Mathematics for Economics - I
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Lecture 20
Extreme, stationary points, first derivative test

Hello and welcome. So, we are going to start with a new topic, a new module today of this course, all Mathematics for Economics Part I.

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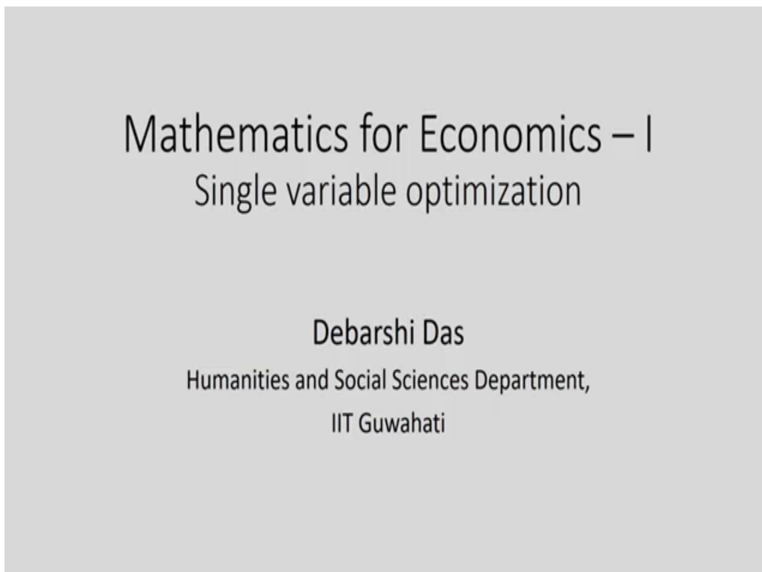

The name of this module is Single Variable Optimization. So, before I set out to explore this particular module in the previous modules that is up till now, we have covered different properties of differentiation and we have talked about applications of differentiation and now what we are going to do is that, we are going to apply the tools of differentiation in a particular context, a very important context which is optimizing a function.

Optimizing could mean finding the maximum value of the function or it could be the minimum value of the function. And we are going to look at how we can do that by using the tools of differentiation in this particular module. So, this is the first slide that you can see on your screen. The topic is single variable optimization.

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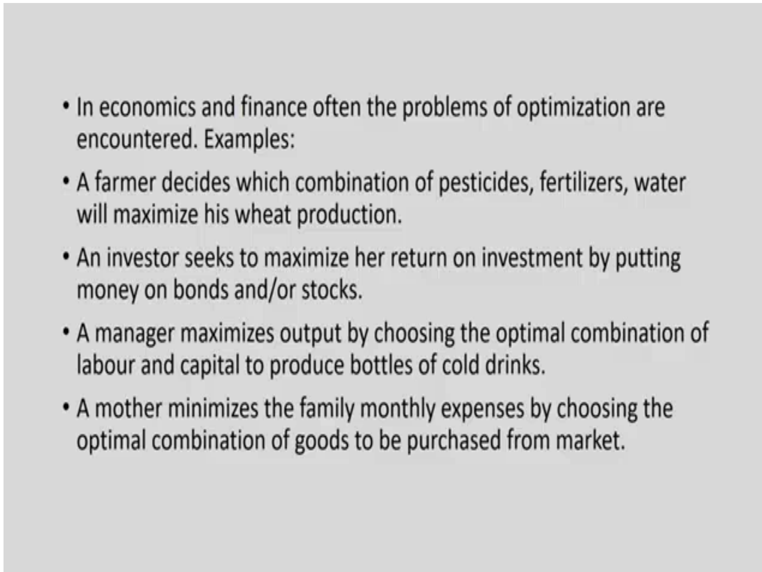

So, what are we talking about in this particular topic? In economics and finance, often problems of optimization are encountered. And here are some examples that I have given. A farmer decides which combination of pesticides, fertilizers, and water will maximize his wheat production. So, here the farmer is encountering a problem of optimization. He wants to maximize wheat production.

And wheat in this particular example is the output of many inputs. These inputs could be pesticides, fertilizer, water, it could be additionally labor and other inputs also. Now, the question is, what is the amount of use of these different inputs that will maximize the wheat production which he wants to do. So, this is one example.

The second example is from the area of finance. An investor seeks to maximize her return on investment by putting money on bonds and/or stocks. So, an investor has some amount of money that he can, in this case she, can invest on either she can buy some bonds. Bonds will give her a fixed return, a fixed amount of return, but it is guaranteed.

On the other hand, she can put the money in buying stocks. Stocks means shares. In case of shares, the return is not certain. It can go up and it can as well go down. So, it can be very high also and it could be very low also. So, it is a risky asset. So, how should the investor allocate her funds into different, these sorts of assets so as to maximize her return? So, this is another problem of optimization.

And then we come to the third example. A manager maximizes output by choosing an optimal combination of labor and capital to produce bottles of cold drinks. So, this is from the area of, let us say, industrial production. So, here the production one is talking about is production of bottles of cold drinks.

Now, cold drinks could be produced by using labor and capital. So, the manager wants to maximize output, let us suppose, then what is the amount of labor and what is the amount of capital that he can use to maximize the output. Here it is possible that he has a fixed amount of money so he cannot spend more than that. So, what is the combination and what are the amounts of labor and capital that he needs to apply.

And now, fourthly, I cite an example of minimization. A mother minimizes the family monthly expenses by choosing the optimal combination of goods to be purchased from the market. So, a mother knows that she has to run a household and she has to buy a certain minimum amount of goods, so as not to hamper the calorie intake or the nutrient intake of the family members. At the same time, she wants to reduce the monthly expenses as much as possible. So, how does she do that? So, this is another example of optimization.

So, as you can see in various works of life, it is extremely common to find economic decision makers making decisions, whereby they want to maximize certain things or maybe they want to minimize certain things. So, in this particular module, we are going to look at how we can analyze those problems through the tools that we have learned and maybe you shall learn something more in this particular module.

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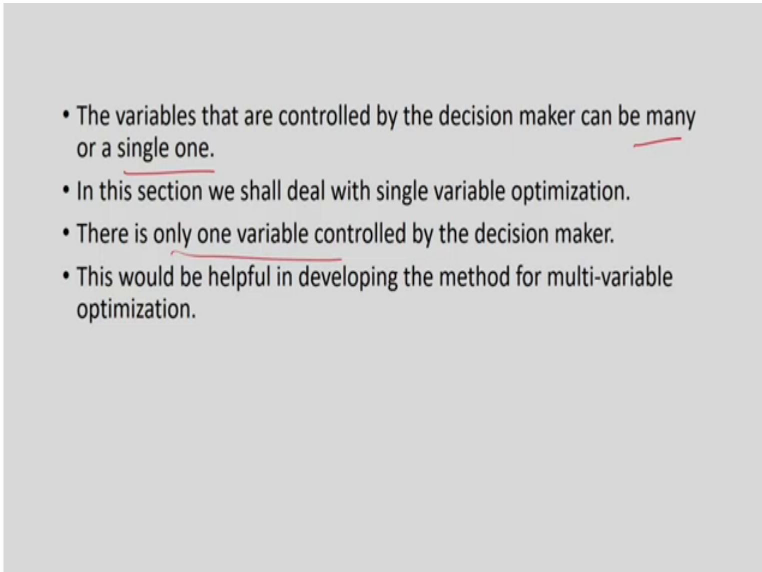

