Mathematics for Economics - I

Professor Debarshi Das

Humanities and Social Sciences Department,

Indian Institute of Technology Guwahati

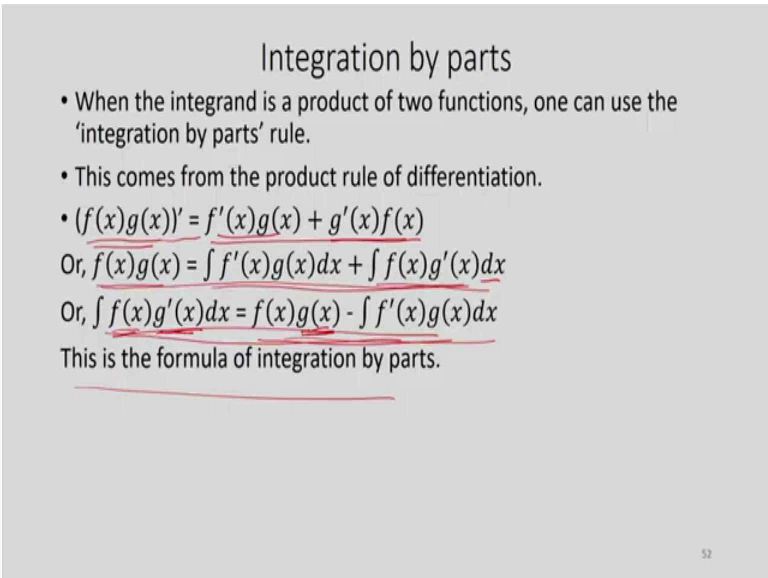
Module: Integration 2

Lecture 28

Integration by parts, substitution, Lorenz curve

Hello, and welcome to another lecture of this course Mathematics for Economics Part I. The topic that we are discussing right now is integration. So, we have covered a few themes within integration. Today we shall talk about certain methods of finding out the integral of a function.

(Refer Slide Time: 01:02)



So, this is the first thing that you can see on your screen integration by parts. So, if the integrand is a product of two functions, one can use the integration by parts rule. Now, what does this rule say? Let us look at that. This rule actually comes from the product rule of differentiation. Now, what does this product rule of differentiation say, on the left hand side you have f multiplied by g, $(f(x)g(x))'$.

So, $f(x)$ and $g(x)$ both are functions of x and here you have a dash sign, which means the product of $f(x)$ and $g(x)$ that product has been differentiated with respect to x and that as we know can be found out by using the product rule. So, on the right hand side you have that application of the product rule, $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$. So, this is the familiar rule of differentiation. We have seen this before.

Now, we are going to work a little bit on this rule. Now, if this is correct, then I can go to this step. So, what is being done here? What we have done here is I have taken the integration of both sides, $f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$. So, here there was a differentiation on the left hand side that is why the dash sign was there. If I integrate both sides, then differentiation and integration will sort of cancel each other, and therefore, you are going to get back the integrand which is $f(x)g(x)$ on the left hand side.

On the right hand side, I have not done anything except to use the rule that if you take the integration of a sum then it becomes the submission of the integrations. So, what I have here is $f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$.

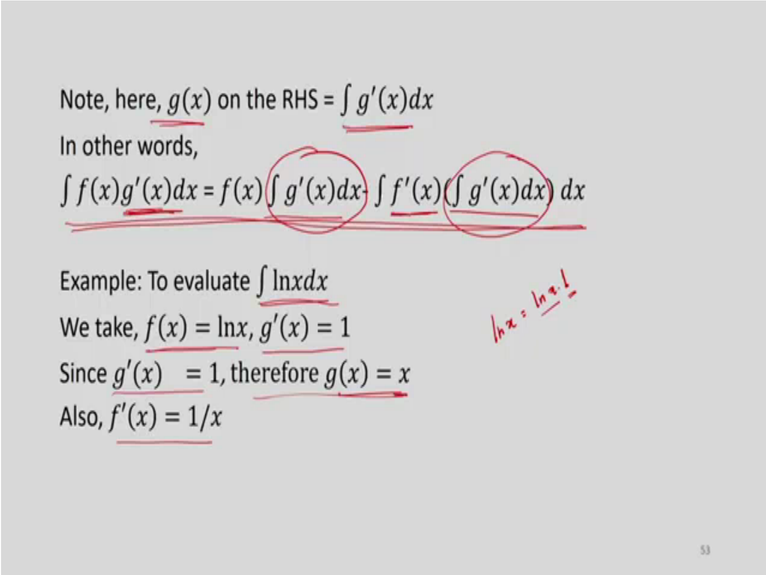
Now, this can be written in this manner, $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. So, here I have taken this term $\int f(x)g'(x)dx$ to the left hand side and that will be equal to $- f(x)g(x) + \int f'(x)g(x)dx$ and then I can multiply both sides by, with a minus sign. So, I will get this expression, $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. Now, this is the formula of integration by parts.

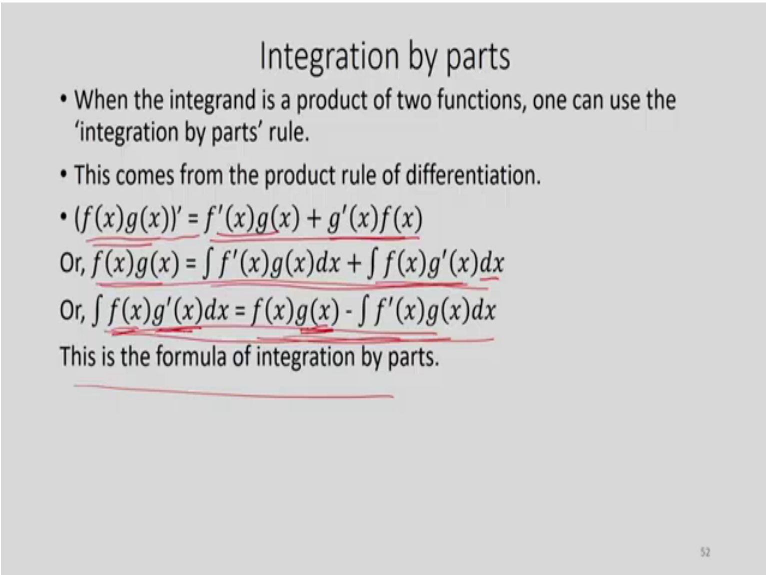
Now, on the first glance it does not seem to be saying a lot. It is just a restatement of the product rule of differentiation. But actually it is very useful as we shall see. The reason being that, look at the left hand side, you have $f(x)$ and $g'(x)$. So, it is not $g(x)$, it is $g'(x)$. Now, with respect to this $g'(x)$, on the right hand side, you do not have any $g'(x)$.

What you have is $g(x)$. $g'(x)$ is not being repeated on the right hand side. What you have is $g(x)$. And what is the relationship between $g'(x)$ and $g(x)$? If you take the integral of $g'(x)$, you will get $g(x)$. So, that is the thing that we are going to use here.

We are going to take these two functions, $f(x)$ and $g'(x)$ and then we are going to use this rule, exactly this rule. But we have to make sure that the $g(x)$ function that we write here is the anti-derivative of $g'(x)$ which is appearing on the left hand side. And that is nothing but saying that the $g(x)$function that is appearing on the right hand side is the integration of this function, which is appearing as the second function in the integrand on the left hand side.

(Refer Slide Time: 06:24)





So, that is what I have written here. Here $g(x)$ is appearing on the right hand side. That $g(x)$ is what? That $g(x)$ is the integration of the $g^{'}(x)$ and $g^{'}(x)$ is appearing on the left hand side. So, in other words, I can write the statement as follows, instead of $g(x)$, I am replacing this by this term, $\int g^{'}(x)dx$and here also I am doing the same thing. So, this is integration by parts. In particular, we are going to be using this rule a lot if we have the integrand to be a product of two functions.

