Introduction to Logic
Prof. A. V. Ravishankar Sarma
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

Lecture - 29
Hlbert Ackermann Axiomatic System

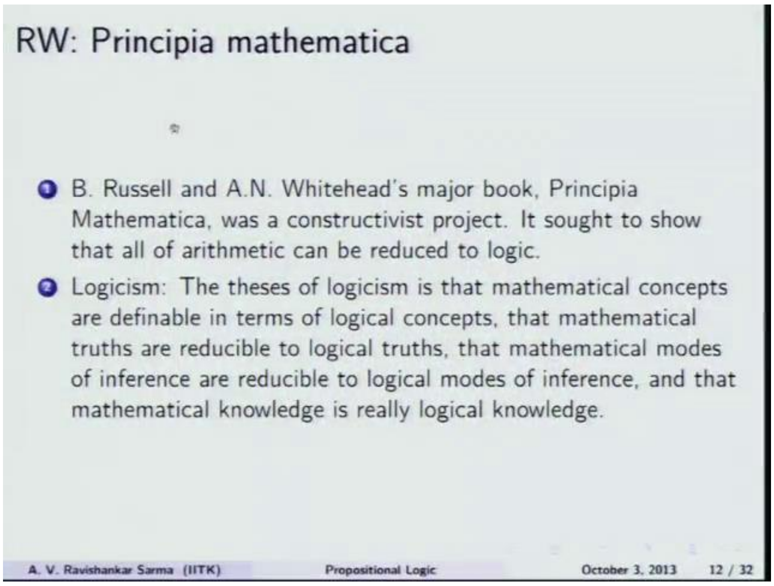
Welcome back. In the last lecture, we just presented 1 axiomatic system. We just introduce what we mean by axiomatic system and what should an axiomatic system should consist of. So, these are the things which we have discussed in the last class. Today, I will be presenting an axiomatic system in the propositional logic, that is, due to Bertrand Russell and Whitehead. So, we will be taking into consideration, a portion of the famous book principia mathematica, where Bertrand Russell and White Head talked about deduction.

So, we will be focusing our attention on that particular portion of that book. And then we will to trying present is axiomatic system and in the best possible manner. So, any axiomatic system should have these thing 3 things at least. So, you should have to begin with, you should have some axioms, which does not require any proofs. So, they are like self provident proofs etcetera. And then these axioms if you use some kind of substitution rules and transformation rules, this axioms transforms into another kind of statement, which you call it as theorems.

So, what we have are a first to start with the axioms and then these are changes into theorems, with the help of transformation rules etcetera. And then only 1 rule that you will be employing here, that is, the modus ponens rule, which is also called as rule of detachment, from alpha and alpha implies beta we can obtain beta. So, now, you will clearly see here, what we are essentially using is as many few rules as possible. And you should note that, these rules are truth deserving kind of rules and the axioms are; obviously, 2 statements which an always we have the absolutely true. And whatever, you substitute in that 1 uniformly and that axioms will generate a particular kind of theorem. And then will you applying the rule, which is also considered to be truth deserving kind of rule. So, it will generate you will generate only theorems. So, that means, what essentially we are trying to do this is that, any given formal language let say l, you are trying to find out proofs for how in the value formulas.

So, now, you know that some formulas are valid, may be just by means of some truth table method etcetera and all. So, now we are trying to generate proofs of some formulas by a means of some particular kind of syntactic method, which is called a axiomatic propositional logic.

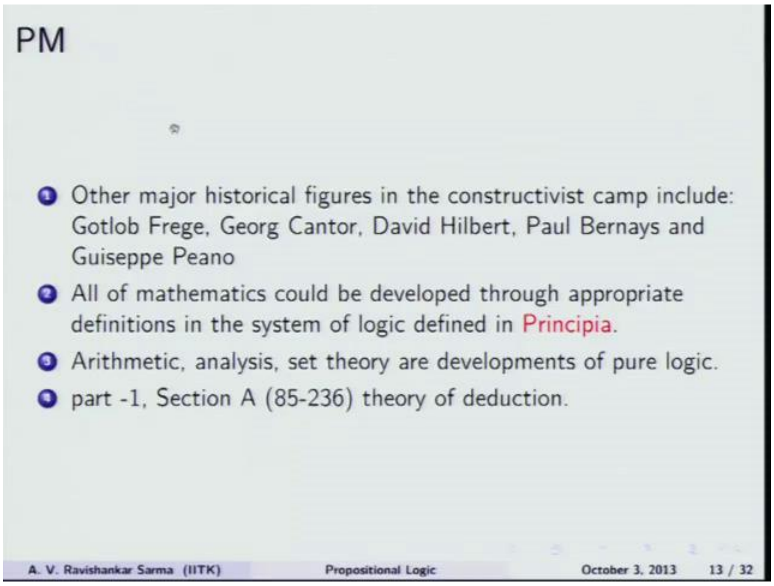
(Refer Slide Time: 02:59)



So, now, Russell and White Head in his famous book principia mathematica, he has presented this particular kind of axiomatic system. So, this grand project is called as constructivist project, his focus is on arithmetic. So, the suit show principia mathematic suit to show that, all of arithmetic can be reduced to logic. So, this is famously popularly known as logicism. So, what is logicism? It is that mathematical concepts we take any concept in arithmetic.

