Introduction to Logic
Prof. A. V. Ravishankar Sarma
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

Lecture - 22
Natural Deduction Method

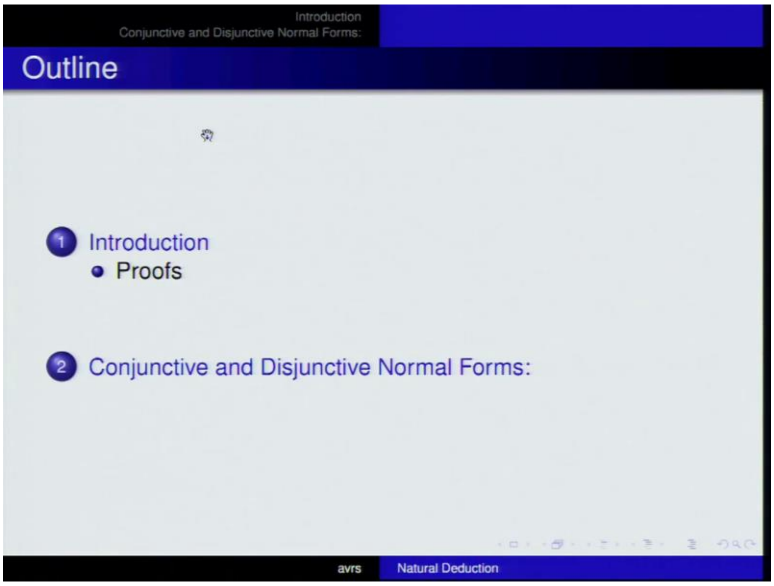
Welcome back, in the last lecture we discussed one important decision procedure method. That is, the semantic tableaux method. Semantic tableaux method, feels better than the others semantic methods, that we have. That is the truth table method. Truth table method will become quite difficult. Especially, when the number of variable increases to four or five or may be more than that.

A computer can easily do it, but as humans, it is very difficult to process that much of information. So, today what will be doing is? We will be talking about one syntactic method, where we will be basically studying about, some of the proofs of some important theorems, in the propositional logic. So, we started our journey with some kind of well formed formulas.

And out of these, well formed formulas some are construct to be tautologies, which are always true and some are considered to be always false. They are considered to be contradictions. And there are some other well formed formulas, which are sometimes true, sometimes false. These are called as contingent kind of statements. So, what will be doing today is, we will be presenting a different kind of methods, which also serves as a kind of decision procedure method.

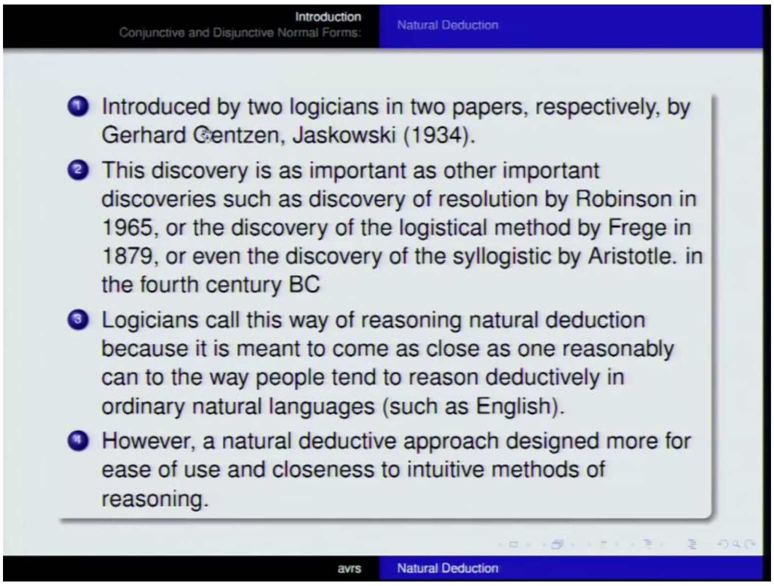
With which we will come to know that I mean, how to prove certain kinds of theorems. So, anything which is there is a one important theorem in proportional logic, which tells us that whatever is provable is obviously true and whatever is true is also provable. So, it is in that sense, if we can prove that something is a theorem. Then, you have already set to shown that, it is a tautology. So, in this lecture, will be trying to study about the Natural Deduction.

(Refer Slide Time: 02:14)



And then we will talk about some of the rules of inference, for proving certain theorems. This is considered with outline of this lecture. First, we will be talking about, what we mean by natural deduction. And then we will consider some examples of some of the important proofs, which are proofs of some of the theorems. So, then we will talk about conjunctive and disjunctive normal forms, which we will talk about little bit later.

(Refer Slide Time: 02:35)



But, first we will focus our attention on natural deduction. So, how we did come into existence? So, it is introduced by two logicians in two different papers. They were working independently to each other. They are not aware of each of their works and all. So, parallelizes two papers are presented. And each one is not knowing about others work and all. So, the first is due to it is attributed to Gerhard Centzen and Jaskowski, is a ((Refer Time: 03:04)) logician.

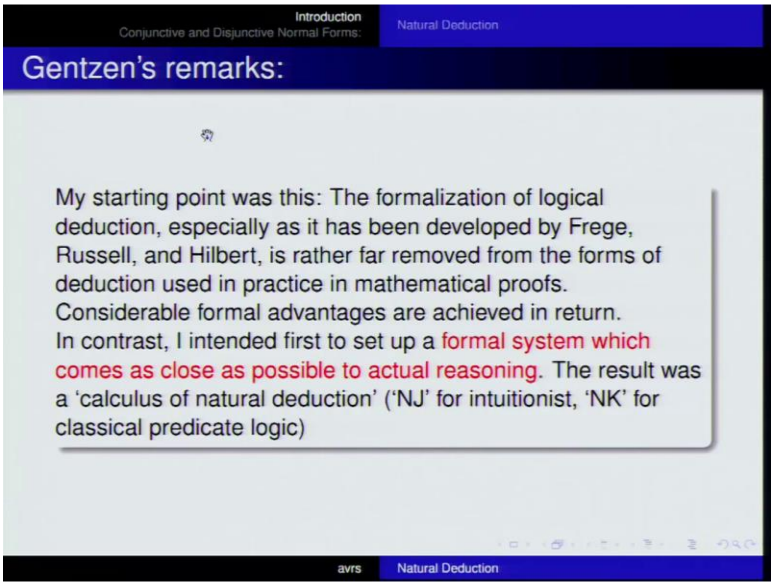
They were working at the same time. And then they came up with more or less similar kind of results and all. So, it is in the year 1934. So, what they are of the view is this thing. This discovery is as important as other important discoveries such as, discovery of resolution by Robinson, which is there in nineteen which has come up little bit later in 1965. There are some of the important results, that are very important in the propositional logic.

So, these are like this or the discovery of logistical method by Gödel of Frege in the year 1879. And even there are some discoveries like path breaking discoveries by the Greek philosopher, such as Aristotle in the 4th century BC. So, this natural deduction method is equally as important as, one of these path breaking kind of decision procedure methods that are discovered in the history of logic.

So, why it is called as natural deduction? So, logician call this way of reasoning a natural deduction. Because, it is meant to come as close as possible to the human reasoning, which they use in the day today discourse. So, as far as possibilities closer to the intuitive reasoning or the reasoning, that is employed in day to day discourse by us. So, that is why it is considered to be a natural deduction method.

See, natural deductive approach is designed more for the ease of use and the closeness to intuitive methods of reasoning. So, that is way it has got some kind of prominence in the literature of in the history of logic.