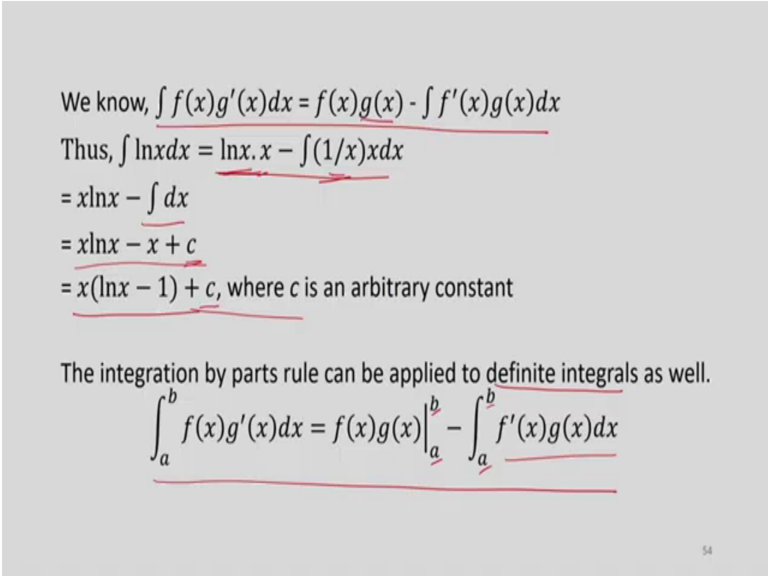
So, here straight away we go to an example, evaluate $\int lnxdx$. Just to recall, we know how to differentiate lnx with respect to x. But so far, we have not seen the $\int lnxdx$. And here we are going to find out how we can do that. So, this can be done by integration by parts. So, here you have as the integrand log natural x, just that.

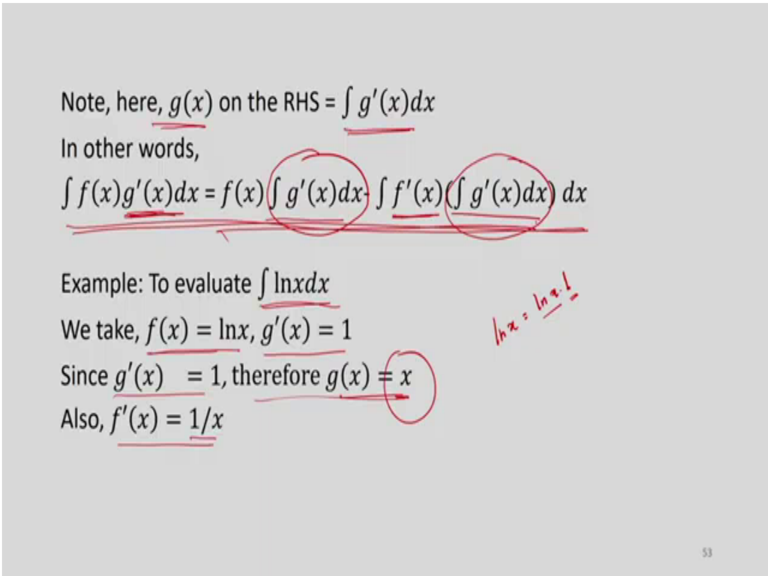
Now, lnx itself can be thought of as $lnx.1$. So that is what I have written here. Imagine this $f(x) = lnx$ and the second term is what 1. And $g^{'}(x) = 1$, which is there in the formula of integration by parts. So, 1 which we have sort of imagined there, of course, any number multiplied by 1 is that number itself. So, $1 = g^{'}(x)$, which is the second function that appears in the formula of integration by parts.

Now, if $g^{'}(x) = 1$, then we can take the anti-derivative of both sides, that is the integration. And if I do that, I get $g(x) = x$. Because what is 1, $1 = x^{0}$. And so I use the formula that if you take the $\int x^{n}dx = \frac{x^{n+1}}{n+1}$. Applying that integration of 1 will give me x.

And remember I have to have this expression also. I have got $g(x)$ which I need on the right hand side. This is $g(x)$ and this is also $g(x)$. So, these things, I have found that out $g(x) = x$, but I also need this term $f^{'}(x)$. That is easy to find. Since $f(x) = lnx$, I can take the derivative and I know what is the derivative of lnx it is $\frac{1}{x}$. So that will be substituted here.

(Refer Slide Time: 10:15)





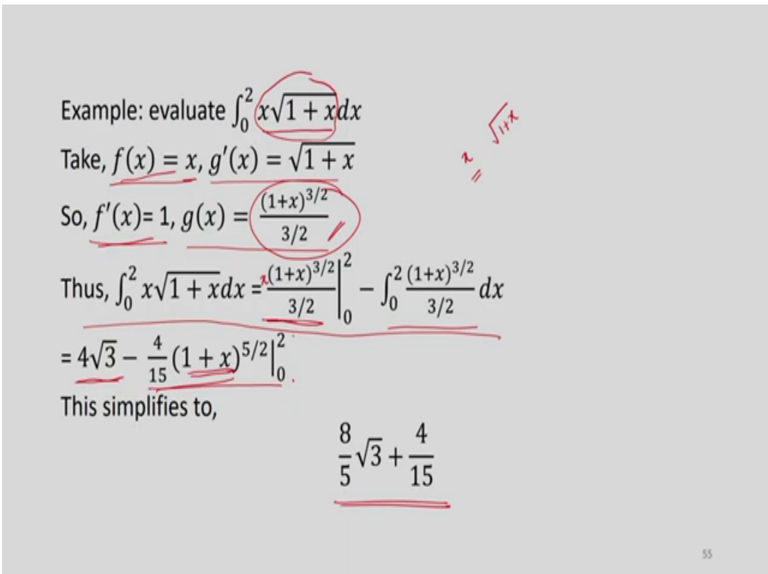
So, I have got everything that I need. I now use this formula, formula for integration by parts. And on the right hand side what do I have, $f(x)$ I know, lnx. $g(x)$ I have just found out, $g(x) = x$ and I put that here so I get x and minus integration of $f^{'}(x)$ that I have just found out $f^{'}(x) = \frac{1}{x}$, so that is coming here, and $g(x)$ as we have found out it is just x. So, I have everything under control here.

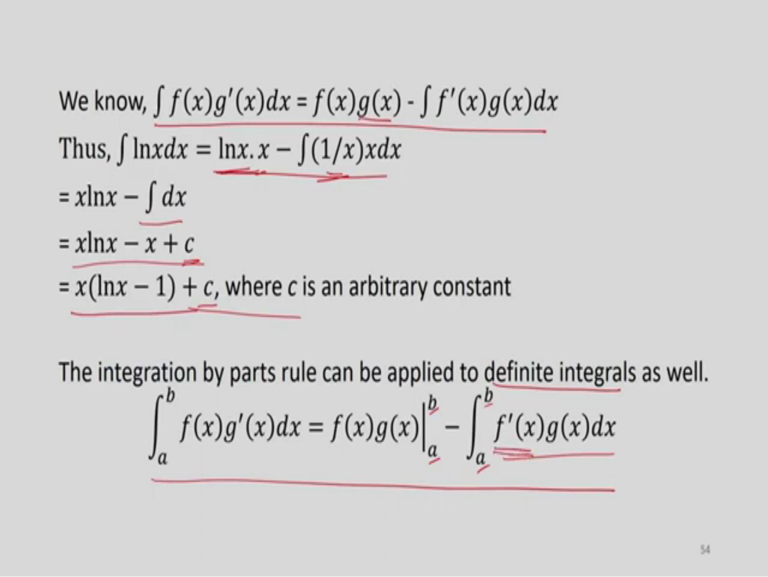
Now, what will happen in the next step? The first term remains the same. It is $xlnx - \int(1/x)xdx$. Those two terms will cancel each other. So, you just have integration of x. And what is the integration of x? It is x. And obviously, there is this, since we are talking about indefinite integral, so there is a plus constant term. So, that constant term is c. So, in short it is $x(lnx - 1) + c$. So, this is basically the integration of lnx.

Now, we can say that we have found out the $\int lnx$ through our method of integration by parts. Now, this was an application where the integral was indefinite integral, but that is why you have this small term, but the same thing can be used if we have definite integrals as well. So, here is that formula. So, here I have the limits of the integration as a and b. So, $\int_a^b f(x)g'(x)dx$, the limits are a and b, a is the lower limit, b is the upper limit.

And remember what was the case for indefinite integral? It was $f(x)$, this was the formula $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. But here the limits are there. So, I have to evaluate whatever I get at these two limits and take the difference. So, that is why you have b and a here and in the second term you have the integration. So, I can use this b and a as the limits of this integration. So, this is the formula for definite integral, $\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$.