So, will be focusing or attention on arithmetic, the same things can be extended to even geometry also. So, it is this is that mathematical concepts are definable in terms of logical concepts. All the mathematical concepts will be find some kind of notation in the logic and also that, mathematical tools are reducible into logical truths.

(Refer Slide Time: 04:25)



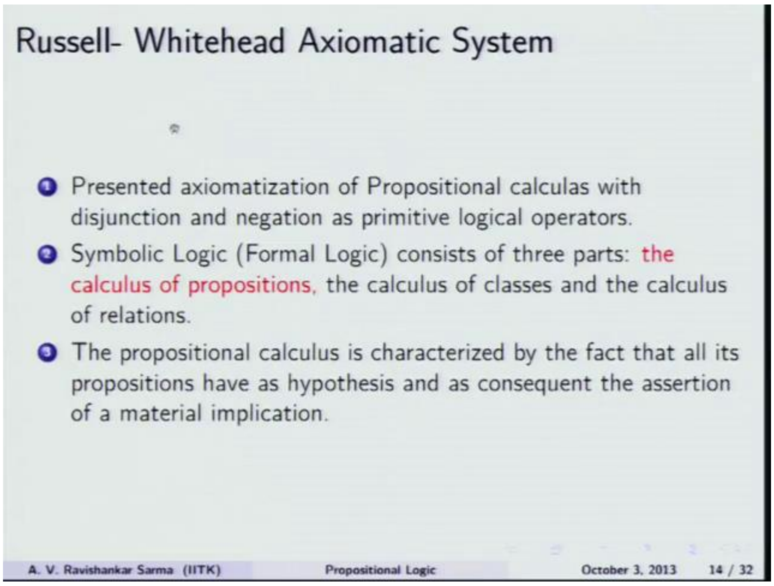
In mathematical modes of inference have also reduced to logical modes of inference. And mathematical knowledge is in that sense, essentially in logical knowledge. So, if you can reduce mathematics and all there is the main thesis statement of logic is a ... Mathematics can be reduce to logic, then in that sense, mathematic will save as a branch of logic. So, there are other kinds of constructivist axiomatic systems, which you will find in the literature of history of logic. So, there due to it initially it start with, we have Gotlob Frege axiomatic system and David Hilbert, Paul Bernays and Peanos arithmetic etcetera. All this are examples of constructivist camp and they belong to particular kind of program called as the logicism or formalism.

So, now, all of mathematics we can we can be developed through appropriate definitions in the system of logic as defined in the principia. Principia the main thing which you will find it, these is the essentially that project is all about reducing arithmetic to logic. So; that means, all the statements of arithmetic etcetera, defined some kind of corresponding translation in the language of logic.

So, arithmetic analysis set theory are in the branches of mathematics, will now become part of few logic. So, now, will not be focusing on entire book of principia mathematica, but will be focusing our attention on part 1 of the book, where he mentions about;

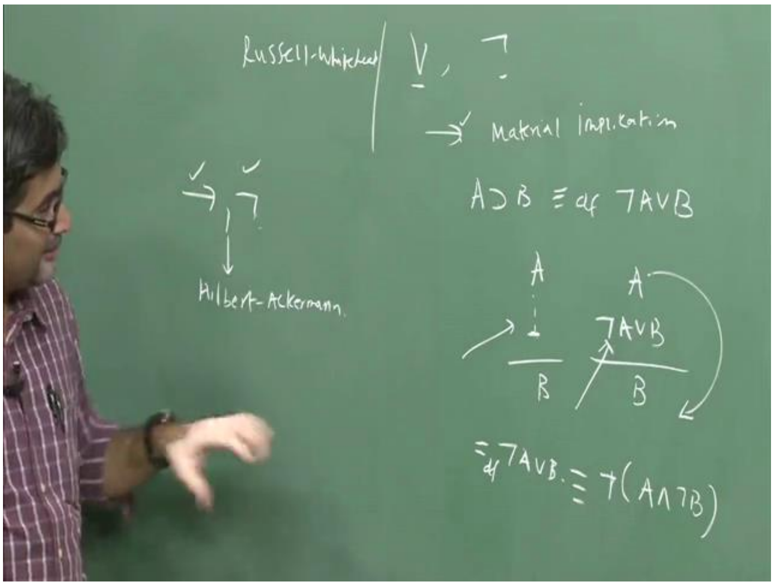
Russell Whitehead mentions about theory of deduction.

(Refer Slide Time: 05:45)



So, Russell Whitehead arithmetic system is like this. He presented arithmetic system of proposition logic with only 2 variables. So; that means, 2 logical constituency.