(Refer Slide Time: 04:49)



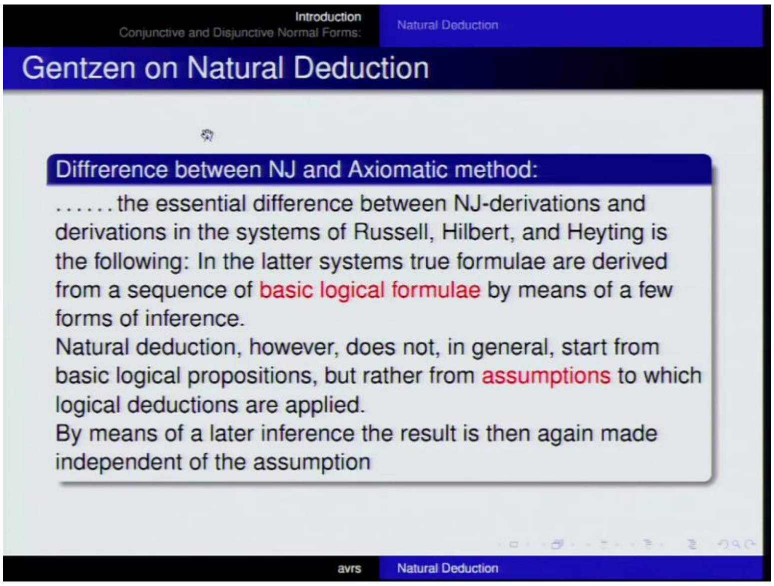
So, what is that we are going to do in this natural deduction method? So, the founder himself says like this, a Gerhard Centzen remarks like this. My starting point was this. The formalization of logical deduction, especially as it has been developed by Frege, Russell and Hilbert is rather far removed from the forms of deduction used in practice, in mathematical proofs. So, we are going to see little bit later about, some of this proofs, which we will be doing today in the context of axiomatic systems.

And you, yourself will note that how difficult it would be to prove. Simple theorem such as, p implies p or log x excluded in the middle p or not p x etcetera. So, these things will become little bit simpler in natural deduction method. So, there of this view that, mathematicians especially when they prove certain theorems and all, they may not be following the method that is adapted by Russell and Hilbert.

So, they went on to say this thing that, considerable formal advantages are achieved in return. In contrast, Gerhard Centzen I intended first to set up a formal system, which comes as close as possible to the actual reasoning. So, actual reasoning is the way we reason in day today discoursed. It is a kind of, some kind of linear process etcetera. So, the result was this thing that a calculus of natural deduction.

So, it is called as the NJ. If you are an intuition is, you name it as NJ or otherwise NK. Specially, a classical logicians belong to classical predicate logic, we call it as NK. It does not matter, in what way you call it. So, this natural deduction method is, I mean closer to the actual reasoning, that the human being employees in day to day discourse.

(Refer Slide Time: 06:44)



And we went on to say that about, this natural deduction that… So, there are two methods with the one method, which we will be talking about little bit later. That is the axiomatic method. So, the difference between natural deduction method and the axiomatic method, according to Gentzen is as follows. That the essential difference between natural deduction derivations and the derivations in the systems of Russell, Hilbert, Heyting etcetera and all.

Heyting is an intuition stick logician, it is the following. In the later system, that means in the Russell, Hilbert, Heyting axiomatic systems, true formulas are. True formulas in the sense, means the theorems. True formulas are derived from a sequence of basic logical formulas. Usually, they are construct to be axioms by means of few forms of inference ruler. So, what they achieved in the axiomatic system, in case of Russell, Hilbert and Heyting is this.

They started with the fundamental or first evident or self evident truths and all. They are called as axioms, which cannot be questioned and all. They are obviously true. And they used very few rules of inference. The only rule of inference, mostly they use is the more responsible principle. That is, if A implies B is the case A and then B follows from that. Only that particular kind of inference rule, very few inference rules they use and the rest of things are only axioms.

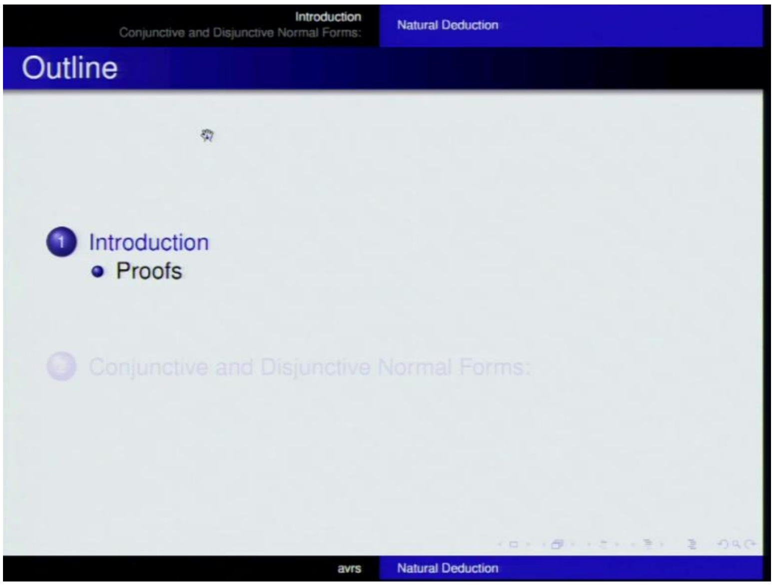
An axioms are also very few in number. Sometimes in Frege, there are five axioms. In the Russell whitehead, there are five axioms, whereas, Hilbert Ackermann has three axioms to begin with. And then they use more respondance and then all the theorems. I mean those statements which are obviously considered to be tautologies are all derived within that formals, an axiomatic system, using some kind of transformation rules etcetera.

On the other hand, naturally deduction however does not in general start from basic logical propositions. They do not start with fundamental principles or self evident truths such as axioms. But, rather from the assumptions to which the logical directions are apply. So, what they do is, they start with there are some obvious rules of inference used in logic. Like, modus ponens, modus tollens, constructive dilemma and all these things, law of conjunctions, law of additions.

So, these are the principle they start with. They begin with these assumptions. They begin with this truth preserving rules plus there are some assumptions which comes from the given formula. And from that, they will reduce some kind of theorems. So, by means of a latter inference, the result is then again made independent of the assumptions.

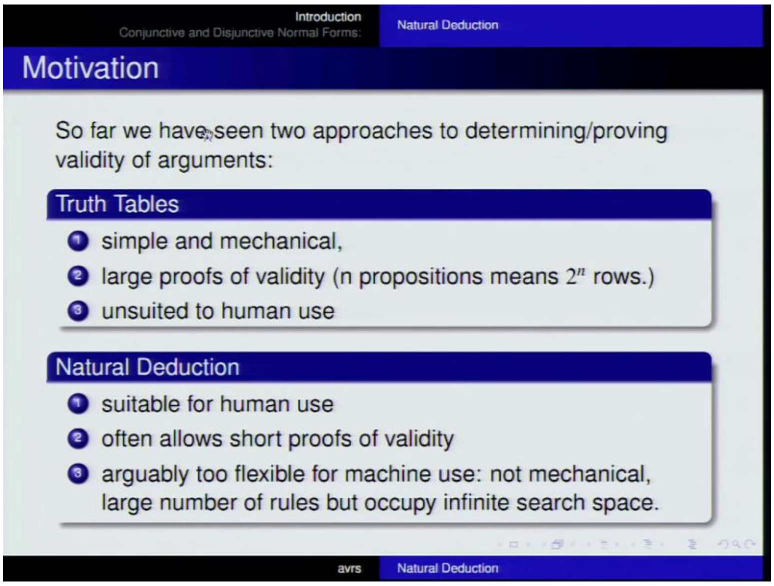
There are no assumption etcetera are used in the formal axiomatic system used by Russell, Whitehead etcetera, except that they used some kind of axioms, which are considered to be self evident rules, which are obviously true etcetera.