(Refer Slide Time: 13:33)





So, again, I come to exercise. So, here I have $\int_0^2 x\sqrt{1 + x}dx$. So, that is the integration I have to do. And the lower limit and the upper limits are 0 and 2. Now, this is the integrand. Natural question that may arise, that this is all right, this is a product of two functions, but which is $f(x)$ and which is $g'(x)$.

Now, there is no hard and fast rule of what you will take as the first function and what you will take as the second function. But practically speaking, since in the formula of integration by parts one has to differentiate one function, this differentiation thing comes in the second term $f'(x)$, so practically it will be easier if I take $f(x)$ to be that function which becomes simpler after differentiation.

Now, out of x and $\sqrt{1 + x}$, which one becomes simpler after differentiation, it is obviously the x, because if you differentiate x it becomes 1. So, that is what I have done here. Suppose $f(x)$ that is the first function is x and $g'(x)$ that is the second function, $g'(x) = \sqrt{1 + x}$.

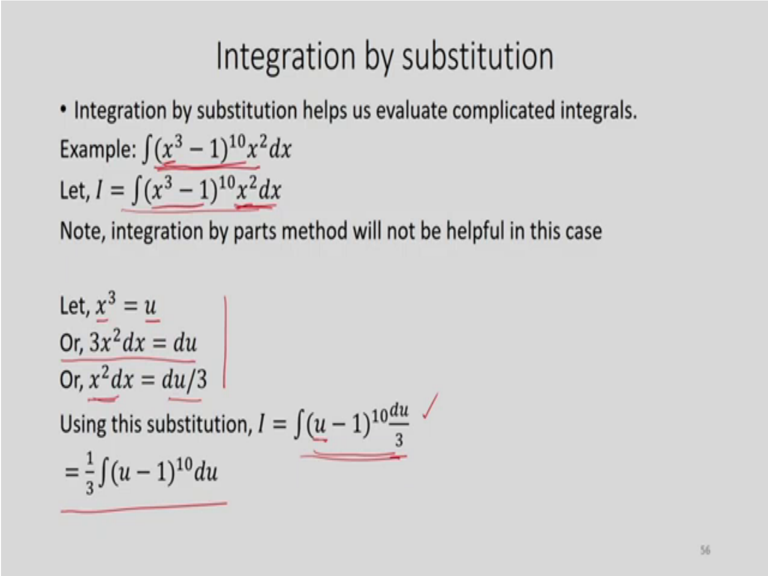
Now, as before I have to find two things one is $f'(x)$that is 1 here and I have to find out what is the anti-derivative of $g'(x)$ that is the $\int g'(x)$and that is not difficult to find. $\int g'(x) = \frac{(1+x)^{3/2}}{3/2}$. I have used the formula that if you $\int x^n dx = \frac{x^{n+1}}{n+1}$.

Now, I use the formula of integration by parts. So, what you are going to have here is the first function that is $f(x)$ which is 1 multiplied by the integration of the second function that is, you are going to get this much, minus integration of the differentiation of the first function that is $f^{'}(x)$ multiplied by the integration of the second function which is this. So, here it is fine, because if I take the derivative of x, it becomes 1. So, there is no x here. But in the first term, there should be an x that is what I have written.

And you see there is one more integration still left to be done and which is done in the second step. And the limits are there, 2 and 0. So, I have to evaluate this first term in these two values that is 2 and 0 and take the difference that comes out to be $4\sqrt{3}$ and the second term is integrated once more, and so I get $[\frac{4}{15}(1 + x)^{5/2}]_0^2$.

And then I have to obviously, here there are xs here, so I have to evaluate this particular term at 2 and 0 and take the difference. And if I have done that, then I will get this expression which is $\frac{8}{5}\sqrt{3} + \frac{4}{15}$. And this should be alright, because we know the definite integrals are numbers and this is a number.

(Refer Slide Time: 18:13)



Now, we come to another technique of finding the integration of complicated integrands and this method is called integration by substitution. Integration by substitution helps us evaluate complicated integrals. So, here is an example, $\int (x^3 - 1)^{10} x^2 dx$. Here the integrand is this $(x^3 - 1)^{10} x^2$. This is the integrand and this is being integrated. So, how do I do that?

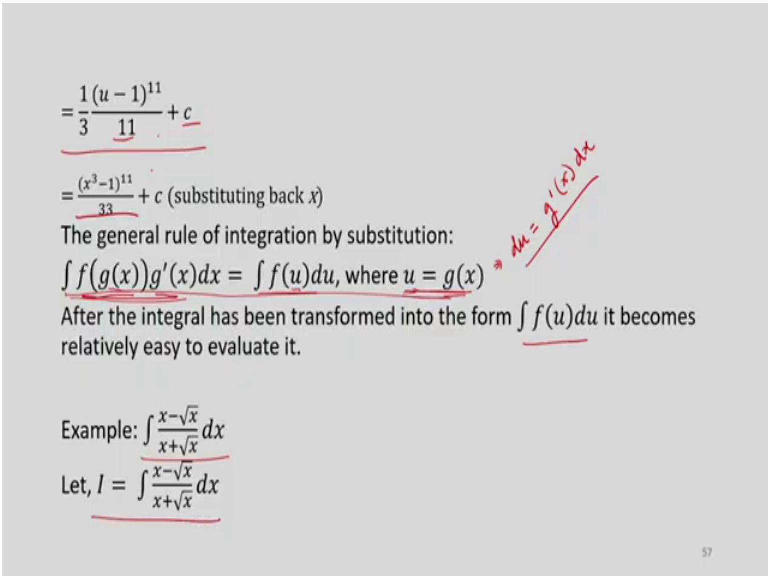
Now, obviously, one can do the following method that you take this expression $(x^3 - 1)^{10} x^2$ and you decompose it, because this is like $(a + b)^n$. And this can be found out using the binomial series. But there are smarter ways to do that, because this will become a very cumbersome exercise. It will have 10 terms here, more than 10 terms actually, and then you are multiplying it by x^2. So, it will become a long series of terms.

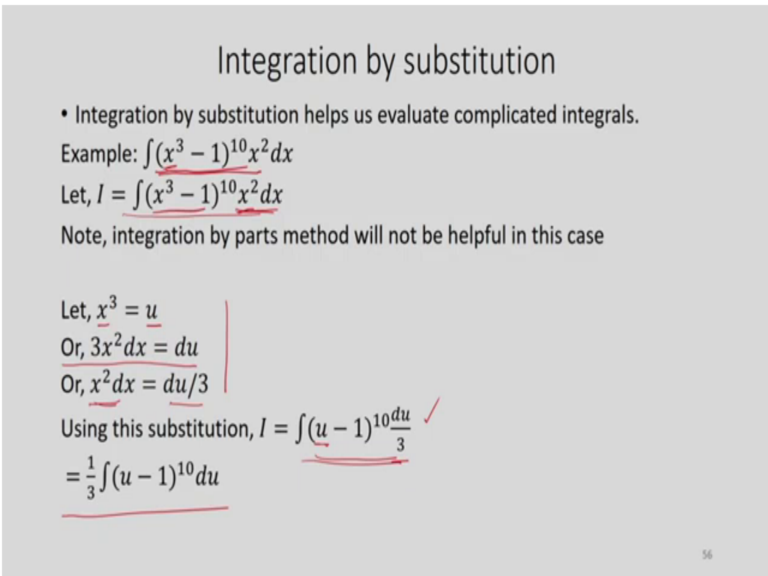
So, a better way is to do it by integration by parts. And what I do here is that I assume that $x^3 = u$. Now, you might ask where did I get this x^3? Well, x^3 is here. So, I have taken that x^3 and assume that x^3 is some other variable, that variable is u and I have taken the differentials of both sides and it becomes $3x^2dx = du \ or \ x^2dx = du/3$. I have divided both sides by 3.

Now, why am I interested to find out what is x^2dx? The reason being, here in this problem I have this expression x^2dx. So, that is why I have found out what is x^2dx from this small exercise. Now, I have got what is x^3, it is u. I have got what is x^2dx, it is $du/3$. So, actually I can get rid of xs in this expression.