(Refer Slide Time: 05:59)



So, different logical systems have different kind of, they use different kind of symbols. For example, in the case of Russell and Whitehead, the only primitive symbols that you will find, the logical symbols that you will find are disjunction and negation. But in the absence that I am going to mention in a y from now, you will find mostly this particular kind of symbol. So, this stands for material imputation. Since, it is easy to write in terms of au implication, so which is better to it is easy to use material implication. So, now, that particular kind of material implication Russell Whitehead uses this particular kind of symbol and using this particular kind of symbol.

So, this by definition is saying as not A r B. So, not A r B means A implies B only. So, how did we come to this particular kind of definition, you must looking for a solution for this particular kind of thing, when thing we say that A materially implies B. So, what kind of substitution 1 needs to make here so that, you can move from A and then some more statements to B? In that sense is of the you that B can be reduced from A.

So, now, he was looking for he was experimenting on various kinds of things here, substituting at implies of the missing here missing implant here. So, when you substitute this particular kind of thing, there is a possibility of B from A to B. It is in this science A materially implies B. So, that is the reason now this particular thing not A r B as holds as definition. So, this can be return using demorgan's laws, as this it is not that A and not B. So, this is same as this.

So, now, in the Russell Whitehead axiomatic system, you will find only this junction and the negation. So, there the minimal kind of logical constant that you will find it in the Russell Whitehead axiomatic system. Another choice could be simply implication and negation. This is what where you will find it in the next axiomatic system that you will be talking about, you will be talking about, which is due to another great mathematician Hilbert and Ackermann. He makes use of this 2 logical symbols in his axiomatic system whereas, Russell choice was this thing this junction and negation.

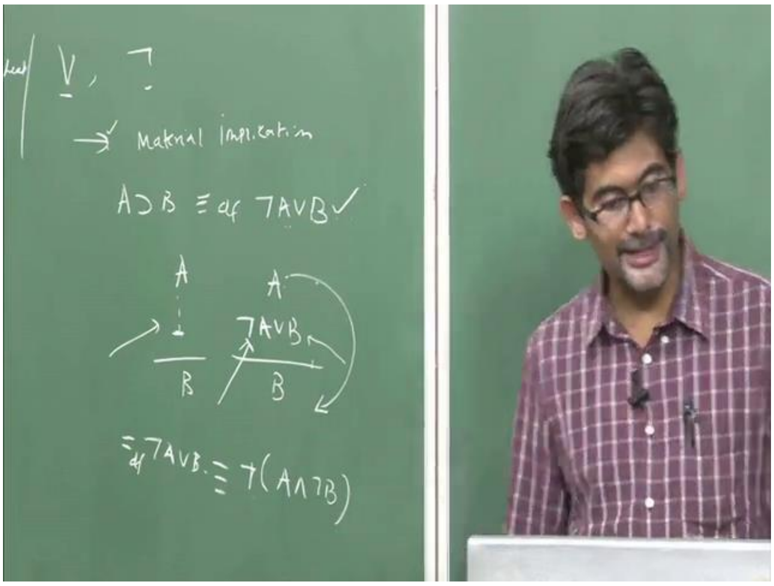
But mostly you will find axiom is form implication form because, it is easy for as to write it an all. So, but actual translation should be in the form of disjunction, you will find you should find only disjunction and negation and that it in all the exits. So, he presented and examination of proportion logic with only disjunction negation as primitive logical operators. Symbolic logic according to him consists of all these 3 things, but will not we focusing or attention all these things. First is a calculus of proposition, calculus of proposition is a is 1 step then the 1 proposition changes to another 1, that is a p plus q plus p plus substitute something into a which will changes to another statement. So, it is in that sense, change of proposition is nothing calculus of propositions.

The other 1 is about calculus of classes, something which you said theory and the other 1 is the calculus of relations. But will be focusing or attention on calculus of relations calculus of propositions. Basically we will be talking about a particular kind of called deduction. What you will be reducing? We will be reducing some theorems based on the axioms that were presented by Russell and Whitehead.

So, what essentially we are trying to talk about what is simply like this is; start with you have 4 or 5 axioms and then you have some kind of transformation rule and you modus ponens. And now you can deduce, whatever you think is a truth in the arithmetic, can be deduced by using the logical notations and you can reduce a truths all these things. That means, you know, deducing truth means, you have proving the particular kind of; obviously, where the valid statements, valid truths that existing in your formal logical system.

So, now, come to Russell, propositional calculus is characterized by the fact that, all its propositions have as a hypothesis and as consequent, the assertion of material implication.

