and that is important, we are assuming that all through these years, all through this T number of years, the consumption is rising by 10 percent, that is an assumption so that we can apply this formula.

If the consumption actually rises by a faster rate, let us suppose after some years, then actually the exhaustion of coal reserves will take place even with fewer years. And on the other hand, if people are more conscious of the environment, then maybe the consumption growth will slow down and in that case the reserve will last for a longer duration. So, this is the answer to the question.

(Refer Slide Time: 64:09)



This is the last I think the sum that we will do today and then we will call it a day. A loan of 1 lakh is to be repaid in equal annual amount of installments over 10 years, the first repayment starting one year from now. The interest rate is 6 percent per year. What is the amount of annual installment?

So, it is as if, that you have taken a loan of 1 lakh, suppose you want to buy a car or maybe a house you have taken some money from the bank. Now, that money you will repay over the next 10 years. And obviously the bank will not give you a loan unless you pay some rate of interest, that, rate of interest is 6 percent per year.

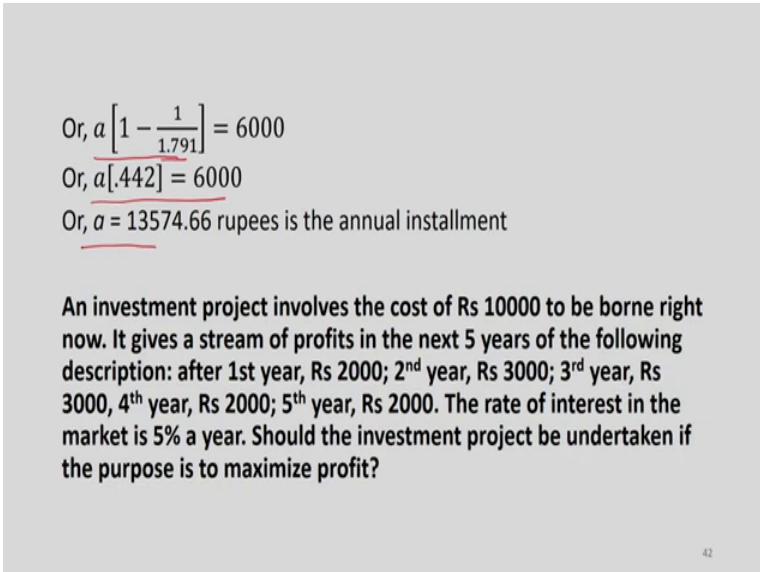
Now, the question for you to solve or maybe for the bank to solve is that, what is the amount of repayment that you will do per year. This is in Indian banking parlance it is called the EMI, equal monthly installment, if you are paying it on a monthly basis. But suppose we are taking the payment per year instead of month then the problem is similar. Then what is the amount that you will pay annually over the next 10 years assuming that the payment per year is the same.

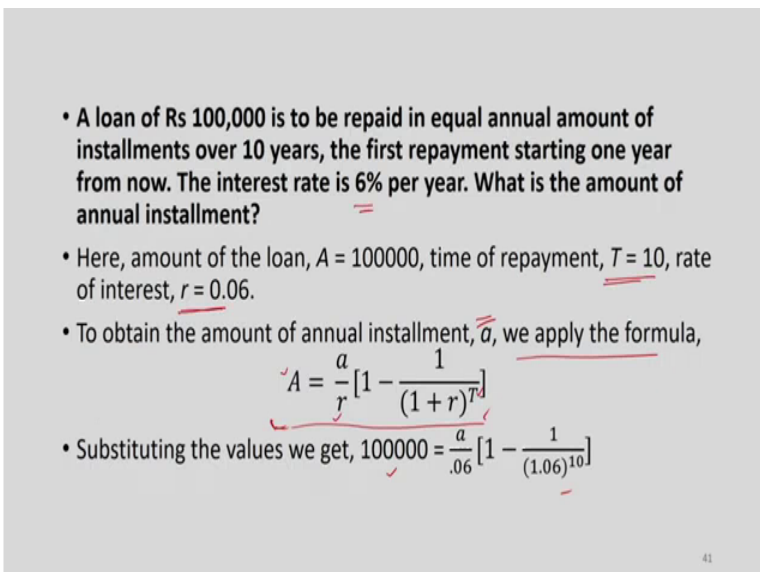
How to solve this problem? Here the amount of loan, suppose that is given by A. A is 1 lakh. Time of repayment, what is the period over which the payment has to be done that is given by capital T, capital T is 10. The rate of interest is here given by 0.06 that is 6 percent and that is, suppose given by small r.

Now, to obtain the amount of annual installment and that is let us suppose that is given by small a, we apply this formula. We have discussed this formula again during the course. $A = \frac{a}{r}[1 - \frac{1}{(1+r)^T}]$.

So, this formula actually is coming from the summation of geometric series, finite geometric series. You just have to review, go back to the books and the notes to remember that this was the formula. Now, I have to just substitute these values of A that we know, r we know, T we know. What we do not know is a, that is the annual installment that we have to solve. So, I have substituted all these values in this formula.

(Refer Slide Time: 67:25)





And in the next stage it becomes a little bit simpler. Again, I am taking the approximate values, three decimal places. And so, this turns out to be this and in the next stage I get $a = 13574.666$ as the annual installment. So, that is the answer.

Now, just to get a hang of this 13,000 is more than 10,000. Remember, if he had paid 10,000 per year for 10 years, the total amount would have been 1 lakh rupees. 1 lakh is the amount of loan. Here each year he is paying more than 10,000. He is paying about 3500 more than 10,000 each year and that is because he is paying the rate of interest. There is a 6 percent rate of interest. So, this additional 3574 is due to the rate of interest component.

If the rate of interest had been higher, suppose instead of 6 percent the rate of interest is 10 percent, then this 3574 would have gone up. Each year he would be required to pay more if the rate of interest had been higher. So, this is one practical application which we often encounter in finance and economics of geometric series. Again, we are going to stop here because of lack of time, but we shall carry on with this series of tutorials in the next lecture as well. So, I hope to see you there. Thank you.