And that is what I have done here. I have substituted all these things, whether it is u or du in this expression. So, instead of x^3, I am writing u and instead of x^2dx, I am writing $du/3$, that way I have got rid of all the x's and dx's. So, I have got this. So, I can take $1/3$ outside. So, it becomes $1/3\int(u-1)^{10}du$.

(Refer Slide Time: 21:36)





And this can be easily found out what is the integration of this. It is $\frac{1}{3}\frac{(u-1)^{11}}{11} + c$. So, it becomes $\frac{(x^3-1)^{11}}{33} + c$. What I have done from here to here is that I have substituted back x, because remember initially I assumed that $x^3 = u$.

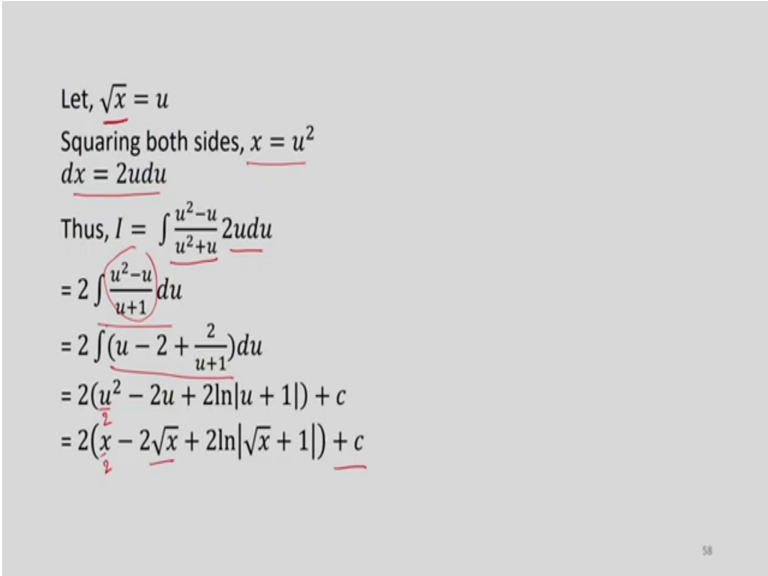
So, the result cannot be in terms of u, because the exercise itself is in terms of x. So, I cannot leave it in terms of u so I have substituted x^3 in place of u. So, I have got this. So, these are the two exercises, but what is the general rule of integration by substitution? So, this is the general rule of integration by substitution. Suppose you have to integrate this kind of a function f which itself is a function of $f(g(x))g'(x)dx$.

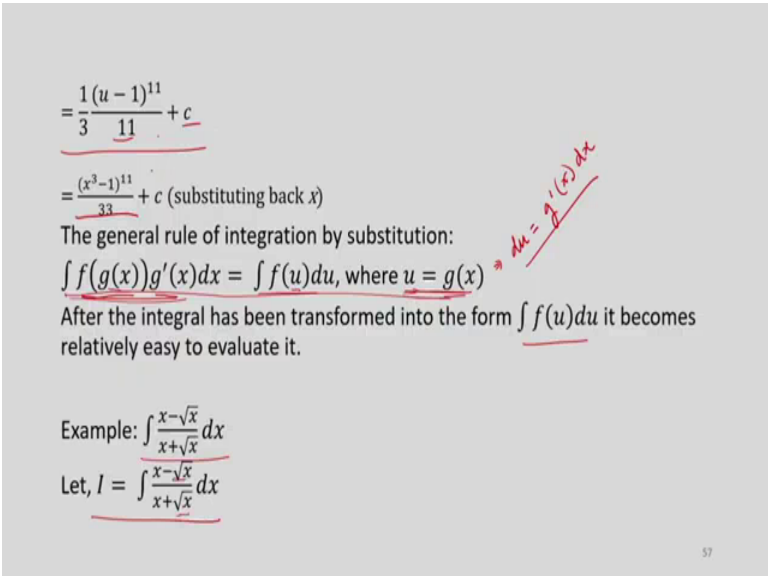
So, this is a composite function and there is another term here which is $g'(x)$ after this. So, what we do here, if we have expressions like this is that we assume that $g(x) = u$. So, whatever there is here as $g(x)$let us assume that to be another variable which is $u$and from here what you are going to get is $du = g'(x)dx$.

And then our task becomes easy, because the original integral that we have here in place of $g(x)$ I can write u, in place of $g(x)$ I am writing just u, so it becomes $f(u)$. And then there is this term $g'(x)dx$ Now, $g'(x)dx$ has been found out to be du. So, therefore, this cumbersome expression becomes simply $\int f(u)du$. And if it is just $\int f(u)du$ then it is easier to evaluate.

So, that is the general sort of method if we are relying on integration by substitution. So, here is another exercise. Example, $\int \frac{x-\sqrt{x}}{x+\sqrt{x}}dx$ and let us assume that this thing is equal to capital I and then we shall do the substitutions.

(Refer Slide Time: 25:04)





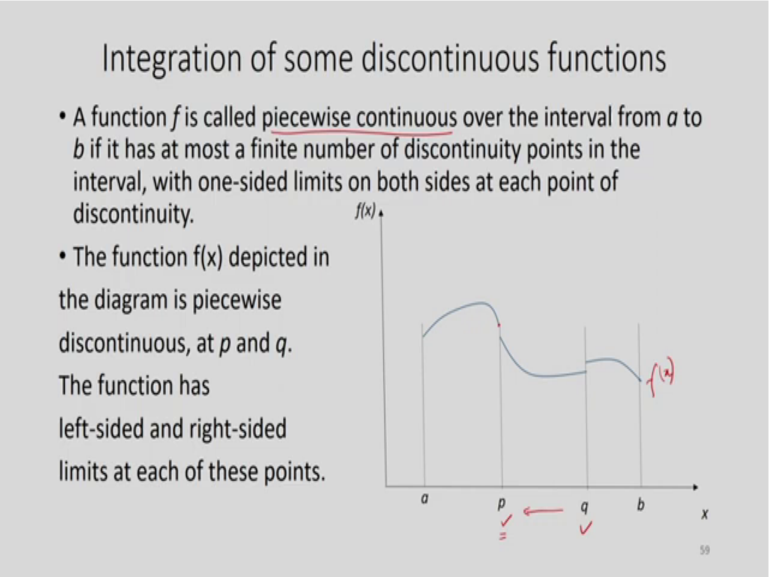
And we are assuming here that $\sqrt{x} = u$. $\sqrt{x}$, x is appearing here. Here is $\sqrt{x}$, here is $\sqrt{x}$. So, I am assuming that $\sqrt{x} = u$. And if $\sqrt{x} = u$, I can take the square root of both sides that will give me $x = u^2$ and then I take the integrals it will become $dx = 2u\,du$. Now, I will then use these substitutions in the original integral.

So, what do I do, $x - \sqrt{x}$, what is x, $x = u^2$. So, instead of x I will write u^2 and instead of $\sqrt{x}$ I will write u, so that will give me this expression, $\int \frac{u^2-u}{u^2+u} 2udu$. And then there is this dx term. $dx = 2u\,du$. Now, what do I do next? I take 2 out, because it is a constant.

So, I have $2\int \frac{u^2 - u}{u+1} du$, because one u is here and that u will get cancelled with the u in the denominator and then I can divide this expression and I will get this term $2\int (u - 2 + \frac{2}{u+1}) du$. So, one has to take the integration of this. And this is not difficult to do. First is the u tern which is $= 2(\frac{u^2}{2} - 2u + 2ln|u + 1|) + c$.

And then you have the coming back of the x's instead of the u's. So, instead of u^2, I will write x and here there is 2. And after that, you have $- 2u$, instead of u I am writing $\sqrt{x}$ and then $2ln\left|\sqrt{x} + 1\right|) + c$. So, this is how it is done.