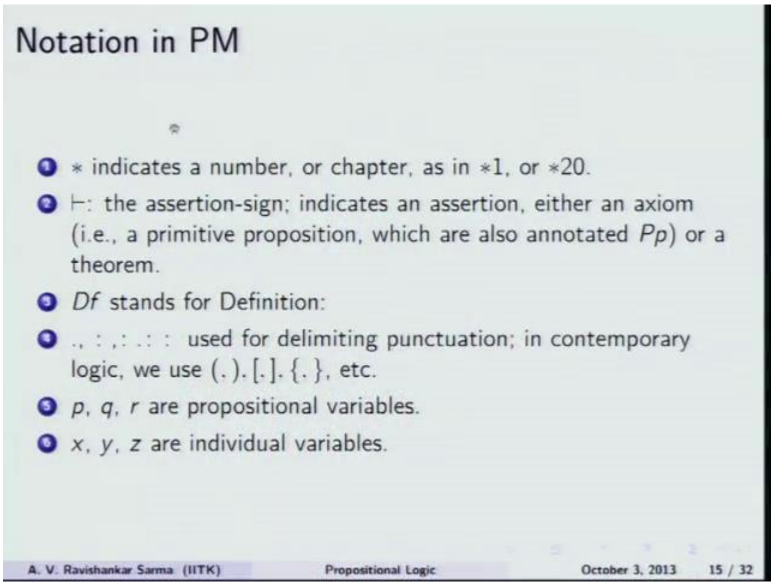
(Refer Slide Time: 11:05)



So, what is central to Russell and Whitehead axiomatic system is; this particular kind of material implication. So, A materially implies B for Russell and Whitehead is like this. In all the case of A are r B is the case. So, this is the way we came up with this particular kind of thing; A material implies B, only when you can make this particular kind of substitution. Off course this substitution is same as this 1; it is not the case of A n not B.

So, what is central to Russell Whitehead arithmetic system is; the material implication. If, this is missing, then there is no way in which we can move from 1 proposition to another 1. So, 1 till any were proof, each step is considered to be part of the proof and all. You cannot move from the 1 step to another step, without invoking this particular kind of concept that is, the material implication. So, now, it is in that sense, all of its proposition has as hypothesis. And its consequent, consequent means next step follows from that particular kind of proposition is considered to be an assertion of some kind of applying material implication.

(Refer Slide Time: 12:50)

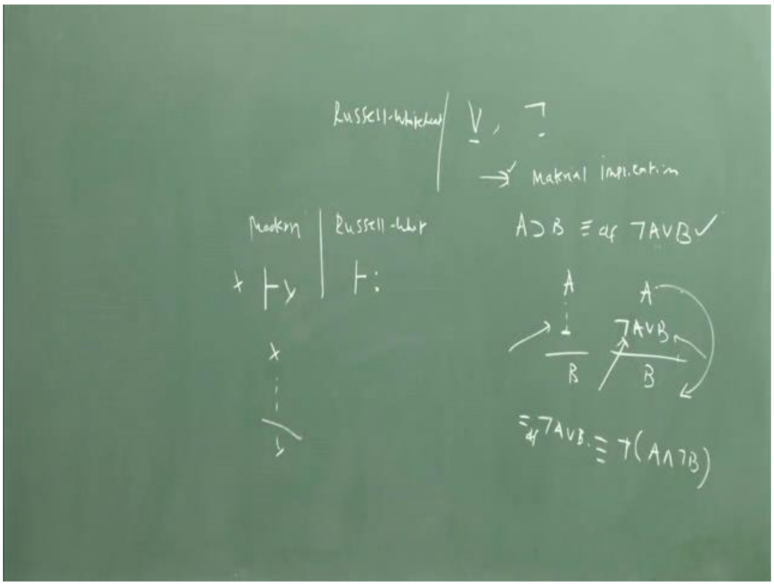


So, the definition of that 1 is that, what is see on the black board. So, when you see the original work of Bertrand Russell and Whitehead the book principia mathematica, which has 3 volumes. The notation would be very difficult to follow, but you were using the different kind of notation, but is more or less, were convening the same kind of information, were anyone interested in the actual notation and all, they should looking to a principia mathematica, but just for the sake of our understanding, I mentioning the notation that is used by Russell and Whitehead in his path making book; the principia mathematica.

Usually, you will find some of the symbols and all, but he might some more symbols. But this movement, I will be restricting our attention on Positional logic axiomatic proposition logic. So, that is why we do not see any quantifies etcetera. So, now, the first 1 is that, when he mentions star, it indicates some kind of number or sometimes which is which is also used as some kind of chapter.

Example: if he says star 1, it is some theorem in chapter 1. If he says 20, then it is in chapter 20 and then some theorem followed by depth. We are using this particular kind of thing V dash. But Russell and Whitehead uses particular kind of symbol which is called as session sign.