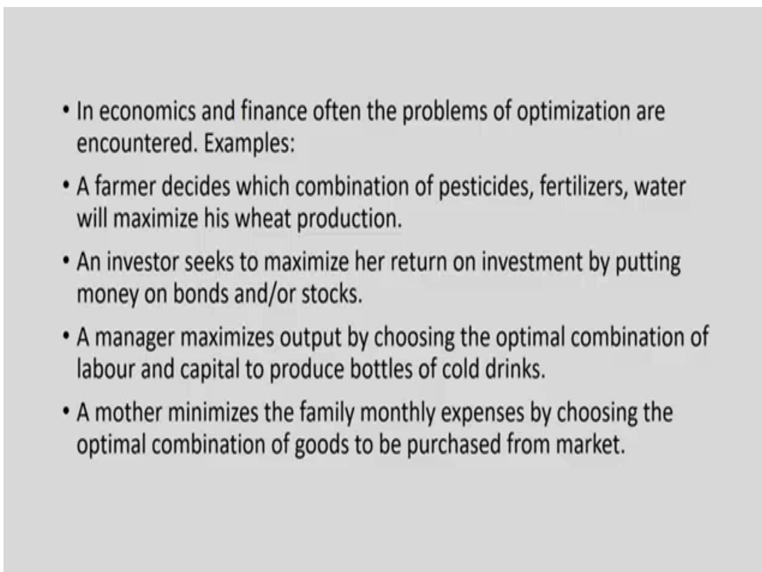

The variables that are controlled by the decision maker can be many, multiple variables might be involved. So, in the examples that I have just cited, all these examples are cases where the decision maker controls many variables. But it is also possible that the decision maker controls a single variable. And that is what we are going to study in this particular module. Later on, in the second part of this of this course, second part means in the part two, one plans to take up the case of multiple variable optimization.

In this section, we shall deal with single variable optimization. So, there is only one variable controlled by the decision maker. But it might seem a little bit limiting in our scope, but this particular section or module will be extremely helpful for you because it will give us valuable insights of how to do general optimization, that is how to do multivariable optimization. So, this would be helpful in developing the method for multiple or multivariable optimization.

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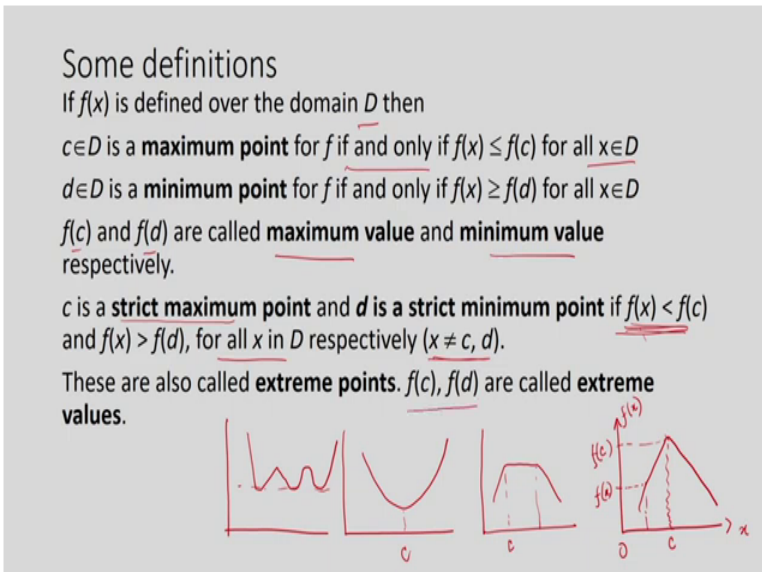

So, we start out with certain definitions, basic definitions. Some of these we have seen before. This is just to recall. If $f(x)$ is defined over the domain D then what is the maximum point $c \in D$ is a maximum point for f if and only if $f(x) \leq f(c)$. In that case, we say that small c is a maximum point. So, we can think about this. This is a maximum point. This is small c. And here you have the independent variable and here we have the dependent variable.

And here you can see, obviously, for any take, any small c in the domain, the value of the function that is $f(x)$ is less than the value $f(c)$. So, this is $f(c)$. So, $f(x) \leq f(c)$. Here in this particular case, it is strict inequality, but one can have a weak inequality as well. So, think about this. Here you will have weak inequality. So, there are multiple maximum points here. So, all these are maximum points. Therefore, you have a plateau kind of thing. Now, this was the case of maximum point.

And similarly, there is an idea of minimum points. Small $d \in D$ is a minimum point for f if and only if $f(x) \geq f(d)$ for all small $x \in D$. So, here the shapes will be something like this. So, here you have the minimum point or it could be a weak minimum, not a strict minimum. So, you have something like this. All these are at same value. So, therefore, you have multiple minimum points. So, these are not strict inequalities.

So, $f(c)$ and $f(d)$, remember, c and d are maximum and minimum points. So, $f(c)$ is the value of the function at the maximum point, $f(d)$ is the value of the function at the minimum point. These are called in short maximum value and minimum value, respectively. These are cases of maximum point and minimum point then we encounter the definition of strict maximum point.

Small c is a strict maximum point and small d is a strict minimum point if you have a strict inequality like $f(x) < f(c)$ and $f(x) > f(d)$ for all x in the domain D, respectively. Now, here if we are talking about strict maximum and strict minimum, obviously, the x that we are considering which satisfy these conditions, $(x \neq c, d)$, cannot be those points. So, that is why I have this not equal sign, because if they are the same points, if $x = c$, then obviously, $f(x) = f(c)$, but then this condition will not be satisfied.

So, therefore, I need to take, for comparison I need to take x which is not equal to c and then this condition has to be satisfied, the strict inequality has to be satisfied. These are also called extreme points. So, in short, if we bunch them together, maximum points and minimum points, they are called extreme points. Correspondingly, $f(c)$ and $f(d)$ they are called extreme values.

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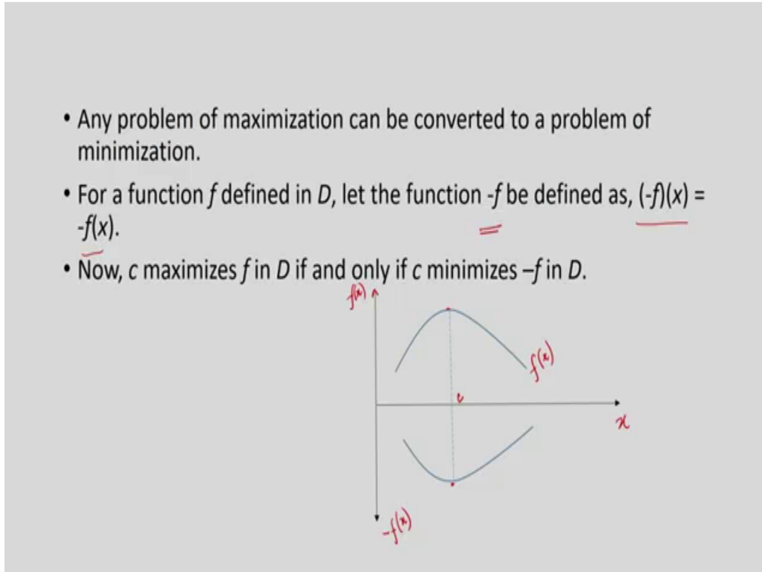

Now, there is a correspondence between maximization problem and minimization problem. Any problem of maximization can be converted to a problem of minimization. For a function defined in the domain capital D, let us define another function which is called $-f$. So, f is a function. Now, we are defining another function which is $-f$. How are we defining it? $(-f)(x) =- f(x)$. So, this particular function $-f$ is defined in such a manner that it is taking the value of the function $f(x)$ and just, we are putting a minus sign before that.

Now, if this is how the function is defined, c maximizes f in D if and only if c minimizes $-f$ in D. So, this is how we are going to establish the relationship between maximization and minimization. We are saying that we are defining a new function which is $-f$ and then we are saying that let us suppose we are trying to maximize f, then if we are maximizing f at c, it must be the case that c is minimizing $-f$ in d.