(Refer Slide Time: 27:59)



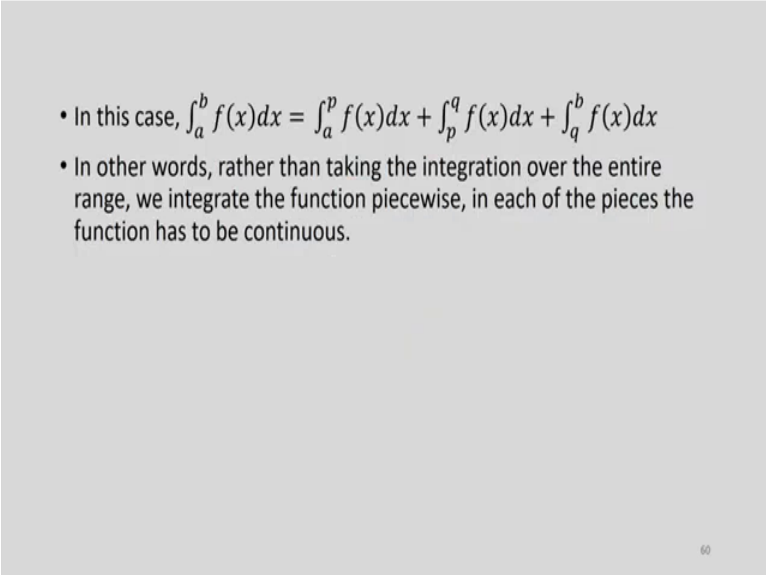
Here is another sort of problem that one might face which is integration of some discontinuous functions. Mind you, one is not saying integration of all discontinuous functions. If some discontinuous functions are there which satisfy certain properties, then we can take the integration of those discontinuous functions. Now, what are those properties? We shall see them in a while.

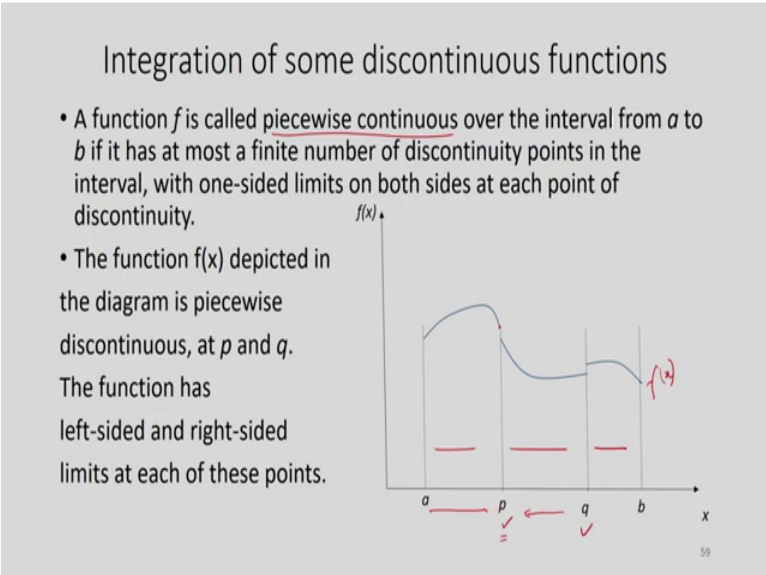
A function f is called piecewise continuous over the interval from a to b if it has at most a finite number of discontinuity points in the interval with one-sided limits on both sides at each point of discontinuity. So, the functions that one has in mind are not any discontinuous functions, these are piecewise discontinuous functions. So, here is an example. So, this function $f(x)$ it is defined from a to b, but you can see it is discontinuous at two points at p and at q. Because you can see the function is jumping. So, there is a point of discontinuity at p and at q.

The function $f(x)$depicted in the diagram is piecewise discontinuous and this discontinuity is at points p and q. The function has left sided and right sided limits at each of these points. So, for example, if you take the point p, the left sided limit is there. So, if x approaches p from the left hand side then the value of the function actually approaches this value.

So, there is no point of discontinuity from the left hand side and from the right hand side also there is no point of continuity. There is a limit at p from the right hand side. But these two limits are not the same. So, that is why at p the continuity is not there.

(Refer Slide Time: 30:36)

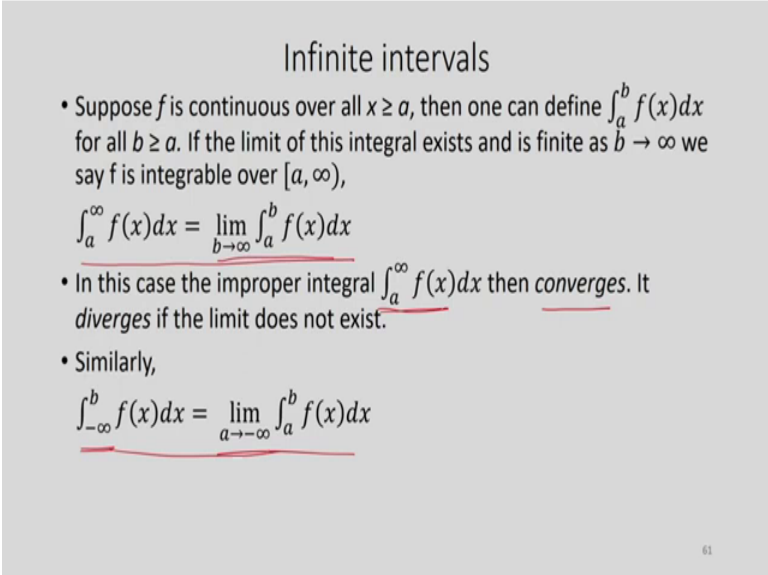




So, in this case, what do we do? How do I find the integral of this function from a to b? I take it piecewise. So, $\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^q f(x)dx + \int_q^b f(x)dx$, so basically, this part plus this part plus this part.

In other words, rather than taking the integration over the entire range, we integrate the function piecewise in each of the pieces the function has to be continuous. Yes, that is satisfied, because between p and q there is no point where the function is not continuous. So, it is continuous, and therefore, I can take the integration of the whole range as the summation of three parts.

(Refer Slide Time: 31:41)

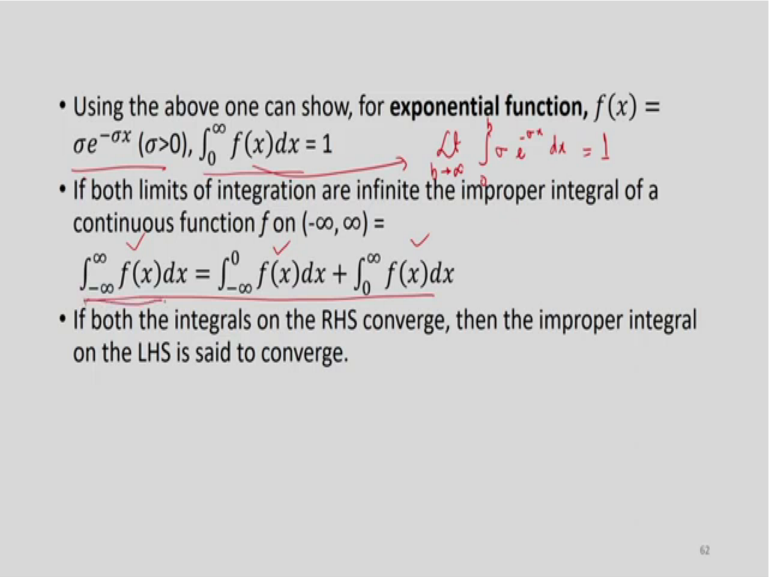


Infinite integrals, suppose f is continuous over $x \geq a$, then one can define $\int_a^b f(x)dx$, where the limits are a and b, for all $b \geq a$. If the limit of this integral exists and is finite, then as $b \to \infty$ we say f is integral over $[a, \infty)$.

So, here what is happening is that the upper limit is b, but b suppose it goes to infinity and the function is continuous over this entire range from $[a, \infty)$. And in that case if the limit exists as $b \to \infty$ then we can say that the function is integrable. And what is the value of that integral, it is the integration of $f(x)dx$ a to infinity and this can be written as $\lim_{b \to \infty} \int_a^b f(x)dx$.

In this case of improper integral, this is the improper integral, $\int_a^\infty f(x)dx$ converges, because the limit exists. It diverges if the limit does not exist. Similarly, the lower limit can also go to minus infinity. Here the upper limit is constant at b, but the lower limit can tend to $-\infty$. So, then we can write it like this. And here also the limit must exist, otherwise we cannot say that the function, that this integral converges.