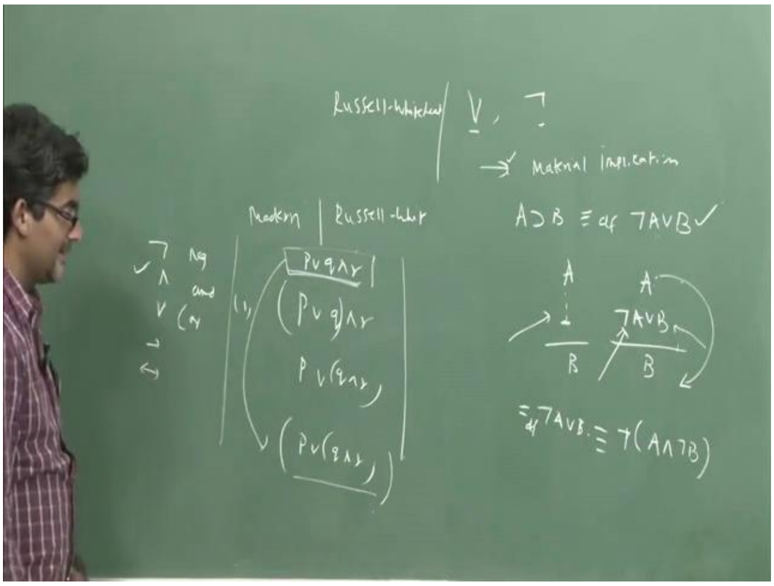
(Refer Slide Time: 13:45)



So, in the modern notation, you will be using this particular kind of symbol. In a modern notation is most convenient, so we use we use this particular kind of symbol. This means suppose 2 variables are there in both sides, this means y is reduce from x. So, Russell and Whitehead in his book, they use this particular kind of symbol. So, this stands for asserting some kind of the preposition. This is also called as assertions obtained by a employee in the usage material application.

So, that particulars anything which follows after this associations and which it means, it has to be either simply and axiom are it can be a primitive proposition, it goes to truth cannot be question and all. So, there; obviously or absolutely true, which is denoted as p t are it should be a theorem. So; that means, if it is an axiom have any proof, if it is a primitive kind of proposition, already true proposition were 2 plus 2 is equal to 4 also does not require any proof or it can be some kind of theorem and all is always true.

(Refer Slide Time: 15:57)

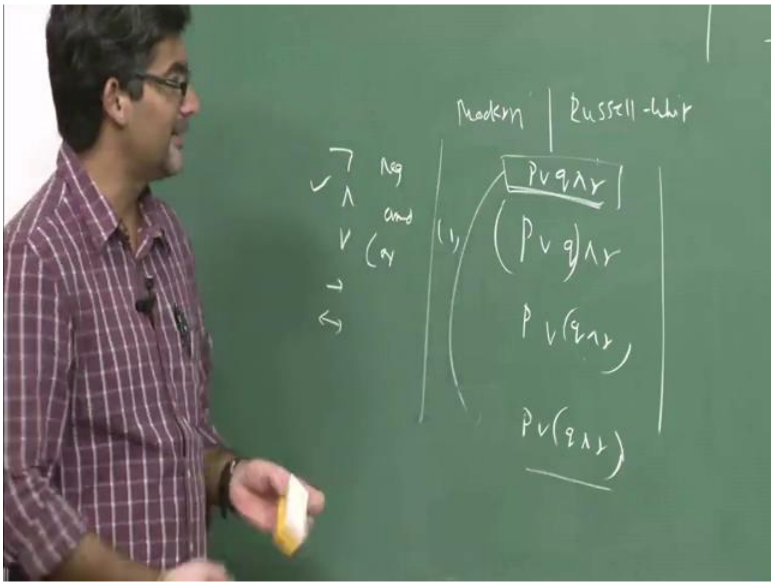


So, in principia mathematic, uses a d f stands for a definition. And then these are the symbols that you will see in that book; full stop, colon, semi colon and there are 2 colons follows each other and a follows. So, that is used for a some kind of punctuation. So, in the contemporary mode and logic text books, it is stands for single colon stands for brackets, are sometimes some other symbols tools colons follows each other may stands for square brackets etcetera.

So, usually they convey some kind of punctuations. For example: punctuation marks are very important in the sense, in the last few classes you have seeing that, for example, if we have p r q and r. So, what do you mean by saying that, it is p or q or r you. There may be some confusion which arise in our mind; whether you should be read as p or q and r are whether it should be read as q and r.