And here is the geometric depiction of that. So, here is you have this x variable along this axis and this is $f(x)$ and here is suppose c and this is $f(x)$. So, this is $-f(x)$. So, you have the mirror image of this graph in the fourth quadrant. And after drawing the mirror image, we can straight away see that at this point particular point small c, f is maximized, but that itself means that $-f$ is minimized. So, this is the point of minimization of this function and this is the point of maximization of this function. So, that is how they are related. So, in other words, if we are maximizing a function then at the same time we are minimizing the minus of that function.

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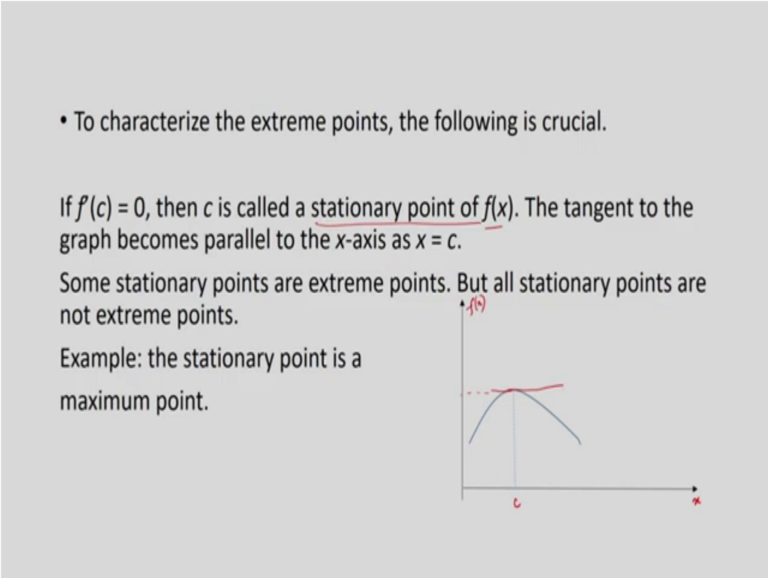

Now, remember, we want to characterize the extreme points, the points where the maximum value or the minimum value is attained. Now, to do that the following property is important. If $f'(c) = 0$ then c is called a stationary point of $f(x)$. I think I have defined a stationary point in some previous discussion. So, I am just repeating that particular definition here. Now, if I have this function f such that at point c, $f'(c) = 0$ What does it mean?

It basically means that at this particular point c the tangent to the graph is parallel to the x-axis and this is very obvious because the derivative of a function at a particular point, that is nothing but the slope of the tangent. Now, if the slope of the tangent is equal to 0 that means that the tangent is horizontal, which means that the tangent is parallel to the x-axis. So, this is how we can geometrically represent this. So, at the stationary point the tangent becomes parallel to the x-axis.

Some stationary points are extreme points. Here you have an example where the stationary point, suppose here is the stationary point, now, here the stationary point is actually an extreme point. It is getting a maximum value here. But all stationary points are not extreme points. Only some stationary points will give you extreme points, but not all. So, here is an example where it is giving you an extreme point.

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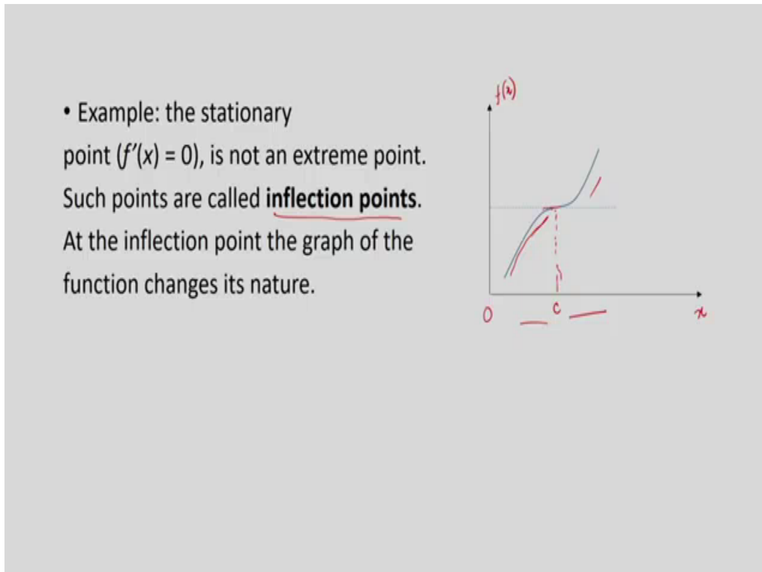

Here is an example where you have a stationary point but it is not an extreme point. So, here you have, suppose this is point c and at this point c the graph is more or less parallel to the x-axis. The tangent, if you draw a tangent through this point the slope of the graph at this point is 0. So, this kind of point where the slope is equal to 0 but the function is not getting a minimum or maximum these points are called inflection points.

Now, the interesting thing about the inflection points is that at the inflection point the graph of the function changes is its nature. So, here you can see that happening. Before c is reached, that is to the left of c, the function is like this. It is kind of dome-like shape. It is rising. But then it is sort of tapering off as it is going up. But to the right of c the function is rising, but it is rising at an increasing rate. It is getting more and more steep to the right. So, that is why I am saying that the inflection points are such that at the particular inflection point the nature of the graph of the function changes quite a lot.

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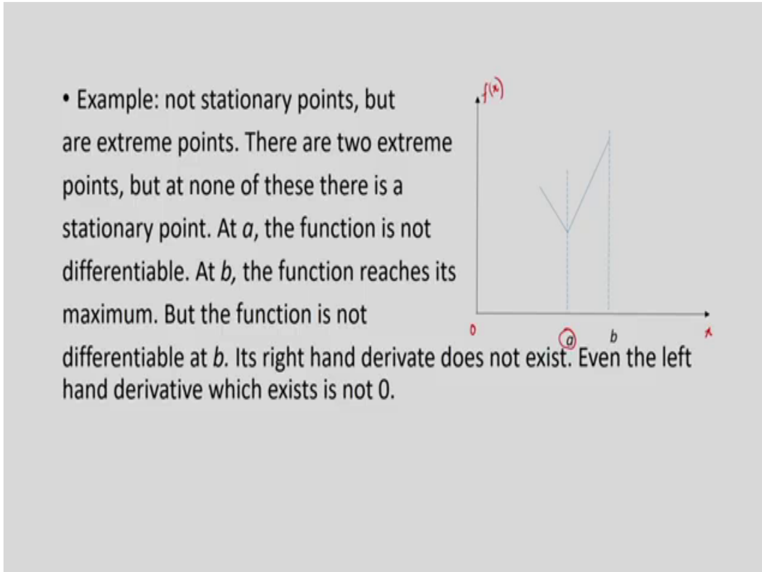

So, we are going to talk more about that in some, later lecture. Now, here is another example, where you do not have stationary points, but these are extreme points. So, in this example there are two extreme points. So, here is one extreme point which is a. a is the point of minimum in this case. It is this point where the function is getting the lowest value. And b is the maximum point, because here the highest value of the function is reached, but neither a nor b is a stationary point.

At a the function is not differentiable. As you can see there is a kink point here. And in the previous module we have just discussed that if you have a kind of kink point then the function is not differentiable there. The left hand derivative and the right hand derivative, they are not equal. So, at a it is not differentiable. So, if it is not differentiable then you cannot talk about $f^{'}(x) = 0$.

b is also a point of extreme value. Here the function is reaching its maximum. But at b the function is again not differentiable. So, here the problem is more severe. The right hand derivative does not even exist because the domain is ending at b. So, you do not get to define what is the right hand derivative. You cannot move to the right of b. The left hand derivative can be defined. It does exist. But it is not equal to 0.

So, remember, what is the idea of differentiability, at that point where the function is supposed to be differentiable, the left hand derivative, and the right hand derivatives must be equal and in this case we wanted the left hand derivative to be equal to 0 only then it could have been a stationary point. But the problem here is, although the left hand derivative exists it is not equal to 0. So, here is the third example where you do not have stationary points, but still there are extreme points.