(Refer Slide Time: 09:53)



So, now we will be considering some of the important proofs within the axiomatic system.

(Refer Slide Time: 10:06)



So, before that we need to, the motivation for the sink for doing this natural deduction method is as follows. So for, we have seen semantic tables method, truth table method, etcetera. In the truth table methods, they are very simple and their mechanical and their large proofs of validity. For example, if there are n prepositions we have 2 to the power of n rows in the team, truth table.

If n becomes large, then it becomes difficult for us to handle. Because, for checking the validity of a given formula. For example, if there are, if n is 6 then that means 2 to the power of 6 entries will be there in the truth table, that is around 64 entries. We need to check, all the 64 entries. And then we need to check for row in which the promises are true, whether your promises are true, then the conclusion is false.

If that is the case and you will say that, the argument is invalid. Instead of checking all these 64 rows and then those rows in which, we have true promises and false conclusions, etcetera for proving the invalidity. See, it becomes little bit difficult for us to handle, when n becomes large. Usually, it is unsuited for human use. Because, the once the number of rows increases then the things with complicity increases.

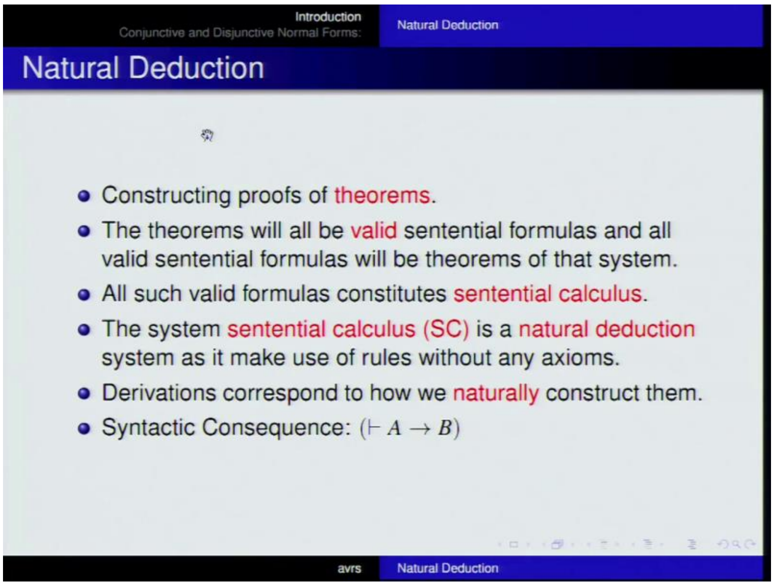
So, it usually unsuited for the human kind of reasoning. Especially, when n is large. The entries in the truth table are large than a computer can process it, but it is very difficult for us to handle. So, there are some other effective methods which we have introduced earlier, that is a symmetric tables method which is considered to be an elegant method, so for. And then now we will going to see the natural deduction method.

It is suitable for human reasoning because, it is in the beginning we said that, it is closer to the actual reasoning. The way we actually reason in actual reasoning course, it comes closer to that. If not, it is completely same as that one. But, it comes as for as possible. It is closer to the human reasoning, that we employee in day to day discuss. It often allows short proofs of validity.

For validity in particular, we can improve it and that is considered to be a true statement. And then all the true statements are obviously valid formulas. So, it allows for shorter proofs of validity, which are going to see. How we are going to achieve these things? And it is also arguably true flexible machines use. It is not mechanical, a large number of rules. It involves large number of rules. But, it also occupies infinite such space. This is another limitation of natural deduction method.

So, what essentially we do in a natural deduction method is as follows. We start with simple truth preserving rules. Like, modus ponens, modus tollens and constructive dilemma. There are valid principles of reasoning and that we employ in logic. So, we begin these rules of inference and then we add these rules of inference to the assumption that we get from the formula, which we are trying to prove. And then from that we will get the desired kind of formula that, we are going to derive.

(Refer Slide Time: 13:23)

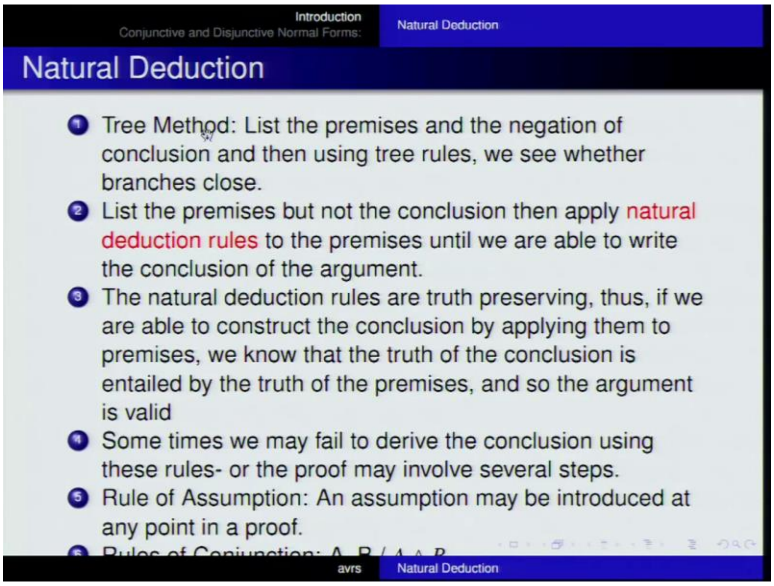


So, natural deduction is used for constructing proofs of theorems. Theorems means, they are obviously true formulas. That also considered to be tautologies, etcetera. They difference from axioms. Axioms are considered to be self evident rules, etcetera. So, theorems will all be valid sentential formulas and all valid sentential formulas, will obviously theorems of that particular kind of systems.

Any formal system you begin with all the valid kind of sentential formulas are obviously true at tautologies, etcetera. So, all such valid formulas constitutes, what we are calling it as sentential calculus. So, what is that essentially we are saying is, this thing. You began with well formed formulas. And then out of that, there are some tautologies. All these tautologies are obviously valid formulas.

And then all this valid formulas constitutes, what we mean by a sentential calculus. So, it becomes a particular system. The system sentential calculus is, it can be viewed as a natural deduction system, as it makes use of rules, without any axioms. So, we are going to see, how we are going to achieve this particular kind of thing? So, the derivations in natural deduction corresponds to, how we actually or naturally construct these theorems.

(Refer Slide Time: 14:42)



So, in the tree method especially, what we have done is simply for establishing the validity of a given formula, well formed formula what we have done is, simply this thing. We listed out all the premises and we included the negation of the conclusion. And then considering the negation of the conclusion leads to the branch closer; that means, it is unsatisfyable. Then, we have said that negation of conclusion is false. That means, the actual conclusion is true.

So, we list the premise, but not the conclusion. In the natural deduction method, we do it in a different way. We list out all the premises. And then using all the truth preserving rules, which we are going to talk about in while from now, use those rules and we have premises, which serves as a assumptions. And from that you will derive, whatever you wanted to derive you know.

So, what we will do here is, list that. We list out all the premises, but not the conclusion. Conclusion is our, which we are trying to derive from this premises. Then, we apply natural deduction rules which are truth preserving. And that will lead us to our destination, that is the conclusion. That is what, we are trying to prove. So, we are able to write the conclusion of the argument, based on the assumptions plus natural deduction rules.