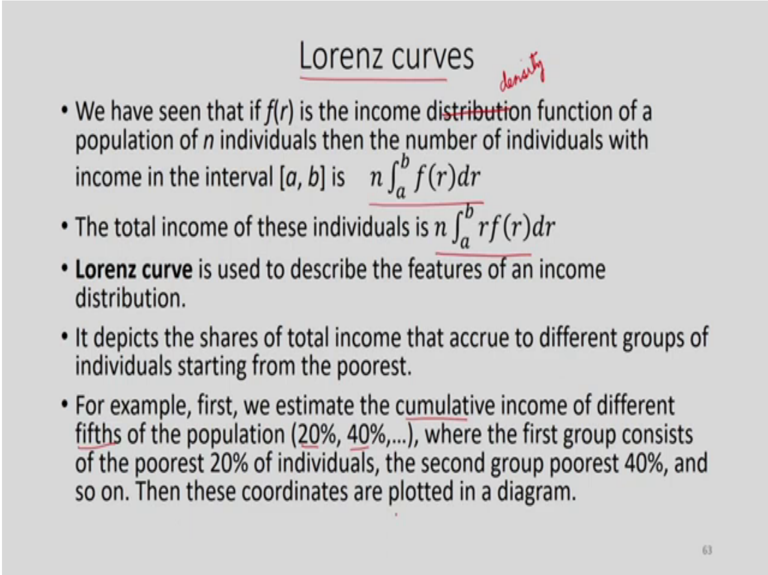
(Refer Slide Time: 33:38)



Using the above rule one can show for exponential function $f(x) = \sigma e^{-\sigma x}(\sigma > 0)$, the integration of this function from 0 to infinity is equal to , $\int_0^\infty f(x)dx = 1$. So, what we are going to do here is that this thing is nothing but $\lim_{b\rightarrow\infty} \int_0^\infty \sigma e^{-\sigma x} dx = 1$. If both limits of integration are infinite, the improper integral of a continuous function f from $(-\infty, \infty)$ can be written as this.

So, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x)dx$. If both the integrals on the right hand side that is this and this both converge then the improper integral on the LHS that is this one is said to converge. So, both of them have to converge for this to be defined.

(Refer Slide Time: 35:24)



Now, we come to one application of what we have learned so far in terms of definite integrals and its various properties. This is the application in what is known as Lorenz curves and it has something to do with the distribution of income in a country, in an economy. We have seen that if $f(r)$ is the income density function of a population of n individuals, then the number of individuals with income in the interval [a, b] is given by this, $n\int_a^b f(r)dr$.

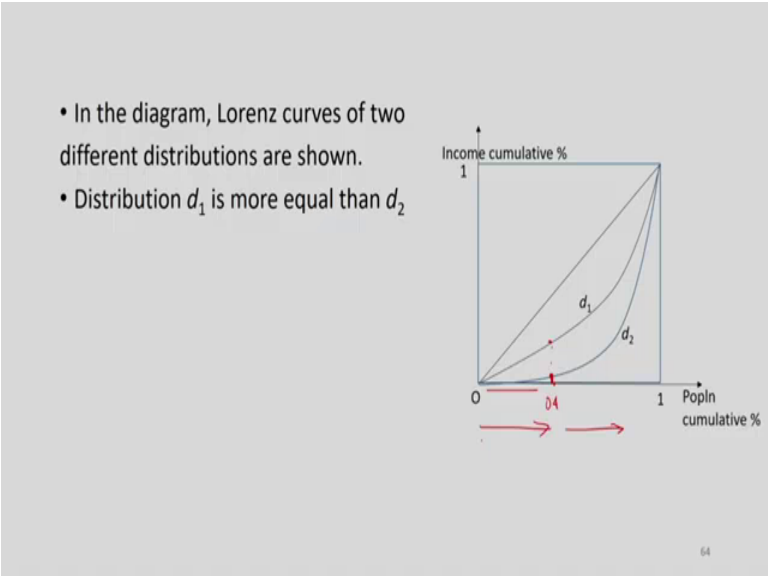
So, this is the total number of people in that country who have incomes between these two limits a and b. And what is the total income of these people, these number of people. The total income of this number of people is given by this expression, $n\int_a^b rf(r)dr$.

Now, Lorenz curve, what is the use of Lorenz curve? Lorenz curve is used to describe the features of an income distribution. It depicts the shares of total income that accrue to different groups of individuals starting from the poorest. The Lorenz curve is a depiction of the different income shares obtained by different groups of people starting from the poorest then the next poorest section then the next poorest section etc.

So, here is an example. For example, first we estimate the cumulative, this is important, cumulative, income of different fifths of the population such as 20 percent, 40 percent like that it will go on. So, if it goes on like that, there will be five different sections; 20 percent, 40 percent, 60 percent, 80 percent, 100 percent, but here we are talking about cumulative income.

So, in the first section the bottom 20 percent is considered, in the second section the bottom 40 percent is considered. So, in that 40 percent the first 20 percent is included, likewise it will go on. Where the first group consists of the poorest 20 percent of individuals, the second group poorest 40 percent and so on, then these coordinates are plotted in a diagram.

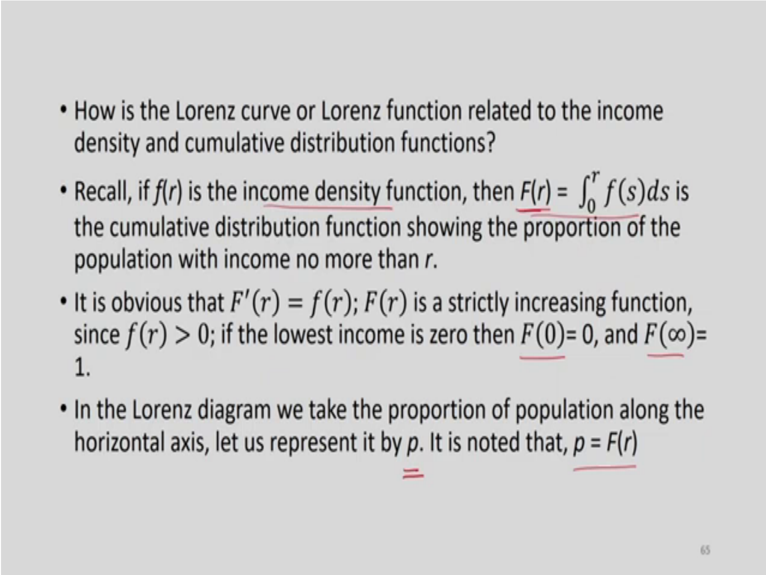
(Refer Slide Time: 38:43)

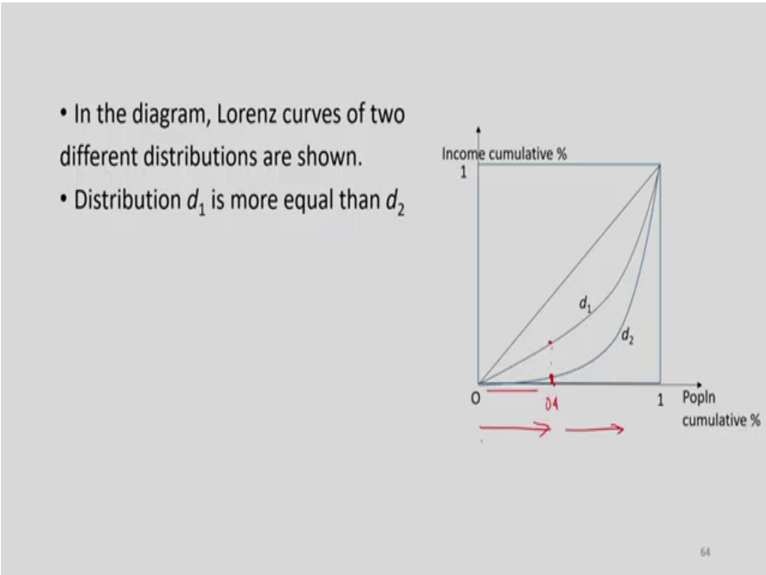


So, here is an example. In this diagram, Lorenz curves of two different distributions are shown as d_1 and d_2. So, on the x-axis you can see the population cumulative percentage, on the y-axis the income cumulative percentage. For example, suppose here is 0.4, so this is the point on the curve, on the Lorenz curve for d_2 and for d_1 this is the point on the Lorenz curve for d1. 0.4 means what, bottom 40 percent, because from here the poorest people are starting and we are going to the right and we are encountering richer and richer people.