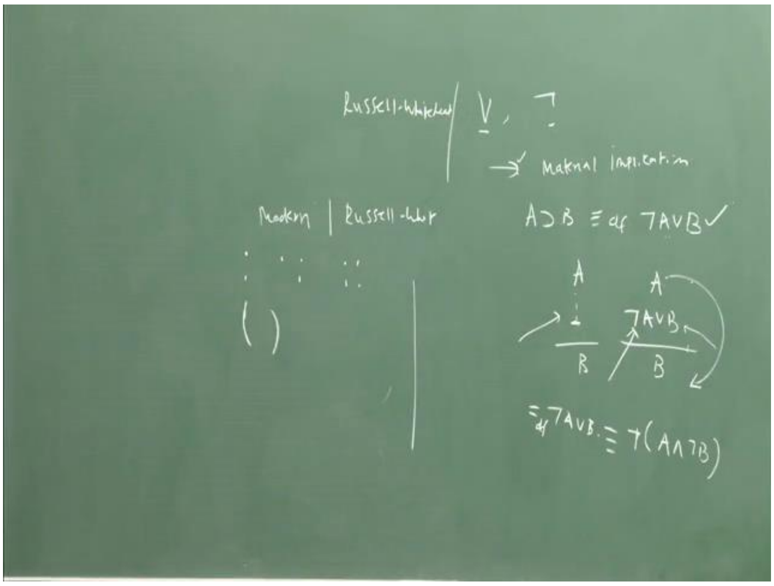
So, now in that sense, we come out of this particular kind of ... In this way that, we give some kind of difference to it is particular kind of logical constants. So, first you will give preference and then or, this is negation. And then you implies in then if an on if... So, in this sense, suppose if there is no bracket which is given and all. And this means, we need to use some kind of that convention of whether from this first we need to bracket there, is in this conjunction is to given first preference and then followed by the whole thing.

(Refer Slide Time: 17:12)



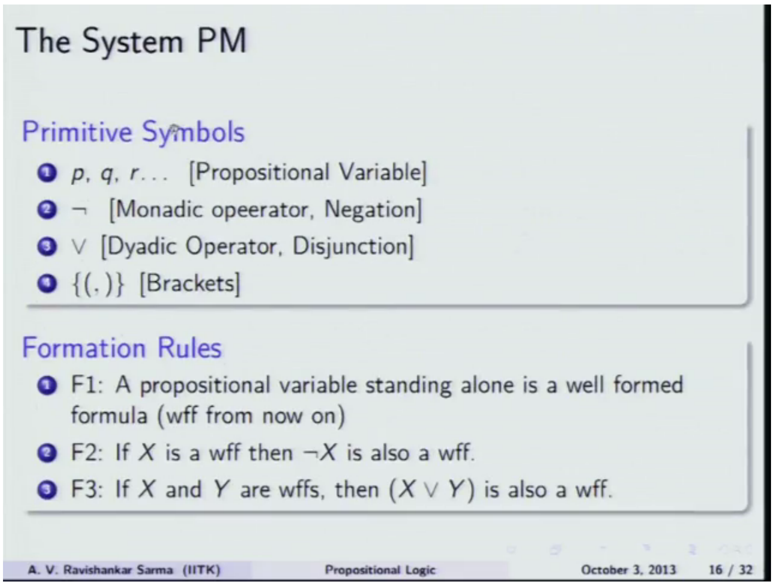
So, now this listen what we mean by this p r q and all. So, now we can eliminate this bracket and still say this thing. Without loss of generally we can even remove the outer bracket also. So, this is what we mean by this 1.

(Refer Slide Time: 17:28)



So, it is in that sense, Russell Whitehead uses this particular kind of symbols; colon sometimes x we uses 2 colons are is 4 dots and all followed by this thing, t is stands for left bracket are if it find both the things may be is like closing by a bracket and all. So, it is, it helps us in dealing with the formulas now. Since we are not doing the way principia way you find it principia mathematic, but we are strike to change our proofs and all, which fits in our convenience.

(Refer Slide Time: 18:34)



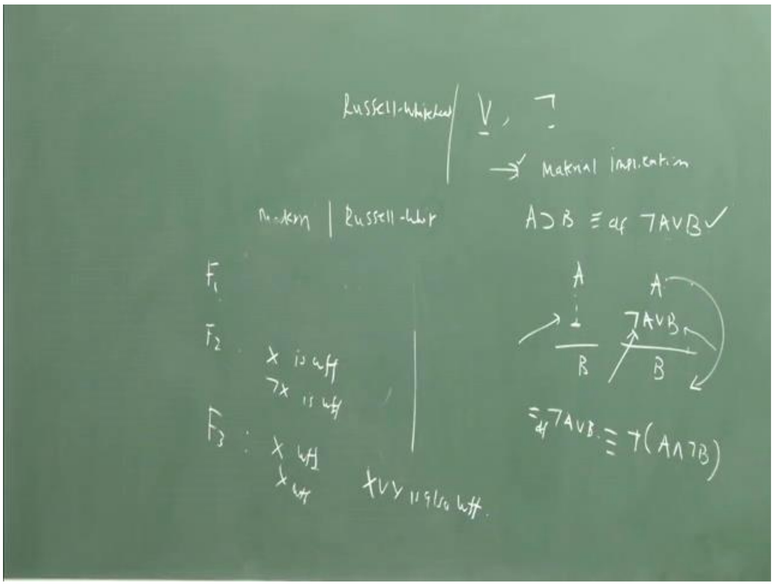
So, which we meet our convenience. So, now as usual p q s r etcetera are propositional variables and any formal languages, one of the same here are infinitely many number of such kind of variable. If p q s are alphabets are exhaust and we can use p 1 p q p 3 etcetera. So, now, there are some individual variables such as x y and etcetera. So, the all the present, propositional variables represents some kind of propositions and variables individual may represent some kind of individual names etcetera.