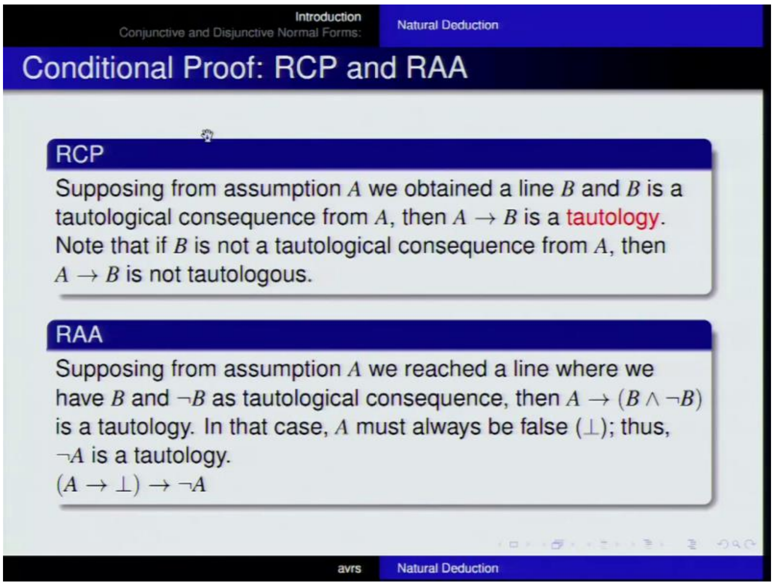
The natural deduction rules are usually truth preserving, thus we able to construct the conclusion by applying them to premises. And we know that, the truth of the conclusion is entailed by the truth of the premises. That is what, we mean by the deductive argument. A deductive argument is the one in which it is impossible for the premises is to be true and the conclusion is false. But, here all the rules or all the steps of your proof are all truth preserving.

So, based on that there is some kind of truth preservation is achieved in this particular kind of proof. And obviously, each step of your proof is true. Then, ultimately the final step which is considered to be a theorem, which is obviously true. And all the true formulas are obviously, they are considered to be tautologies. And all tautologies are obviously valid formulas. But, the problem with this method is, which we talk at the end of this lecture.

It is that, sometimes we may fail to derive the conclusion using these particular kinds of rules. Your derivation go on and on and all. So, then that sense, it may not be a kind of an effective method. So, for that what we will is, we follow another kind of method within the natural deduction. That is called as Reductio ad absurdum. So, what we will do is, we list out all the premises and we negate the conclusion.

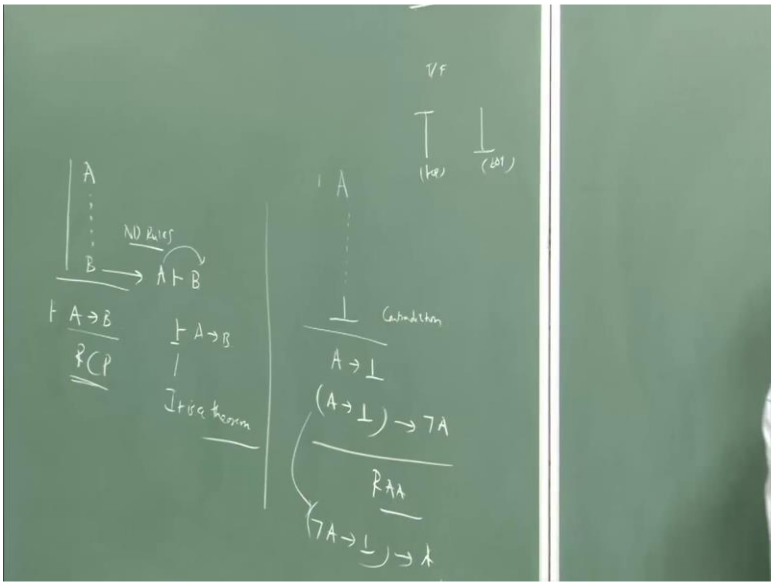
And then we will derive a contradiction. Contradiction means that, a and not a. And that is the case. Then, you cannot derive a conclusion. That means, the conclusion, denial of the conclusion leads to the contradiction. That means the actual conclusion is true. So, now before doing natural deduction, there are some rules which we will be following. These rules are like this.

(Refer Slide Time: 17:50)



So, these are basically two different methods that we will be using. First one is RCP conditional proof. So, what we will do here is, this thing. So, we have some kind of assumptions. A which, usually comes from the given formula. And we obtained a line, where you found B. So, now B is a tautological consequence of A. If that is the case then A implies B is considered to be a tautology. So, what actually happens here is, this thing.

(Refer Slide Time: 18:27)



So, you have A here. And then obviously after few steps, after applying natural deduction rule you got B. So, that means A is derived from B. And B is derived from A. If that is a case, then we can simply write like this, A implies B. You draw a line like this and then say that, A implies B is derived from some kind of assumptions. And natural deduction rules which you employ, that helps us in moving from A to B.

That means, A implies B is the one which you have derived. So, this is called as conditional proof. So, that means in your derivation, suppose if have a step A and from that you got B. Now, you draw a line like this. Then, you see that you obtained A implies B and all. Because, this says that B is derived from A. So, that means if this goes to the right hand side, then this becomes A implies B.

Suppose, if I say this is a single turns state, this means that, it is a theorem. This should be read as like this thing. It is a theorem that, A implies B. A implies B is a theorem, within that formal. This is in that system. That is a natural deduction system. So, this is what we called as RCP, that is conditional proof. ((Refer Time: 20:00)) So, now the second thing which will be using in the natural deduction method is this thing, which is called as Reductio ad absurdum.

So, in this Reductio ad absurdum, you started with some kind of assumptions. It can be premises and it can be hypothesis. You can call it as hypothesis. And in all the way down here, in your proof you got some kind of contradiction. So, usually it is mentioned in this way. So, there are two symbols which we are trying to use. This is the symbol, which we used it for a formula which is always true.

And there is something which is called as formula, which is always false. If this is stopped and this is false. This is different from p and f. p and f are truth files. So, now what happens here, this is that. From A, all the way down somewhere else here. We got a contribution. So, that means A implies this one. So, from A you got contradiction and all. So, that means, if A implies a contribution, then that means, it is not simply A and all.

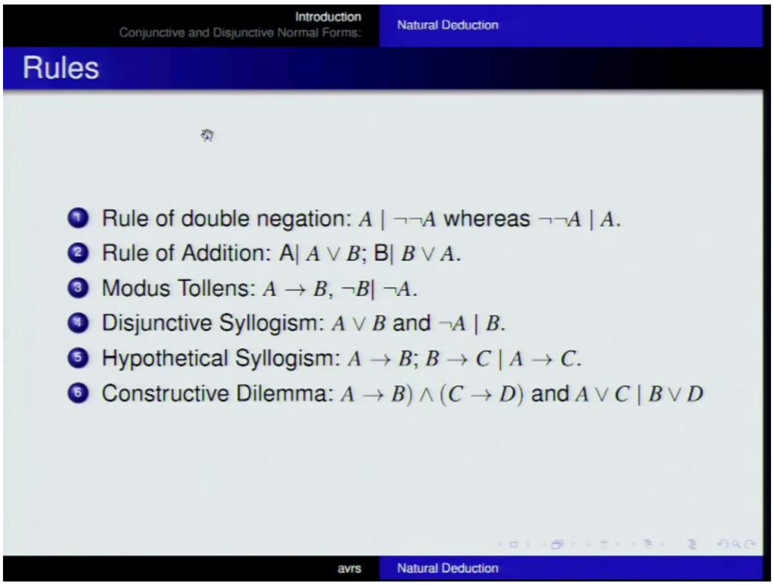
It should be not A. So, this particular kind of thing is called as Reductio ad absurdum. It essentially say that, given some assumptions which leads to contribution. That means, let us see if your assumption is A and that laid to contradiction. Then, the actual assumption is false. That means, it should be not

A. Suppose, if you begin with not A and began to contradiction and that should be A.

So, this can be also written in this way. From not A, it led to contradiction. Then, it should be not not A, that is A. So, these are the two important methods, that we will be using in the natural deduction. So, the idea of Reductio ad absurdum is that, if you derive the contradiction from a given assumption, then that assumption is false. That means, it has to be not A. So, that is what we have set here.