So, in this group, the poorest 40 percent of the population are counted. What is their share of income in d_2, it is just this much. I do not know, maybe 0.05 that is 5 percent. Here it is a little bit higher. I do not know, maybe it will be close to 0.2. That is 20 percent of the income. d_1 is more equal than d_2. The intuition being that in d_1 you see for the bottom 40 percent in d_1 they are getting a greater proportion of the income. In d_2 the proportion of income that this bottom 40 percent gets is much more lesser. That is why it is being said that d_1 is more equal than d_2.

(Refer Slide Time: 40:27)





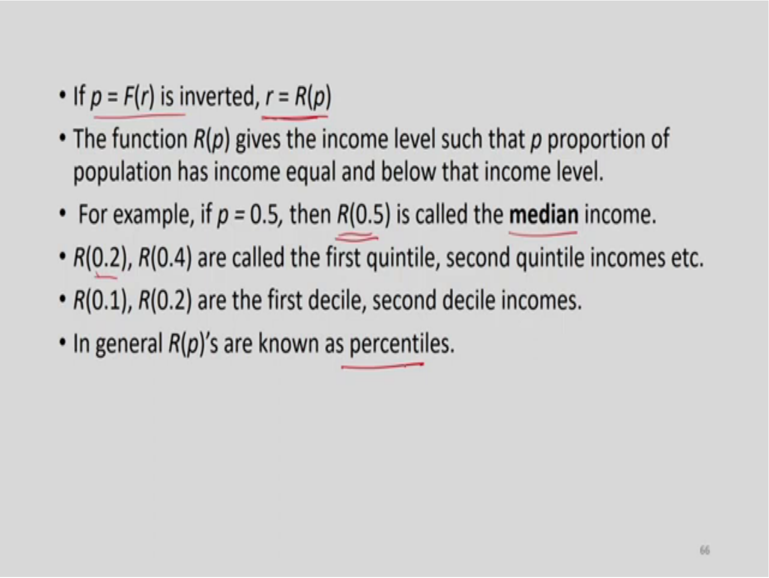
Now, the question that arises is mathematically how these two things are related. How is the Lorenz curve or Lorenz function related to the income density and cumulative distribution functions? Recall if $f(r)$ is the income density function, then capital $F(r) = \int_0^r f(s)ds$ is the cumulative distribution function and it shows the proportion of the population with income no more than r. So, r is the level of income, $F(r)$ shows the proportion of the population whose income is not greater than r.

Now, it is obvious that $F'(r) = f(r)$, it is coming from here, $F(r)$ is a strictly increasing function since small $f(r)$ is always positive. So, that is again obvious. If the lowest income is

0, so in that case capital $F(0) = 0$, because below the lowest income there is no population and $F(\infty) = 1$. So, if you take the highest income going to infinity, then all the population will be included in your population group. So, what is their share of the total population, it is 100 percent.

In the Lorenz diagram we take the proportion of population along the horizontal axis that we have done here, the proportion of population here 0.4 means 40 percent, let us represent it by p. It is noted that small $p = F(r)$. What is $F(r)$? Remember, this is a proportion of the total population who have income less than r. So, I can write $p = F(r)$, because p is the proportion.

(Refer Slide Time: 42:39)

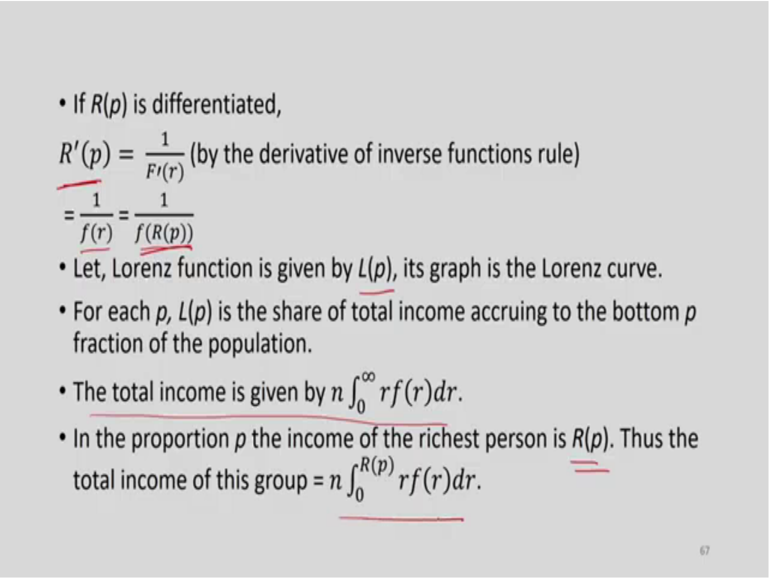


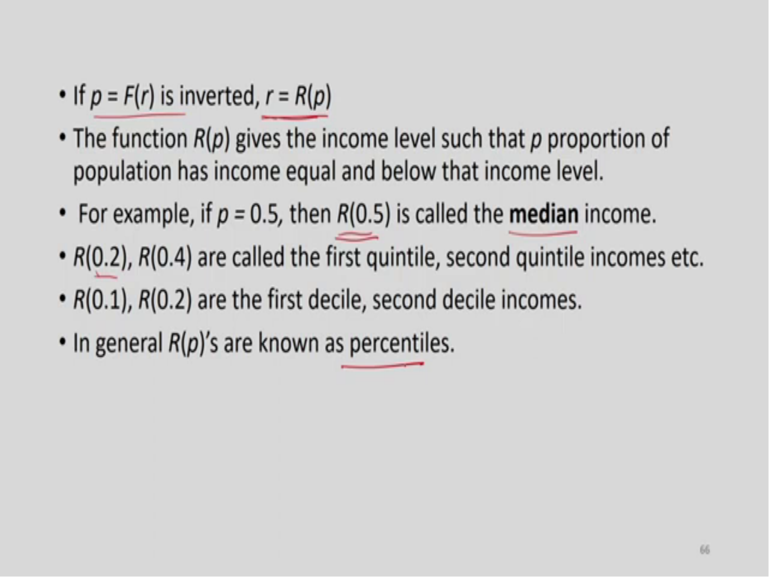
Now, if it is inverted, $p = F(r)$, I get $r = R(p)$, let us write that as $R(p)$. Now, what does this function signify? This function $R(p)$gives the income level such that p proportion of the population has income equal to or below that income level. So, you give me one fraction p,p is a fraction, and I will give you that income level such that$p$ proportion of the population earn their income below that r.

For example, if $p = 0.5$ then capital $R(0.5)$ is called the median income. So, below that, this value 50 percent of the population is earning their income. Or in a different way, the population of the country is such that 50 percent of the population is earning their income above $R(0.5)$.

Similarly, $R(0.2)$, $R(0.4)$, they are called the first quintile, the second quintile incomes etc. Quintiles has to do with one-fifth. Here 0.2, 0.4, if we add them up, if you add five such terms, you will get to 100 percent. Similarly, $R(0.1)$, $R(0.2)$ are called the first decile, second decile incomes, etc. In general, $R(p)$ are known as percentiles.

(Refer Slide Time: 44:33)