So, this is the axiomatic system due to Russell and Whitehead and there this is also called as PM this is principia mathematica. So, you have to call this some name and all. So, we are calling it as PM. So, is principia mathematica consists of first of all propositional variables like this is a duster, chalk piece etcetera and all these is all these things are represent by some kind of proposition variables. And he makes is up only 2 logical constants that is negation, which is a monadic operator. It operates only 1 particular kind of proposition. And the next 1 is a dyadic operator, we choose on monadic operator and 1 dyadic operator, it operates on 2 propositions because at least 2 propositions.

So, disjunction is a 1 which he has to use. And the other brackets for him like; colon and semicolon etcetera and all, are 2 colons following each other. So, this stands for some kinds of brackets, which is very important for punctuation. And this is the formation rules, not any kind of well form formula that cannot be any kind of formula which is generate, is not a well form formula.

So, the formation rules are like this. A propositional variables stand in the law to write just p q r etcetera and all simple propositional variables itself is formula are, suppose if x is a well form formula not x is; obviously, is going to be well form formula.

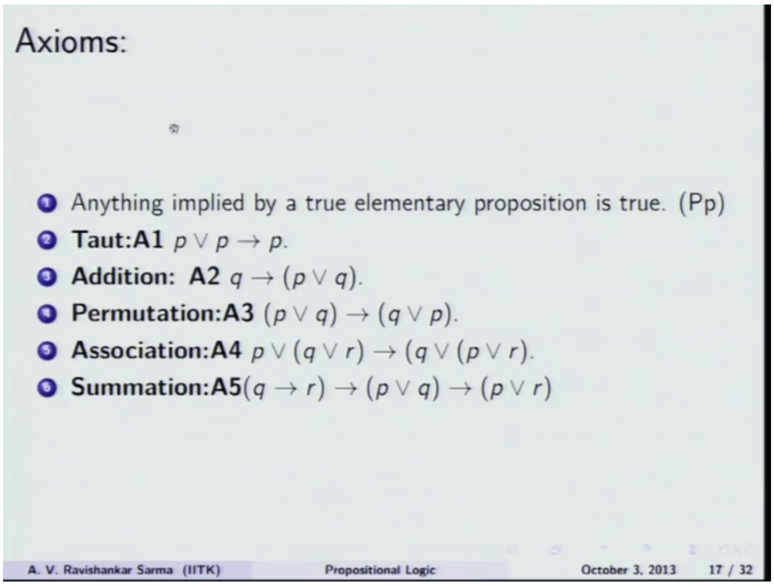
(Refer Slide Time: 20:12)



But these are formation of rules and all. Suppose if X is a well form formula and then; obviously, not X is also well form formula. Suppose you will, you not suppose to write like this; X following negation and all, this is not a well form formula. So, this is the first thing, very simple kind of rule which we have discussed, then we have introduce the language of propositional logic. In the same way, first we need to define our language that is, a syntax.

So, now, suppose X is a well form formula and Y is also a well form formula, then he make use of only 1 particular kind of logical constant, that is a disjunction. So, X are not Y is also a well form formula.

(Refer Slide Time: 21:25)



So, since he has used only in a negation and disjunction, there are no other formula such as, all the other things can be defined with the help of negation disjunction, by using material implication. So, these are the only thing formation rules, non other formula is a well form formula, if is dell does not follow 1 of this 3 rule. You can add fourth rule, its states that you might to follow this 3 rules ridiculously.

So, now, any axiomatic system should have to begin with axioms, followed by that, we need to have some kind of transformation substitution rules. And third 1 is as minimal rules, we inference when going to be present in a axiomatic system. In this case, we use only material implication. All the other rules will come as an outcome of this axioms etcetera and all. It is like a caption and then we are trying to derive everything from particular kind of. So, everything is hidden and this particular kind of caption consists of 5 axioms.

So, now, in the first 1 is pretty obvious and all. So, that is the start with tautologies and you will end with tautologies all. This now in which, you will get if he start from tautology, this is always a tautology in the proportional logic, is considered to be a statement, which is a always under all possible interpretations. So, you start with the tautology and you will transform it to some other thing, it is going to be a tautology always. If you using inform substitution truth preserving rule etcetera on that, you will generate any tautologies. Some tautologies you will generate tautologies, all is now in which you will get contradiction.

So, there is the first statement; anything for implied by a true elementary proposition; obviously, it has to be true. So, that is a primitive kind of fortuity calls. So, now, these are the 5 axioms, with which we can talk about entire arithmetic in a, all the statements of arithmetic, true statements of arithmetic are valid from arithmetic, will find proofs by using only one of this, anyone of this theorems are may be more also.