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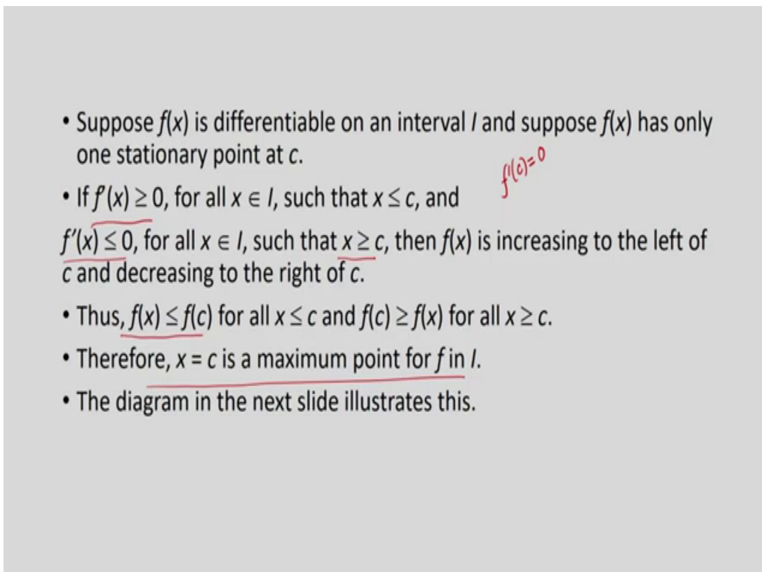

Now, suppose $f(x)$ is differentiable, so the problem of differentiability is not there, suppose on an interval I and suppose $f(x)$ has only one stationary point at small c. So, just one stationary point is there. So, that basically means that $f^{'}(c) = 0$. Now, if $f^{'}(x) \geq 0$ for all $x \in I$ that is that interval, such that $x \leq c$ and $f^{'}(x) \leq 0$ for all x in the intervals capital I such that $x \geq c$, then $f(x)$ is increasing to the left of c and decreasing to the right of c.

So, just imagine that before c is reached this is satisfied. Now, if this is satisfied then the function is an increasing function to the left of c. And similarly, if you are talking about $x \geq c$ then this is satisfied, $f^{'}(x) \leq 0$, which means that the function is decreasing. Thus, $f(x) \leq f(c)$ for all $x \leq c$, because the function was increasing there and $f(c) \geq f(x)$ for all $x \geq c$, because the function is decreasing there.

So, therefore, $x = c$ is a maximum point for f in I. So, this is important. So, if these conditions are satisfied, then actually you can get a maximum point at c. So, this is a kind of test for finding a maximum point.

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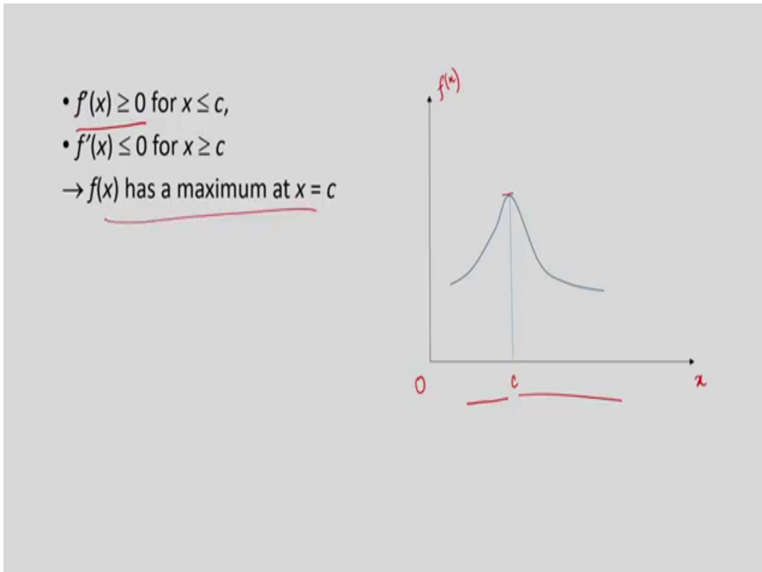

And here is the diagrammatic example. So, here is one illustration. So, here is c and at c we know that there is a stationary point that is $f^{'}(c) = 0$. The function is such that the tangent at that point is parallel to the x-axis. And if you take points to the left of c, this is satisfied, $f^{'}(x) \geq 0$. In this case, actually it is strictly greater than 0. Or if you are taking points to the right of c then $f^{'}(x) \leq 0$. In that case, we are getting a maximum at $x = c$. So, this could be one way to find out the maximum.

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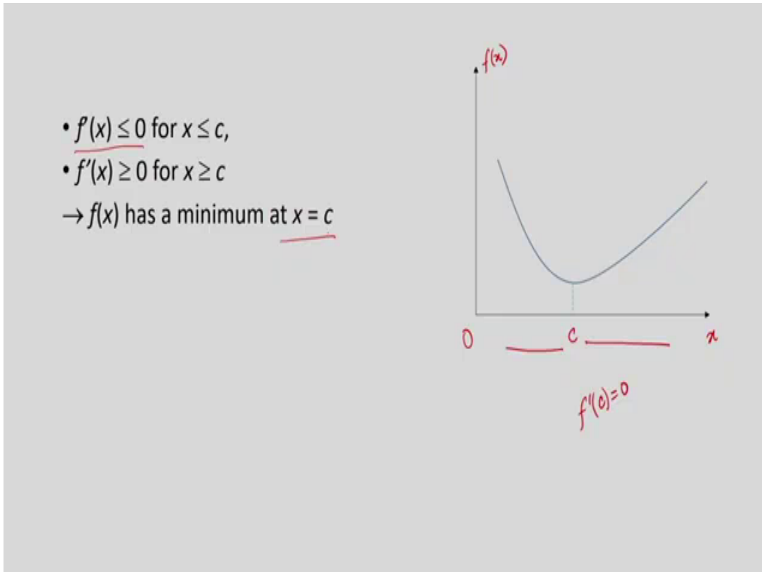

Similarly, just in the **opposite manner, we can talk about** the minimum also. So, $f(x)$ is this graph, has this graph, this blue line, at the point small c, $f^{'}(c) = 0$. So, the tangent is parallel to the horizontal axis. To the left of c you have $f^{'}(x) \leq 0$ the function is decreasing and to the right of c, $f^{'}(x) \geq 0$ that is, it is an increasing function. That means that the function must have reached a minimum at $x = c$.

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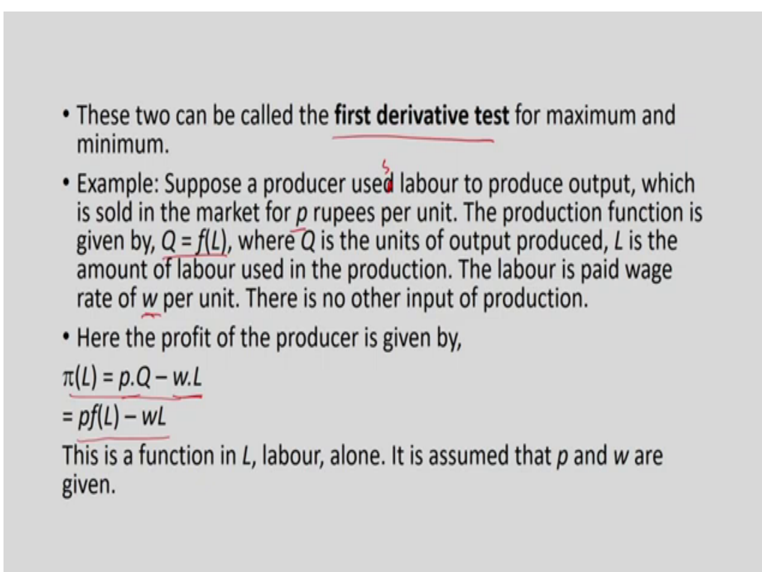

These two can be called the first derivative test for maximum and minimum. So, as I was telling you, these criteria can be put together to find out whether a function has a maximum or a minimum at a particular point, in particular, at the stationary point. So, this is called a first derivative test, because why is it called the first derivative test, because we are depending on the first derivative.