So, these are the things which we follow. But, the actual founder Genzean. He proves the theorem in a totally different way and all. some mostly, only in few logic text books, you will find the proof which are given by Genzean. But, usually we follow just closely keeps proofs in particular, which seems to be little bit simpler and simpler to understand. But, both the proof can be used.

(Refer Slide Time: 22:47)



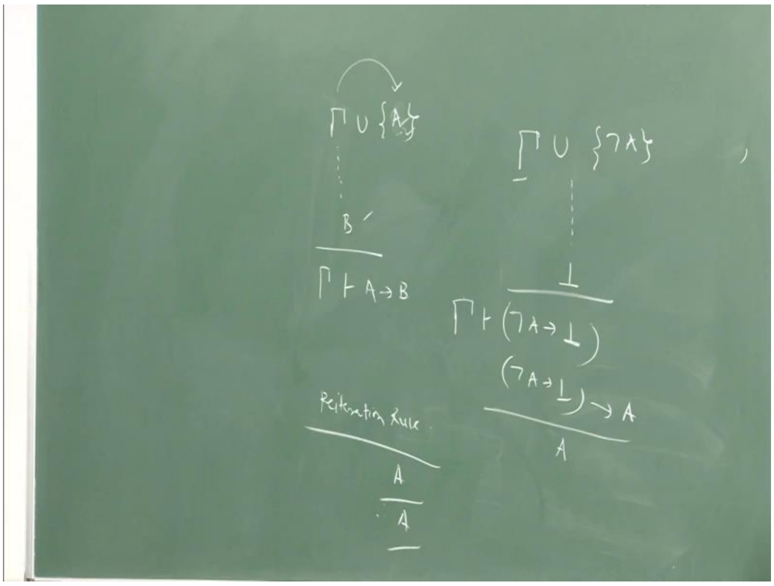
So, now what are the rules that we will be using in this natural deduction method. So, this can be compared with playing a game and all. For example, let us say if you playing cricket. You need not know, how to know and when it is according to existence etcetera. You know, how to know the foundations, principles etcetera. What you exactly need to know, is the rules of the game.

You have to know about wide ball and we have to know about no ball, etcetera and all. So, once you know these rules, then things would be relatively easier. So, then there will not be any defects in your game or mistakes which can happen. In the same way natural deduction, you start with assumptions and then you have some kind of rules. And these rules are like, this particular kind of thing.

So, the some rules are considered to be discharging some kind of assumptions. And then some of the rules you will introduce, some kind of implications etcetera and all. So, these are some of the rules that we will be using in the natural deduction method. You might ask you have to remember all these rules etcetera and all. It comes to know through practice. We will not be much to remember in all these things, obviously follows.

So, these are some of the rules that, we will be using in the natural deduction method. So, first let me list out these rules. And then we will talk about something about these rules. And we will move on to some other, I mean moving certain kinds of theorem.

(Refer Slide Time: 24:36)



So, these are let us say gamma. It is some set of assumptions that we have. And from this, let us say union. That is the formula B. And from this you got. So, this is A. And from this, you got B. This says that, gamma is something like set of formulas which is setting at the background. So, it can be some kind of natural deduction rule or principle or whatever it is.

And these things, which you are adding it to the assumption which occurs in a given formula and from this, you got B. So, now what you will do is, your assumptions are A and B and all here. Now, you discharge these assumptions and then you will write simply this thing. From gamma, you obtained A implies B. So, now earlier it was only a proposition A and proposition B and all.

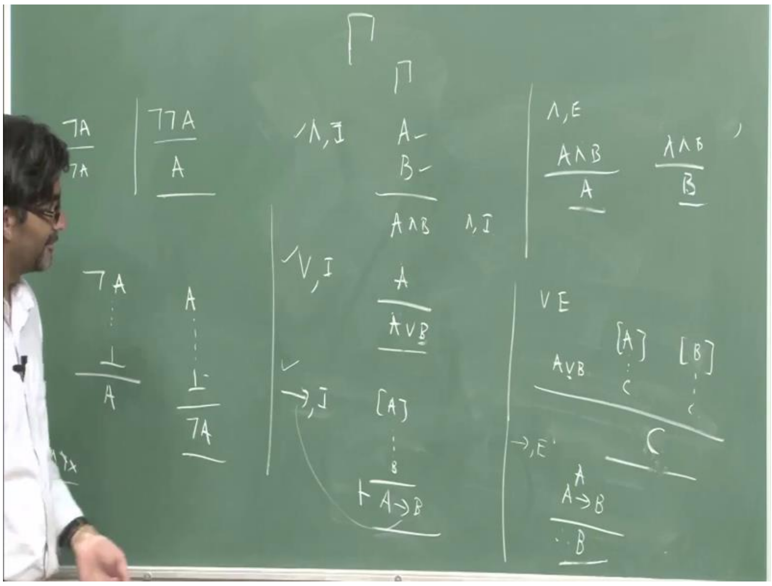
Now, you discharge those assumptions and all. Instead of saying that, it is A and B and all, we say now A implies B is the case and all. So, now you discharge this assumption and all. So, this is one of the rules of which come under the category of discharging the assumptions and all. In the same way, if you have formula a like this thing, not A. So, you might ask what is this gamma and all.

Gamma involves all these truth preserving, principles and all. It can be this thing, A implies B in which we will be talk about little bit later. So, there are all truth preserving principles and all. Or it can be consider as a kind of tautology and all. But, it is not axioms in the case of natural deduction method, something which is true. And now, from this let us say, if you got something called the contradiction.

So, that means what you got is this thing, not A plus contradiction. So, this is what we got. So, that means not A implies. This is a case and it should be A. So, now you simply write A. So, this is one thing which we will be using. There is something called reiteration rule, which is come from some kind of redundant kind of rule and all. Suppose, if you have a formula A and then you can simply say it is A.

So, there is nothing great about this particular kind of formula. If we have A, you can derive obviously A on here. So, this is a reiteration kind of rule. So, now there are some other kinds of rules, which are there for conjunction dissention etcetera. So, when I am talk about some kind of rational for these particular kinds of rule.

(Refer Slide Time: 27:39)



So, in all these things, gamma is already there, something which is true and all. So, now this stands for conjunction introduction, I stands for introduction. So, when you will do that particular kind of thing, you obtained A and you obtained B, in your proof. Let us say gamma and then all the way down, in your proof. You got A and you got B. Then, you will simply write. You introduce the conjunction and then you will write A implies B.

So, now this rule is called as conjunction introduction. So, now conjunction elimination. So, each natural deduction rule comes up in some kind of phrase and all. One is introduction another one is the elimination. So, each connective will have this particular kind of rule. For the conjunction, if we have A B, you proved A and you also proved B. Then, A and B can also be proved from this particular kind of, using particular kind of truth preserving rules.

Conjunction elimination is like this. If you have a formula A and B, you can simply write A. It is like, it is raining and the grass is wet. From that, you can prove grass is wet or you can even prove, it is like raining. So, what we have done here is that. If you eliminate this conjunction and then in your proof, you substituting it with something simply A. And in this way, in the sense A AND B, B also can be derived.