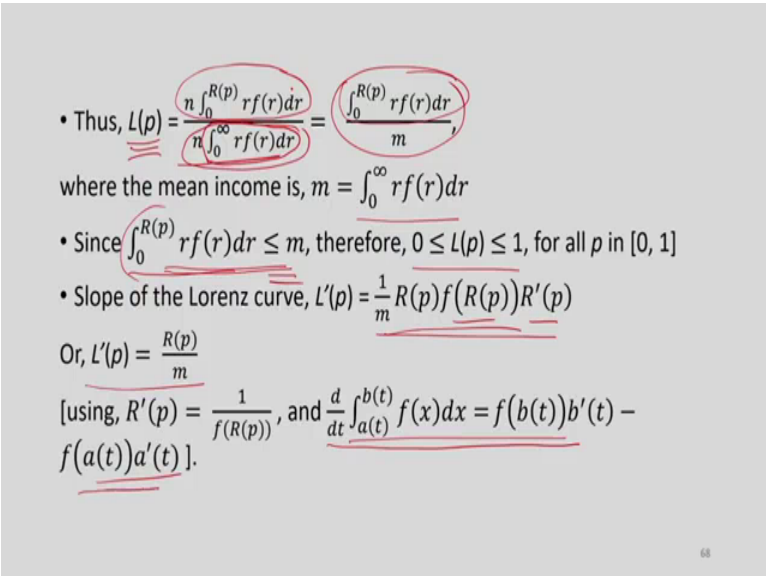


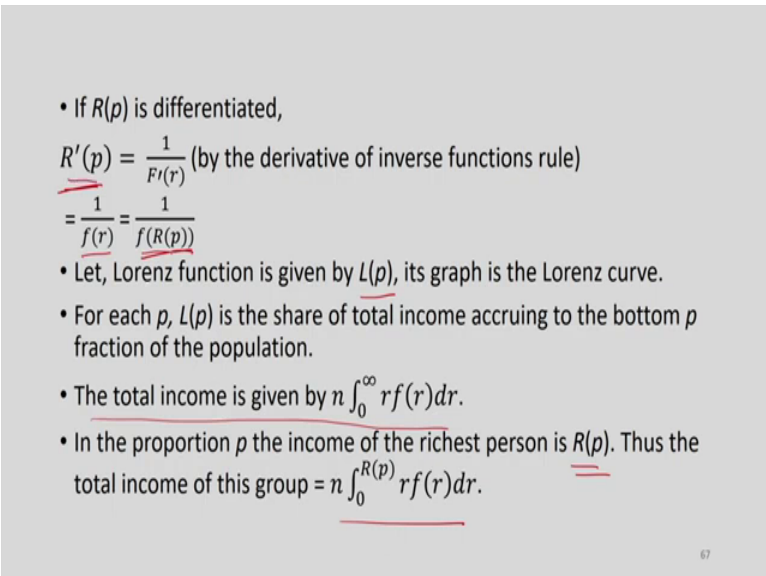
Now, if I take this $R(p)$ function, R is a function of p, and if I differentiate that, so this is the function that I am talking about, and if I differentiate both sides with respect to p, $R'(p) = \frac{dr}{dp}$. And what is $\frac{dr}{dp}$, I can use this. And so I will get $R'(p) = \frac{1}{F'(r)}$. And what is

$F^{'}(r)$, it is small $f(r)$ and which is nothing but $f(R(p))$. So, this is an important thing that we are going to use later on, $R^{'}(p) = \frac{dr}{dp}$.

Let the Lorenz function is given by $L(p)$. Its graph is the Lorenz curve. For each p, $L(p)$ is the share of the total income accruing to the bottom p fraction of the population. Now, we know that the total income is given by $n\int_{0}^{\infty} rf(r)dr$. In the proportion p the income of the richest person is $R(p)$. Thus the total income of this group that is $R(p) = n\int_{0}^{R(p)} rf(r)dr$.

(Refer Slide Time: 46:05)





Thus what is $L(p)$. $L(p)$ is the income share. So, on the top that is on the numerator we have the income of this group of people from 0 to $R(p)$ and on the denominator I have the total income of the population. So, $L(p)$ will give me the proportion. And n and n will get cancelled and I am going to get this, where m is occurring in the denominator. It is nothing but this term. And this is nothing but the mean income.

Since this term $\int_0^{R(p)} rf(r)dr \leq m$, therefore the $L(p)$ that is the value of the Lorenz function is always between 0 and 1. Why is this less than 1. Well, because when you are taking r to p, you are taking a population group which is not the entire group, but some part of the population which is a little bit poorer than the entire group, because we are taking the bottom sections of the population remember.

So, if I take the entire population in this group, then it will become equal to m, but in general it will be less than m. Now, what is the slope of the Lorenz curve that also we can find out if I take the derivative of this $L'(p) = \frac{1}{m} R(p) f(R(p)) R'(p))$. And this simplifies to $L'(p) = \frac{R(p)}{m}$.

Here I have used this relationship that $R'(p) = \frac{1}{f(R(p))}$. So, that is why these two terms actually cancel each other. And I have also used this formula to get from here, from, to get the derivative of this expression as this. This formula I have used. So, $L'(p) = \frac{R(p)}{m}$. So, this is the slope of Lorenz curve, $\frac{R(p)}{m}$.

(Refer Slide Time: 48:42)

- Since $R(p)$ is a rising function, the Lorenz curve is a convex curve, its slope rises gradually from 0 to infinity.
- At a particular p, $m = R(p)$; at that point the slope of the Lorenz curve will be equal to 1. It is given by, $p = F(m)$
- The Lorenz curve can be used to define G, the Gini coefficient, a measurement of income inequality.
- Geometrically G is twice the area lying between the 45 degree line and above the Lorenz curve. But it can represented by

$$G = 2\left[\int_0^1 p dp - \int_0^1 L(p)dp\right] = 1 - 2\int_0^1 L(p)dp$$

- G approaches 0 as the Lorenz curve approaches the 45 degree line. The income distribution is more equal.
- G approaches 1 if the curve moves away from the 45 degree line.

- Thus, $L(p) = \frac{n\int_0^{R(p)} rf(r)dr}{n\int_0^{\infty} rf(r)dr} = \frac{\int_0^{R(p)} rf(r)dr}{m}$,

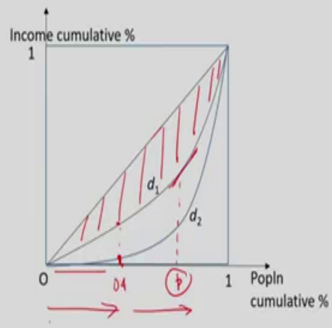
where the mean income is, $m = \int_0^{\infty} rf(r)dr$

- Since $\int_0^{R(p)} rf(r)dr \leq m$, therefore, $0 \leq L(p) \leq 1$, for all p in [0, 1]
- Slope of the Lorenz curve, $L'(p) = \frac{1}{m} R(p) f(R(p)) R'(p)$

Or, $L'(p) = \frac{R(p)}{m}$

[using, $R'(p) = \frac{1}{f(R(p))}$, and $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = f(b(t))b'(t) - f(a(t))a'(t)$].

- In the diagram, Lorenz curves of two different distributions are shown.
- Distribution d_1 is more equal than d_2



Now, $R(p)$ is a rising function, because remember, what is $R(p)$, $R(p)$ is the income going to the p proportion of the population. As the population proportion is rising, the income will also be rising. Therefore, $R(p)$ is a rising function. The Lorenz curve is a convex curve. Its slope rises gradually from 0 to infinity. The denominator, remember, is constant. Mean income is constant, $R(p)$ is rising. So, therefore, the slope is rising all the time, which means that it is a convex function.

At a particular p, where $m = R(p)$, the numerator and denominators are same, at that point the slope of the Lorenz curve will be equal to 1. It is given by $p = F(m)$. Now, let us look at the diagram how it looks like. Suppose I am talking about d_1, at some point the slope is just equal to 1. This slope and this slope are the same. So, that point has a special interest. And that point can be solved by using this relationship, $m = R(p)$. It can also be solved by this, these two conditions are the same.

The Lorenz curve can be used to define G. What is G? It is Gini coefficient, a measurement of income inequality. Geometrically, G is twice the area lying between the 45 degree line and above the Lorenz curve. It can be represented by this expression. Again let us look at that geometric thing. So, let us suppose I am talking about d_1, then the area between the 45 degree line and the Lorenz curve is this shaded region.

What is being said is that, Gini coefficient is twice this area. And it can be represented by this by using the area under the curve interpretation of definite integral. It will be $G = 2[\int_0^1 pdp - \int_0^1 L(p)dp] = 1 - 2\int_0^1 L(p)dp$, this is the area under the 45 degree line and this is the area under the Lorenz curve. So, I am taking the difference between the two. So, I am getting the area between the 45 degree line and the Lorenz curve. And this simplifies as this, $1 - 2\int_0^1 L(p)dp$. $L(p)$ is the Lorenz function.

And it can be seen that as the Lorenz curve approaches the 45 degree line that is this becomes similar to this, the income distribution becomes equal and the G approaches 0, because these two terms are becoming same. On the other hand, as the curve moves away from the 45 degree line, the Gini coefficient approaches 1. So, in this case $L(p)$becomes simply 0, because this becomes like this line.

So, you have this line as the Lorenz curve. So, $L(p) = 0$, therefore the value of the Gini coefficient is equal to 1 in this case. Now, I think we are done about the topic of integration. I can start with the last topic of difference equations in the next lecture. So, I shall see you in the next lecture. Thank you for joining me. See you later. Have a nice day.