At the point of interest that is extreme point the first derivative is equal to 0 and then we are looking at to the left of that what is the value of the derivative, first derivative, is it positive or is it negative, and depending on the pattern of the derivatives we are reaching a conclusion whether you have an extreme point and what kind of extreme point is that.

Here is an example from economics. Suppose the producer uses labor labor to produce output which is sold in the market for p rupees per unit. The production function is given by $Q = f(L)$. So, f is the production function. It is a function of labor, where capital Q is the units of output produced, L is the amount of labor used in the production. The labor is paid at a wage rate of w per unit. There is no other input of production. So, this is the setting.

The producer is producing output for the production function. The production function is a function of only one variable which is labor. And after producing he sells the goods in the market at p rupees per unit and he has to pay the laborers and so w is the wage rate per unit. Here the profit of the producer is given by this by π. Let us suppose the profit function, it is a function of labor $\pi(L) = p.Q - w.L$. So, this is revenue minus cost, $p.Q - w.L$. And we know $Q = f(L)$. So, we are substituting that here, $p.f(L) - w.L$. So, it is clearly a function of L only, because p and w are given to us. They are parameters.

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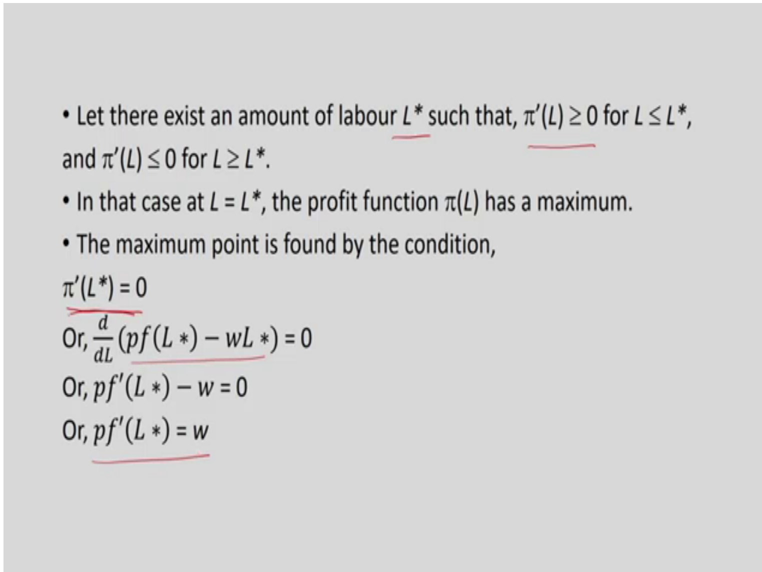

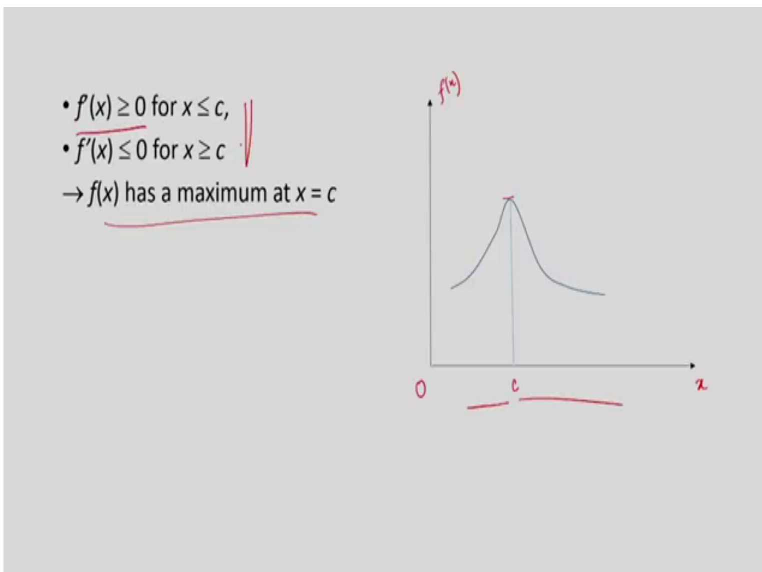

Now, suppose, there exists an amount of labor L^* such that $\pi'(L) \geq 0$ for $L \leq L^*$ and $\pi'(L) \leq 0$ for $L \geq L^*$. So, this is the condition that we have seen before also, I mean, when we are talking about the first derivative test. This is the condition, similar condition.

Now, in that case at $L = L^*$ the profit function has a maximum. And how can we characterize this particular L. We know that at that point you have a stationary point and the stationary point is having a property like this $\pi'(L^*) = 0$. And then we substitute the profit function which we have found already and we say that the first derivative of this profit function is equal to 0 at $L = L^*$ and thus we are obtaining this condition $p.f'(L^*) = w$.

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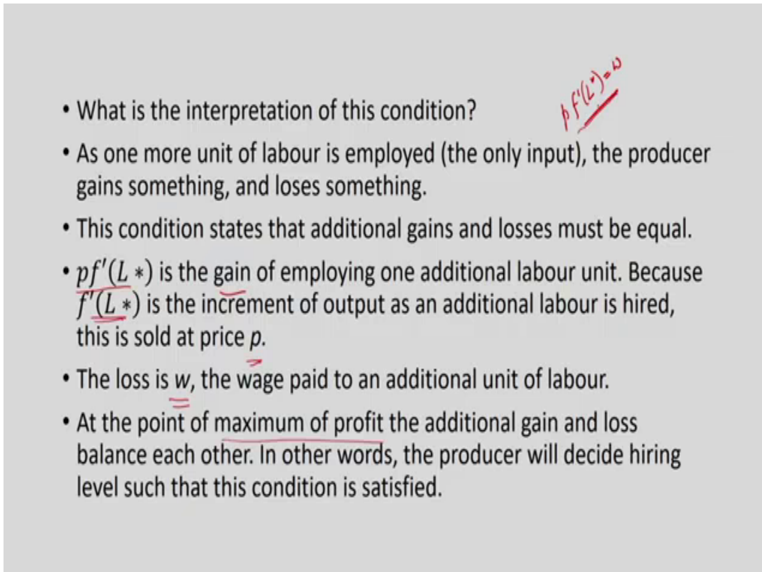

What is the interpretation of this condition? This condition means $p.f'(L^*) = w$. This is the condition. As one more unit of labor is employed, labor is the only input, the producer gains something and loses something. This condition states that additional gains and losses must be equal. So, how can we say that the condition is indeed stating that the additional gains and losses must balance each other?

The left hand side term that is $p.f'(L^*)$ is the gain of employing one additional labor unit, because $f'(L^*)$ is the increment of output as an additional labor is hired. As the employer is employing one more unit of output, remember, we said before when we talked about the definition of derivatives that $f'(L)$, let us suppose, can be interpreted as the change in the output as labor input changes by 1 unit. It is not precisely equal to that, but roughly equal to that.

And this $f'(L^*)$, this is the change in the output, an increment in the output let us suppose, if this is positive, then it will be sold in the market. And what is the price? It will be sold at p per unit.

So, $p.f^{'}(L^{*})$ is that additional money that the employer gets by employing one more labor input. So, that is why I am saying this is the additional gain.

On the other hand, if one more unit of labor is employed, there is a loss, and that loss is given by w. What is this w? W is the wage paid to the additional labor unit. You are employing one more unit of labor and the cost of labor is w. So, the cost is w. So, you have to deduct that. At the point of maximum profit, the additional gain and loss balance each other. In other words, the producer will decide the hiring level such that this condition is satisfied, this condition.