So, this is called as conjunction elimination rule. So, now this stands for disjunction introduction. Suppose, if you have a formula A. That means A is a true. And you can safely add any other thing without disturbing the truth value of this one. So, in your proof, each and every step has to be true. So, without disturbing the truth value of your propositions, you can safely add another B because, if the b value even it is true or false, it is not going to affect the truth value of the compound statement.

Even if B is t and this all become true. Even if B is false, then A OR B is again true only. So, you have to ensure that in your proof, all the steps are obviously true. Then, only the final step is going to be true. That is the theorem. So, in that sense, that is going to be a tautology and obviously, that formulas going to be valid. So, our effort is this that, ultimately we generated kind of valid formulas.

And then we are trying to prove these valid formulas here. And for proving, there is some kind of condition method, that is the natural deduction method. So, this is what is the case. The disjunction elimination is like this. So, now we have shown that A OR B is true. And then from A independently you proved this thing C, that is what you write it here. And then from B also you got C.

That means, C is reduced from A and C is reduced from B and A OR B is already there. If that is the case, then you remove this disjunction and you will simply state here. So, it simply says C. So, this is what is called as disjunction elimination. So, you have A OR B, which is already true. And then from A independently you proved C. And from B, you proved C. And that means, from all these things you can prove C.

You can eliminate this disjunction and then show that, this is the case. Suppose, if these are not there, then you cannot say that A OR B, c can be reduced. So, this is another rule. The rational of this rule is that one, which we have discussed. So, now as per as this implication is concerned, again we have introduction rule. So, especially from A, let us say and you proved B. Then, what will you do is, you will introduce this implication.

And you will say that, A implies B is reduce from this thing. So, this is what we mean by introduction. So, there only assumption A and assumption B is there. And then now we are introducing this thing, because B is reduced from A. So, that is why A implies B will come as theorem of this one. So, now this is, what is the introduction of implication? And in the same way of the other side, introduction elimination, when I write e stands for elimination, I means introduction.

This will be like this. For example, if we have a formula like this. This is usually ((Refer Time: 32:51)) and all. So, now you will eliminate this implication and you can say that, it is simply B. So, what is that we are trying to do is, we are simply making use of some kind of rules. So, what are these rules? These are truth preserving and all. You see any one of these rules. Obviously, for example if you take this one.

If A is true and A implies B is absolutely true, 100 percent true, then obviously B has to follow from this. This is the principle of valid reasoning. So, instead of starting with axioms etcetera and all, we start with only these rules. But, the problem here is that, one has to remember this, some of these rules and all. Of course, with some kind of practice some kind of strategies which you can used with that, we can say you can usually remember this simple rules and all.

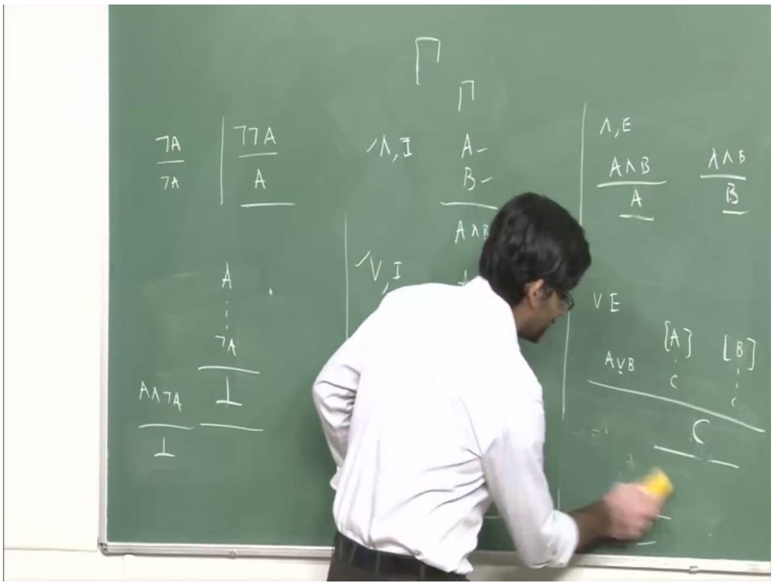
Or if you do not know some these rules, you can derive these rules from some other kinds of things which you have already know. So, this is one thing. And then we are talking about conjunction, disjunction and implication. And the negation of this one, is like this. For example, if you have a formula A and you can derive only this one. And then negation elimination is like this. If you have not not A, and then you negate this elimination and you simply write A in your proof.

So, that is what is with respect to negation. So, now there are some other rules. For example, if you have not A and then from this, you got a contradiction then this means A. It is not A, but it is A. In this same way, if you have a formula A and then you got contradiction. Contradiction sense that, you got x and not x. That is a case, it is called as contradiction. Then, since A let to contradiction, it should be not A.

So, that is a rational of this particular kind of rules. It is just like, you know in playing. While playing cricket, you will be discussing about what you mean by no ball, what you mean by wide ball or when you will get five runs etcetera and all. All these things, you will be discussing. What you mean by a boundary or sixer or something like that. All these things are some kind of rules, which the game.

In the game subsequently, we involved with this particular kind of rules. Just like that, logicians while practicing with number of theorems and all. Proving number of theorems, they came up with this particular kind of rules. These are all truth preserving rules is knowing, which you can saying that if this is true, this is true then it cannot be in case that A AND B can be in false side. So, that is the case deductive argument and all. So, these are some of the rules which we have. And then there is one more thing, which is important.

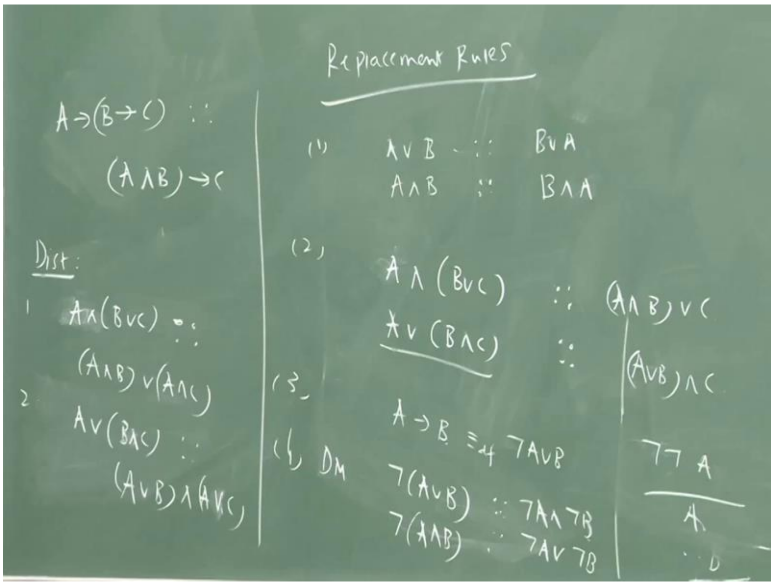
(Refer Slide Time: 36:00)



So, whenever if you have A, and then if give up with not A. And then so this let to this contradiction. We derived A and you derived not A also. So, obviously that is a contradiction. We cannot simultaneously say, that it is raining and it is not raining. So, then this leads to this one. What is happening here is, again you removed the conjunction and then it is actually A and not A.

Now, we can simply remove this conjunction and then you say that, it is simply the contradiction. And in the same way, this is what we have already discussed now. So, these are some of the rules that, we need to remember before solving some kind of problems and all. And then there are some obvious rules such as modus tollens etcetera and all. There are already there, setting at the background.

(Refer Slide Time: 37:04)



So, there are some other rules such as replacement rules. So, what are this replacement rules. So, for example we have a formula A AND B. By commutative property, it will become B OR A. And this happens for conjunction also. This double colon stands for equivalents and all. These two are logically equivalent to each other. It can be B AND A. Suppose, if you find A AND B in your proof, you can easily substituted with B AND A.

So, these are some kind of replacement rules. They come in place and all, conjunction and disjunction. It is commutative kind of rule. So, now we have another kind of thing, A AND B OR C. Suppose, if you find something like A AND B OR C, then replaced it with this particular kind of thing. This is A AND B OR C. You can write it, in this particular kind of way.

So, in the same way if you have A OR B AND C and this is same as particular kind of thing, A OR B associative law and then C. That means, whenever you have this particular kind of formula, you can simply replace it with this and then what it says. And then third set of rule is you see. Suppose, if you have formula A implies B and you can. This is by definition is not A OR B.

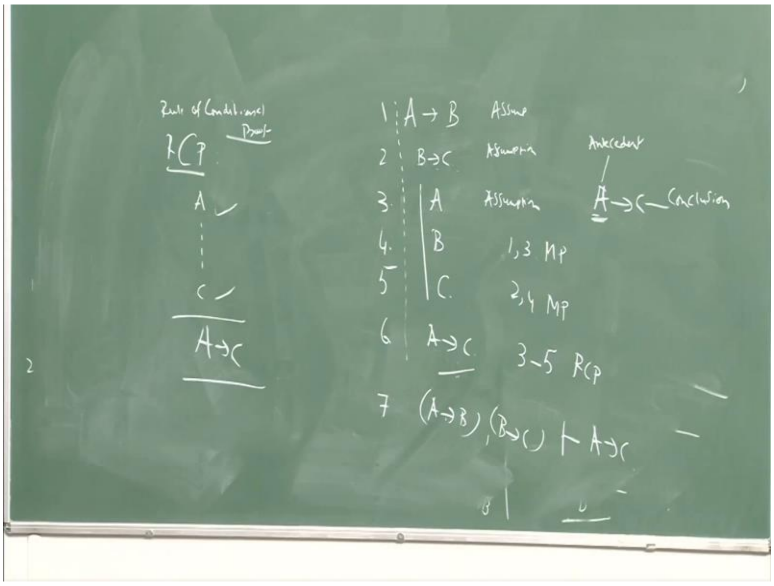
You can simply substitute, for A implies B not A OR B. And then there are some kinds of De Morgan rules, which are obviously we know. For example, if we have A OR B, negation of A OR B. And you can simply substituted it as negation of A AND negation of B because, negation of disjunction is conjunction. In the same way, negation of A AND B is same as negation of A and then negation of conjunction is disjunction of B here.

So, like this we can list out all these rules and all. Another rule is not not A is there. In your proof, you substituted it as simply A. And then there are some other kinds of rules, like A implies B implies C. If you have this particular kind of thing, you can simply substitute as A AND B implies C. That means, whenever you come across A implies B implies C, that can be replaced by this particular kind of thing.

That is why these are all called as replacement rules. And finally, without A AND B OR C this is same as. This is the distribution law, you know. Distribution law says that, if A AND B first one OR it is A AND C. This is the distribution law used on conjunction. And distribution law, in the same way can be used for disjunction law. And suppose we have this thing and this is same as this particular kind of thing, A OR B AND A OR C.

So, now you might be asking, writing so many rules etcetera and all. So, now what is going to do this particular kind of thing? So, now we try to prove some simple kind of formulas. And then we try to see how we can prove this particular kinds of theorems using the rules of inference we have seen so far.

(Refer Slide Time: 41:21)



So, now this is the one which we are trying to begin with obvious things, A implies B and B implies C. And all this is what we know that, which comes as an outcome A implies C. So, now what we will do in the actual deduction method, is simply we write these things assumptions and all. This is the assumption one and this is another kind of assumption two. So, this is what we are going to get.

So, what we will do here is the thing that, you add principles of natural deduction. The one which we have seen, the rules of natural deduction and ultimately you will generate A implies C. So, now how do we get this A implies C is the one, which you are trying to see. So, now we start with these assumptions and you will work out, till you generate this particular kind of thing using the principles of natural deduction.

So, now what we will do here is, you will assume the antecedent point of the conclusions. So, this is antecedent and this is the conclusions. So, now what we will do here, you will assume the antecedent of a conclusion. Again this is an assumption. In some text book, this assumption is written as premise and some other written as hypotheses. It does not make big differences. So, now this is the thing.

So, now we assume these things. Now, we need to use rules of inferences, truth preserving rules. So, now observe one and three. One and three is modus ponens.

What is modus ponens? If A implies B is a case in case B and then you can get B. So, that is truth preserving rule, which we will be using. A implies B and A, you get B. So, this is one and three modus ponens. So, now observe two and four.

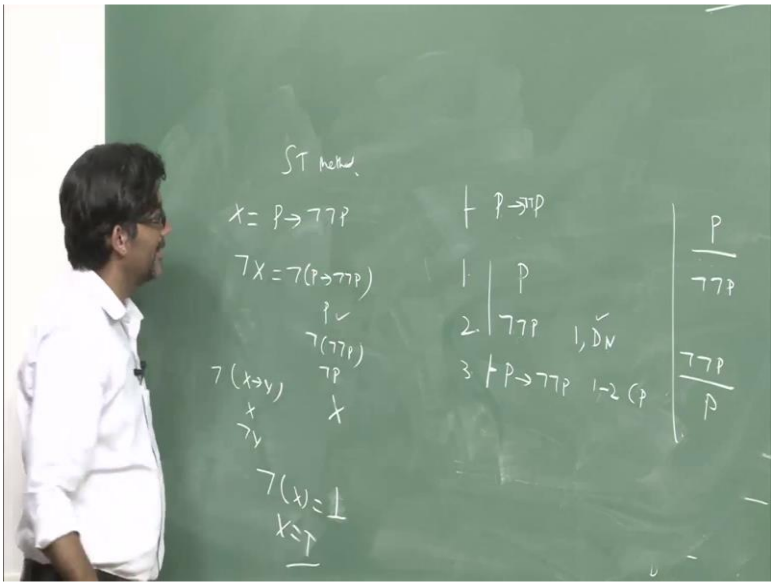
Two and four, modus ponens will give us C. B implies C and C is a case. So, now we have used some truth preserving rules. And now, from A you got C. So, now what is that? There is something called, in the beginning we discussed about RCP, the Rule of Conditional Proof. This is, what is RCP? So, from A suppose you got C, that is what is the case C. From A, you got C.

Then, you can discharge these assumptions. You give up this A and give up C and all. And then ultimately you say that it is A implies. So, for that one what you do here in the natural deduction method. It is that, you draw a line like this from A to C. And then in the sixth step what you will see is, from three to five RCP, according to the rule of conditional proof, which is there. Then, you simply write A implies C.

So, now A implies C is the one which you wanted to derive. So, now you draw another line, here from the whole thing. A implies B and B implies C, you got A implies C. So, now what we will do here in the step two, A implies B and B implies C and from this, you got A implies C. So, this is what we have reduced. So, like this one can derive many formulas. For example, let us try to talk about some simple things and all.

Like, which will be quite difficult in the axiomatic system which we are going to see little bit later. So, now we are trying to prove p implies p l. So, it is involves in axiomatic system you take in to consideration, it involves at least four five steps.