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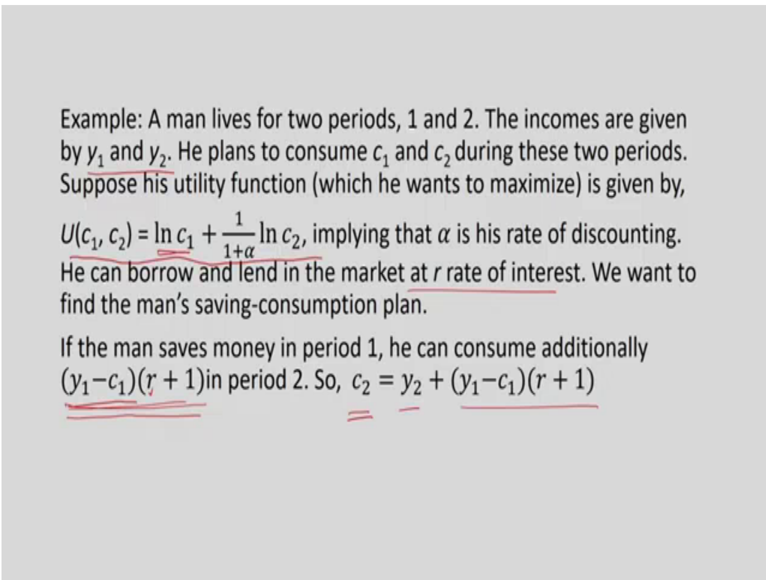

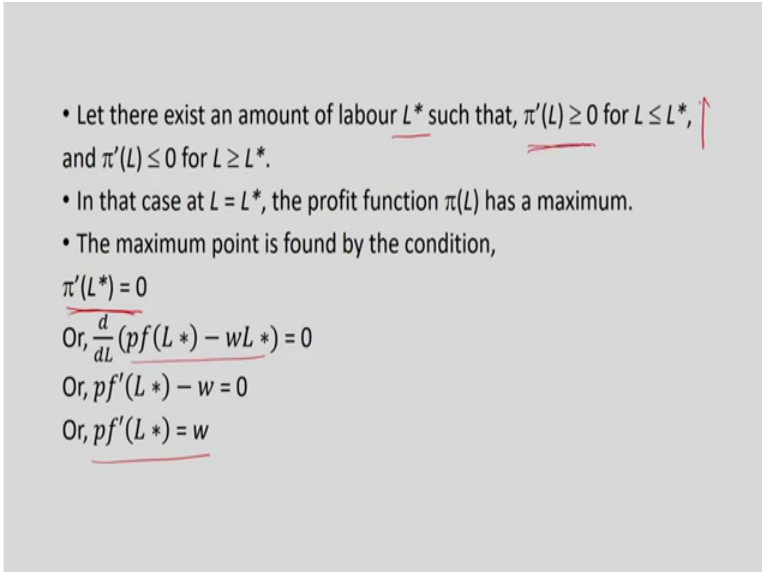

Here is another example from economics. So, here we have shown that if this condition is satisfied, the first derivative condition that before this L^* f' function is a function is giving us positive value or weakly positive value and after this L^* the f' is giving us weakly negative values then at L^* the maximum is reached and the maximum can be found out by this method.

Now, in this another example, the problem is somewhat different. A man leaves for two periods 1 and 2. The incomes are given by y_1 and y_2. He plans to consume c_1 and c_2 during these two periods. Suppose his utility function which he wants to maximize is given by this. So, $U(c_1, c_2) = lnc_1 + \frac{1}{1+\alpha} lnc_2$. Both these logs are natural logarithms. Implying that alpha is his rate of discounting. Remember, we have talked about the rate of discounting before. This is as if c_1 is the consumption.

So, lnc_1 is his utility in this period, lnc_2 is the utility in the next period. But what is his total utility? He is taking this period's utility and he is taking the next period's utility, but as we know the future values have to be discounted. So, his rate of discounting is given by α, which is why he is multiplying the $\frac{1}{1+\alpha} lnc_2$. So, this discounting, the idea of discounting we have talked about before. So, I am not going to spend time on that.

He can borrow and lend at the rate of interest r in the market. So, r is the rate of interest which is there in the market, prevailing in the market. At this r rate of interest he can lend money. He will get the rate of interest r or he can borrow money, in that case, he will have to pay this rate of interest. We want to find the man's saving-consumption plan.

That means, if these are the incomes y_1 and y_2 and if this is the utility function, r is the rate of interest, then how much will he consume in the first period and how much will he consume in the next period, what is going to be the optimal c_1 and c_2 that is the thing that he is controlling here. And additionally, can we say in which conditions he will borrow money or in which conditions he will lend money.

Remember, borrowing will mean then that in the first period his consumption, optimal consumption that is c_1 is greater than y_1. He can afford c_1 in the first period which is greater than y_1 only if he borrows. On the other hand, he will be lending money if his consumption in this period that is the first period is less than the income in the first period, in that case the amount of money that he has saved he will lend it out and on that money that he has lent out he will earn some rate of interest and that rate of interest he will learn in the next period. So, in the next period in that case his total income will be something more than y_2. It will be something extra, because he has earned some rate of interest.

So, we also want to find out suppose that under what conditions he will lend money and under what conditions he will borrow money. If the man saves money in the first period that is period 1, he can consume additionally $(y_1 - c_1)(r + 1)$ in period 2. This is not difficult to understand. $(y_1 - c_1)$ is his savings in this period which he will maybe save, he will lend out on which he will get this rate of interest. So, the total amount of money that he will be getting back in period 2 will be this much, $(y_1 - c_1)(r + 1)$. And obviously, he will spend that money, because there is no third period, remember, so he leaves for only two periods.

So, therefore, consumption in the second period will be the income of the second period plus the money that he has got back which is $(y_1 - c_1)(r + 1)$, i.e. $c_2 = y_2 + (y_1 - c_1)(r + 1)$. So, this is the case where he is saving money. What happens if he is borrowing money in period 1?

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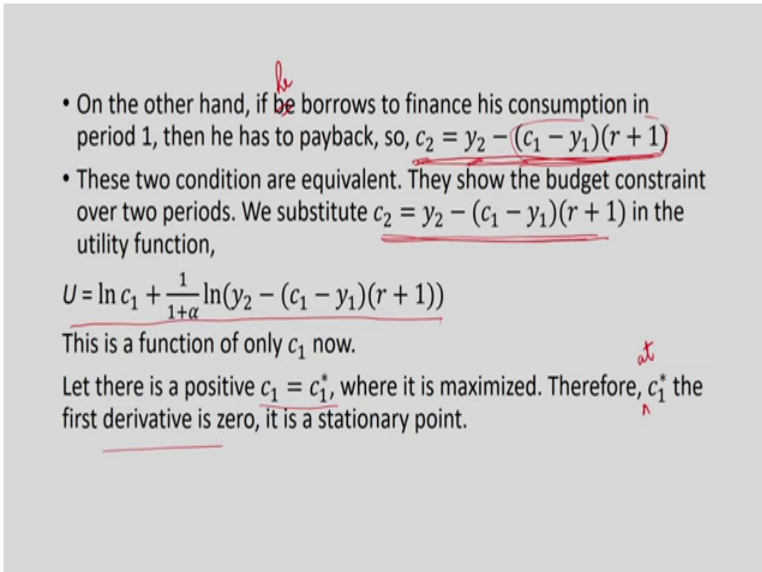