(Refer Slide Time: 45:52)



But, in natural deduction it is quite simple. So, this looks very strange for us. Actually, this is considered to be the proof. So, now we started with the assumption p. What you need to prove is p only. So, now you write again p only. So, why you written this thing? If p is there, so you can reduce p automatically. That is reiteration. You can remember this time, you can n number of times you can write this same p again and again.

It does not make any sense and all. So, this is reiteration. Suppose, p is reduced to be true, I mean p is derived and again p can also be derived. Now, you draw a line like this and you will say that p implies p is derived and all. So, this is the most simplistic kind of proof that, one can used. It does not make any sense and all. But, it involves according to the rules of inference and natural deduction.

It involves only two methods, two steps. In the same P implies p, in the given axiomatic system. Sometimes, it may take eight steps or may be nine steps and all. If you can start with the axioms and then you can use the modus ponens and rules of transformation. And then may be after six or seven steps, you will get the desired things P implies p. In the same way, if you want to prove p implies not not p.

So, if you want to derive p implies not not p, again in the natural deduction method you start with an assumption p. Now, so two whenever, you have p you can substitute this particular kind of thing. In the same way, in not not p, you can substitute p. So, now this is not not p. One and double negation, so d n stands for double negation. So, p is there.

You can substitute not naught. You can substitute four not also. It is going to be p only. So, now since you have derived p from not not p. You draw a line like this. And the third step, you will say that p implies not not p. That is one to two conditional proof, because not not p is reduced from p. So, that why, p implies not not p. You write it like this, to say that this particular thing is consider to be a theorem.

So, you must take care of this particular kind of justification. These are the steps and all. And the right hand side whatever used, you find it here. It is a justification for writing this particular step. So, how we are justifying this thing? Based on the truth preserving rules, that we already have. So, this says that one, we have applied double negation do it and this is we got.

And then since not not p is reduced from p. Using the conditional proof, you can say that it is p implies not not p. So, this is the derivation of theorem, that is p implies not not p. The same thing, which you can do it in semantic tableaux method. In semantic tableaux method, what you will do is... So, in this what you will do is, you take this as x. And then you negate the conclusion, that is p implies not not p.

And then you will see whether the branch closes or not. So, now this is p. The first one, this is not of x implies y is simply x and not y. So, you write the same thing and then you substitute another not p now. So, not of not not p is p, in the negation of p. So, this is not p. So, now you derived p and not p and all. So, hence this branch closes, because of this particular kind of ((Refer Time: 50:18)) there p. And you have not p. So, the branch closes.

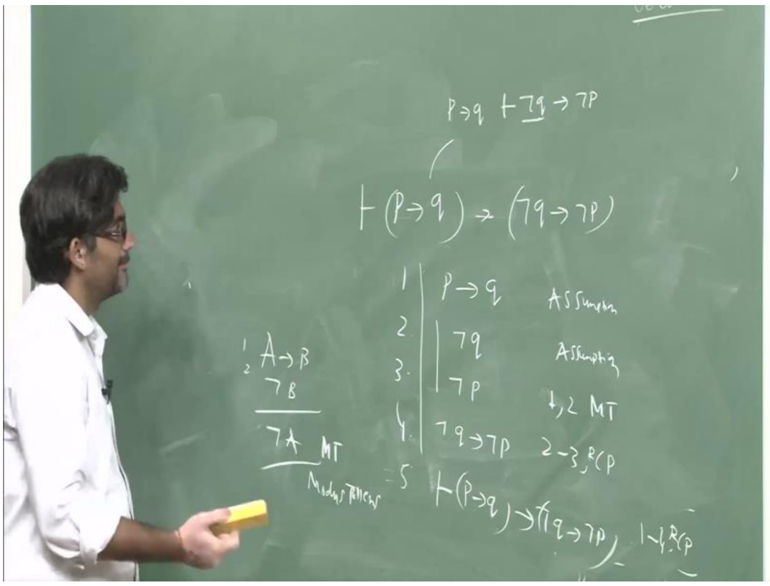
So, that means what we have say not of x is false. Usually, you write it in this way. So, x has to be t. The original formula should be t. That means, it is the valid formula. This is another way of showing it, using semantic tableaux method. But, in the natural deduction method of course, things are little bit simpler. You just use one double negation rule and then you got the answer.

But, here little bit of some more rules which we are using. Of course, this is an very efficient method, which also involves one two three, three steps. So, these are considered to be an effective kind of methods for proving this particular kind of formula, is a valid formula. So, however we showing that this particular thing is a valid formula, because it started with a truth.

And then we used principle of natural deduction which is obviously, truth preserving kind of rule. Truth preserving, truth preserving are ultimately, conditional proof also preserves truth and all. So, obviously the final step of your theorem is obviously true. So, that means all the tautologies are obviously valid formulas. So, there is one important theorem which tells us that, something provable.

So, like in this way in the natural deduction method and that has to be true, that is the tautology. In all tautology, they are obviously valid formulas. So, compactness soundness etcetera takes care of particular kind of thing. Proposition logics are all complete, sound and even consistence as well. These are some of the things, which will be doing. And then some more proof which we will try to ((Refer Time: 52:10)). So, the more and more you solve these problems. The more and more, we will get expertise this particular kind of method.

(Refer Slide Time: 52:22)



So, we all know that, this is law of contra position from p implies q, you will get not q implies not p. So, this is the one which we to prove. First, you will write, the assumption p implies q. This is the assumption. And then this can be written as p implies q. And then not q implies not p. Now, you take the antecedent kind of this conditional not q. As your assumption, this is also an assumption.

So, now three there is a rule again in the natural deductions. That is, if A implies B is the case, then not B in the case then it should be not B. That means, you denied the consequent here. In the 2nd step, you have denied the antecedent also. This is called as modus tollens rule. So, now with these two things you can derived not p implies q. And you denied the consequent, you denied the antecedent also.

So, now this is what is the case? So, now you draw a line like this. What did you get? From q, you got not p. So, not q and not p. So, now you need to write the justification for this one. One into modus tollens, you will get this. You have to write justification. Otherwise, nobody will be able to understand this particular kind of steps. And usually, that is not considered to be an effective proof and all.

The justification is to be given. After all, logic is used as a justificatory two. So, now not q implies not p, is the one which we got. So, now you draw another line like this and say that from p implies q. So, how did you get this one. not q implies not p, 2, 3. From steps 2 to 3, conditional proofs. Rule of conditional proof, you wrote this one.

And now, you say that in the 5th 1, you write like this because, it is the theorem. So, from p implies q you got this one, p implies q you got not q implies not p. So, this is what we wanted to prove. So, how did we get this one? So, one four conditional proof or you can write rule of condition proof, you got this. So, this is the way in which you can show that, the law of contra position can be derived in your natural position method.

So, let us talk about law of excluded. So, in your formal axiomatic system you should ensure that, at least all these laws the fundamental laws such as p implies p. That is the law of identity and p or not p is a law of excluded middle. And law of contra position, law of non contradiction, all these thing should come as. You should be in a position to derive these things first.

And the rest of these things obviously follow. Rest of the complex theorems etcetera will automatically follow and all. So, in the next class we will be talking about some more complex proofs, based on natural deduction. And then we will see how this method can be considered as an effective kind of method for proving certain theorems and all. One of the advantage of this method is that, your proofs can be singular and it might involve very few steps and all.

So, in this lecture we have presented natural deduction method. Natural deduction method is based on, some kind of truth preserving rules. And we stated those two preserving rules. And we discussed rational for using its truth preserving kind of rules. And then we proved some simple theorems, such as p implies p and law of contra position etcetera. In the next class, we will begin with we will consider some more complex proof and all, based on natural deductions. So, that we will become well equated with this particular kind of methods.