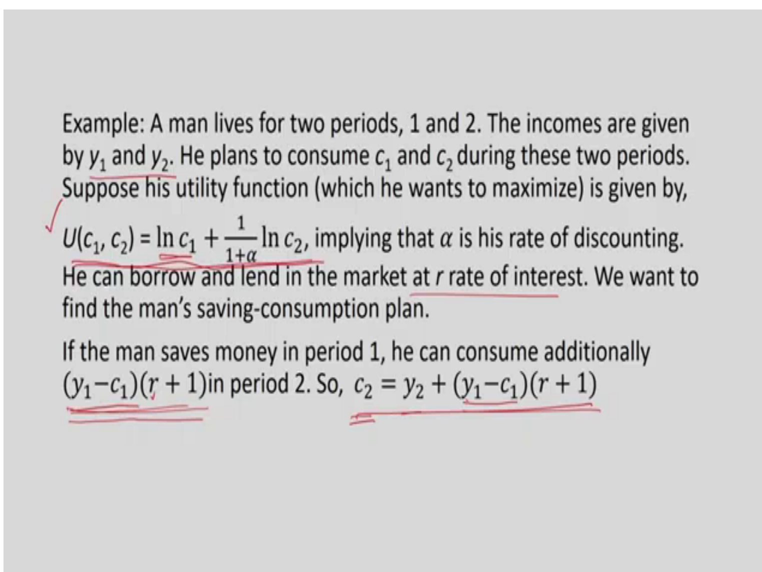

On the other hand, if he borrows to finance his consumption in period 1, then he has to pay back the loan. Therefore, in that case, in period 2, the consumption will have to be less than his income in period 2. So, you have this, $c_2 = y_2 - (c_1 - y_1)(r + 1)$. And what is the amount of this minus term?

Remember, in period 1 he has consumed c_1, whereas his income was y_1. So, this $(c_1 - y_1)$ was the amount of borrowing that he did in period 1, on that, there is this rate of interest. So, this $(c_1 - y_1)(r + 1)$is the total amount of money that he will have to pay back in period 2, . So, that has to be subtracted from his income. So, therefore, this is the consumption that he can afford in period 2.

Now, interestingly this condition here is the same as this condition. There is no difference. You can just take the minus sign out and it will become the other condition. Minus sign means from here you can take the minus sign out then it will become $(c_1 - y_1)$ and that is what is there in this condition. So, these two conditions are the same. And what do these conditions tell us? They show the budget constant over the two periods. So, we can use either of these two conditions.

We have talked about budget constant in a different context before where we were talking about a static utility maximization this for only one period. But here you have two periods. There also could be budget constraints over time. Now, this budget constraint we are using, that $c_2 = y_2 - (c_1 - y_1)(r + 1)$. This is the budget constant we will be using. And what is the thing that he wants to maximize, he wants to maximize the utility function, this. So, this is his optimization problem. He wants to maximize this.

So, here we will just substitute c_2, because this is a function of two variables c_1 and c_2, but we can get rid of c_2 by using this budget constraint, $c_2 = y_2 - (c_1 - y_1)(r + 1)$ and that is what we have done here. So, we have done the substitution. And you can see it is a function of $c_1$only, $U = lnc_1 + \frac{1}{1+\alpha} ln(y_2 - (c_1 - y_1)(r + 1))$. Now, let us try to see whether you can solve this.

Let there be a positive $c_1 = c_1^*$ where it is maximized. So, this is just an assumption. Therefore, at c_1^*, the first derivative is 0, because we are assuming that it is an interior point that is it is not at the border and the function is obviously differentiable. So, we can very well say if it is a maximum then the first derivative has to be equal to 0. So, that basically means it is a stationary point.

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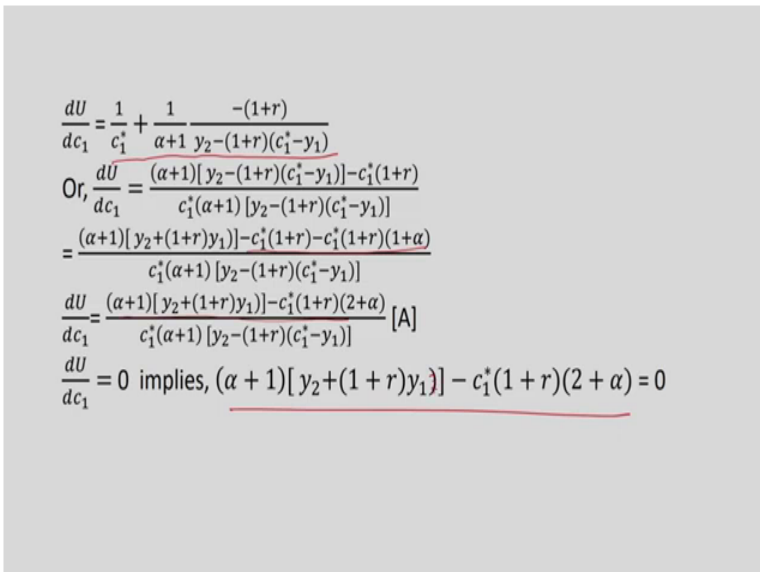

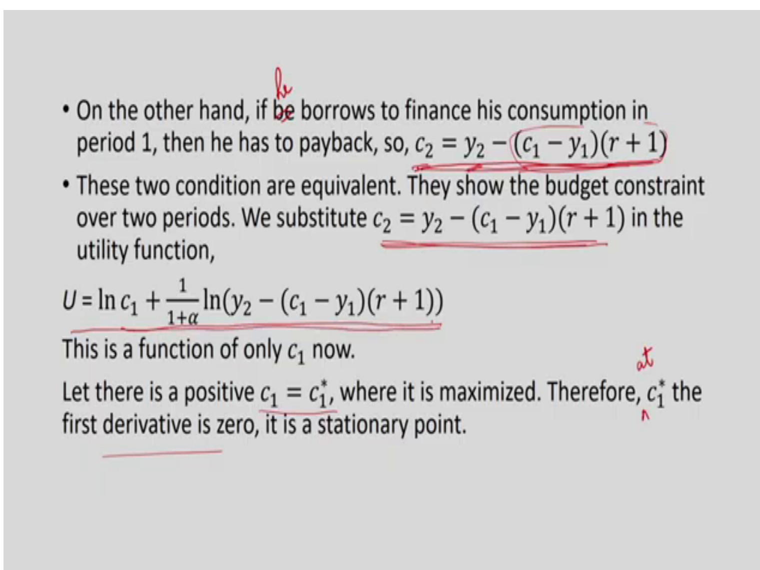

So, I have taken the first derivative and on the right hand side I have written what we get if we take the derivative of this function and I am assuming that we are evaluating this function at the point of maximum which is c_1^*. So, every c_1 has been replaced by c_1^*. So, the next, in the next step I have just taken these two things in the denominator. And it looks like a complicated expression, but it is not. And you see I have put together the terms with c_1^* and in the first term there is no c_1^*.

And in the next stage, I have written these terms by simplifying this by taking c_1^* common from the second and third terms. So, it becomes $\frac{dU}{dc_1} = \frac{(\alpha+1)[y_2+(1+r)y_1]-c_1^*(1+r)(2+\alpha)}{c_1^*(\alpha+1)[y_2-(1+r)(c_1^*-y_1)}$ $[A]$.

Now, we know that this is a stationary point that c_1^* is a stationary point, so this has to be equal to 0, which means that the numerator has to be equal to 0. So, this is what it means, $(\alpha + 1)[y_2 + (1 + r)y_1] - c_1^*(1 + r)(2 + \alpha) = 0$. Here is an additional bracket which should not have been there.

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Or, $c_1^* = \frac{(\alpha+1)[\,y_2+(1+r)y_1)]}{(1+r)(2+\alpha)}$

This is the optimal consumption in period 1.

Two observations:

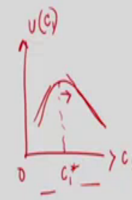
1. If $c_1 > c_1^*$, then $c_1 > \frac{(\alpha+1)[\,y_2+(1+r)y_1)]}{(1+r)(2+\alpha)}$

Or, $(\alpha+1)[\,y_2+(1+r)y_1)] - c_1\,(1+r)(2+\alpha) < 0$

From [A] above, we see, $\frac{dU}{dc_1} < 0$

Similarly, if $c_1 < c_1^*$, $\frac{dU}{dc_1} > 0$

So, c_1^* indeed maximizes the utility function, which was our initial assumption.



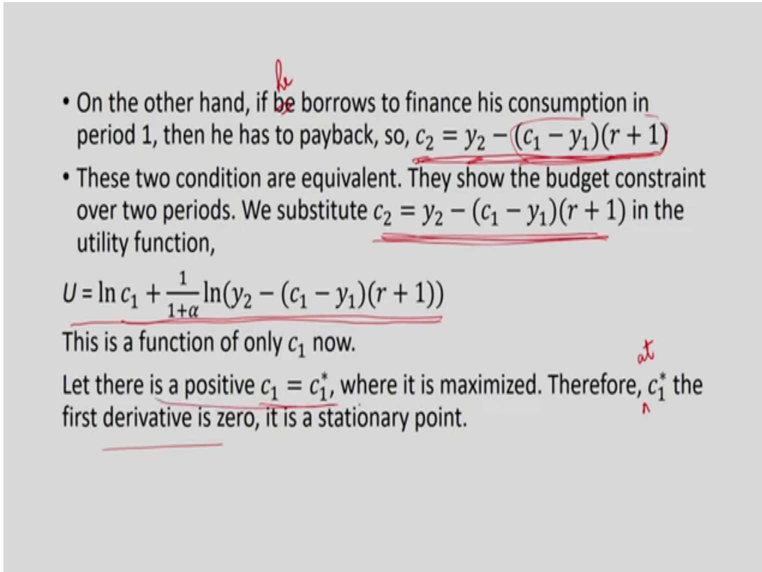
$$\frac{dU}{dc_1} = \frac{1}{c_1^*} + \frac{1}{\alpha+1}\frac{-(1+r)}{y_2-(1+r)(c_1^*-y_1)}$$

$$\text{Or, } \frac{dU}{dc_1} = \frac{(\alpha+1)[\,y_2-(1+r)(c_1^*-y_1)]-c_1^*(1+r)}{c_1^*(\alpha+1)\,[y_2-(1+r)(c_1^*-y_1)]}$$

$$= \frac{(\alpha+1)[\,y_2+(1+r)y_1)]-c_1^*(1+r)-c_1^*(1+r)(1+\alpha)}{c_1^*(\alpha+1)\,[y_2-(1+r)(c_1^*-y_1)]}$$

$$\frac{dU}{dc_1} = \frac{(\alpha+1)[\,y_2+(1+r)y_1)]-c_1^*(1+r)(2+\alpha)}{c_1^*(\alpha+1)\,[y_2-(1+r)(c_1^*-y_1)]} \text{ [A]}$$

$$\frac{dU}{dc_1} = 0 \text{ implies, } (\alpha+1)[\,y_2+(1+r)y_1)] - c_1^*(1+r)(2+\alpha) = 0$$



And then we simplify this condition and we solve for c_1^*. And c_1^* becomes this expression, $c_1^* = \frac{(\alpha+1)[y_2+(1+r)y_1]}{(1+r)(2+\alpha)}$. This is the optimal consumption in period 1. Now, here we have found the optimal consumption. If c_1^* is given to us, we can straightaway find the c_2^* also from this. I just have to substitute c_1^* here and c_2^* that is consumption in period 2 can be found out. So, c_1^* and c_2^* have been found out.

Now, there are two observations. Number one, if $c_1 > c_1^*$, then what happens? Then I am just using what is c_1^* here. I am substituting that on the right hand side, $c_1 > \frac{(\alpha+1)[y_2+(1+r)y_1]}{(1+r)(2+\alpha)}$ and this boils down to this condition, $(\alpha + 1)[y_2 + (1 + r)y_1] - c_1(1 + r)(2 + \alpha) < 0$. And just recall what is A, this is A, $\frac{dU}{dc_1} = \frac{(\alpha+1)[y_2+(1+r)y_1]-c_1^*(1+r)(2+\alpha)}{c_1^*(\alpha+1)[y_2-(1+r)(c_1^*-y_1)}$ $[A]$, which is the derivative. Derivative of the utility function with respect to c_1. And look at the numerator here. It is the numerator that we are getting here and this is negative.

So, basically, what is happening is that, if you cross that c_1^* then the derivative of the utility function with respect to c_1 is negative. So, it is like this. Here is the c_1^*. So, this is c_1 and this is your utility function which is a function of c_1. So, at this point, c_1^* is there. If you are crossing, then the slope is negative. This is what you are getting.

Similarly, if you take the opposite, that is $c_1 < c_1^*$ on the left hand side, you will find that the slope of the utility function is positive. So, actually c_1^* is indeed maximizing the utility function. This we had assumed earlier. Remember, we had assumed that there is a positive $c_1 = c_1^*$ where the utility is maximized. Now, actually, we have found that indeed the assumption is valid that we have found a $c_1 = c_1^*$ which is positive, where the maximum point is there.

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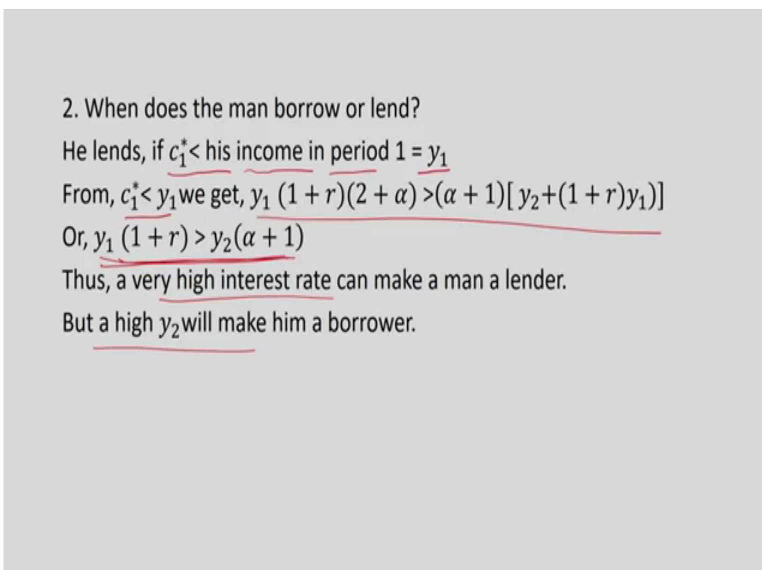

The second observation is this. This is just trying to answer the first question we had started with. When does the man borrow or lend? Now, he lends if his optimal consumption which is c_1^* is less than his income in the first period. What is his income in the first period, it is y_1. Now, this condition is therefore $c_1^* < y_1$. So, I am just elaborating that here. And from this we are going to get this condition. Or if we simplify this, it becomes easier thing to handle. It is $y_1(1 + r) > y_2(\alpha + 1)$. So, under this condition, very simple condition, he will lend.

Now, let us try to get the intuition of this. If there is a high rate of interest that is r is very high, so in that case the left hand side becomes high that is $y_1(1 + r)$ becomes high. In that case there is a greater chance that this condition will be satisfied. So, basically, a high rate of interest makes a man prone towards saving money. And the intuition is very simple that if the rate of interest is high then you want to save money so that you can take the advantage of that high rate of interest.

So, in that case you will consume less in the first period and save more so that you can afford more consumption in the second period. So, that is why rising r will make you a lender that is saver and a lender.

On the other hand, if y_2 is high, then the intuition is that that if there is a greater income that has to be obtained in the next period then you will rather be a borrower, so there is so much income to be had in the next period that, to take the advantage of that, you will borrow money in this period, consume a lot and pay back that money in the second period, because you can afford to do that in the second period your income that is y_2 is very high. So, we shall stop here and call it a day. I will carry forward this discussion of optimization, single variable optimization in the next lecture. Thank you.