So, the first principle says that, this will be axiom has is name and all, its is laws of axiom related to tautology, p or p it is raining and or it is raining. If it is not raining, we are not say anything great about is. Now the addition; if we have q we can say if there is already true, we can add another true, we can add another p to it without disturbing the truth valuing of that.

So, now, you have to note that, here implication is au somehow serving as kind of deduction here. But later it was questioned by C A Louis this work survey of symbolic logic, the questions this particular kind of whether or not, material implication would serve as what Russell Whitehead thought that thought of as deduction. Deduction according to C A Louis a later works you will find it that. So, this is somewhat different from the material implication and all.

So, in that context, C A Louis come out with another kind of implication which calls it as a swift implication and the swift implication has led to model logics etcetera. So, early that led to non classical logic that is not we are trying to talk about. So, this principia mathematica has sound as a starting find for many other kinds of non classical logic etcetera. But how did this principia mathematica come into existence? They were some problems which related to serious axiomatic system because, further axiomatic system is based on set theory and set theory by paradox such as, we need to talk about that particular kind of paradox, it is the Russell's paradox.

So, in order to avoid for avoid this particular kind of paradox, Russell Whitehead has come off with a grand axiomatic system, which you find it in the principia mathematica. It is considered to be a grand kind of program, which tries to reduce mathematic, that means, arithmetic's to logic. So, now, all the arithmetic statements can be translated into 1 of this axioms etcetera and all. And from that, you will generate lots of theorems, that reflects or other statements 2 statements of arithmetic.

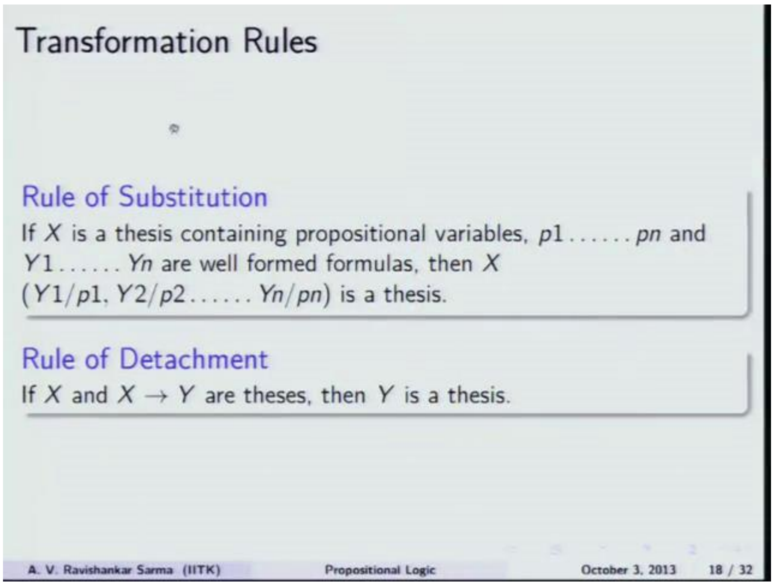
This p s q s etc stands for tools of arithmetic. Or if you are if you are not happy with this particular kind of thing and sure interested in analyzing simples stitching distance circuits, it is p's q's etcetera, you mean some kind of stitching circuits and all. So, when I represent, suppose I can represent p means p is closed if it is closed, but not p means p is which is open. It is in that sense, 1 thing can you this particular kinds of theorems particular kind of formulas.

So, permutation is like this is, like some kind of commutative property p or q plus q are p association p or q are all implies a q r p r or. And the summation good, which was greater question by a famous logician Paul Bernays. And he showed that, this axiom can come as an outcome of 1 of this 4 axioms which we have state, that is, from 2 to 5. Using 1 of these things, 1 can reduce the sixth 1 that is a summation axiom. So, it is in that context stay later and the later works of Russell Whitehead axiomatic system, we will not find this particular kind of axiom because, at anything which is deduce from some other kind of axiom, which will lose this axiom status and all. So, it will no longer surveys an axiom.

So, now some of the simple axioms, but you must note that coming off with these axioms is a most difficult part. So, one of the important characteristics of the axiom this is that, whatever you substitute for p q etcetera and all uniformly, you will generate only tautologies because there is; obviously, tautology. And if you all tautologies with uniform substitution are transformation, will you to tautologies all. It is a machine that generates tautologies.

So, there is the reason why logicians are always interested in tautologies in the sense that all tautologies are construct are construct to be valid formulas. And all valid formulas are obviously, have to have proof.

(Refer Slide Time: 27:47)



So, these are some of the 5 axioms and all. These are some of the important transformation rules. So, now, in what way this axiomatic system is different from the 1 is which we have presented earlier, there is a natural deduction etcetera and all. In natural deduction system, just like when you are playing some kind of game, we need to know you need to familiarize yourself with all the rules of the game and all. Just like them, in the natural deduction, we have familiarized our self with all the rules etcetera. I think there are truth deserving rules etcetera and this truth preserving rules are added to some of the hypothesis and assumptions are also assume to be true. And then we have generated various kinds of truths.

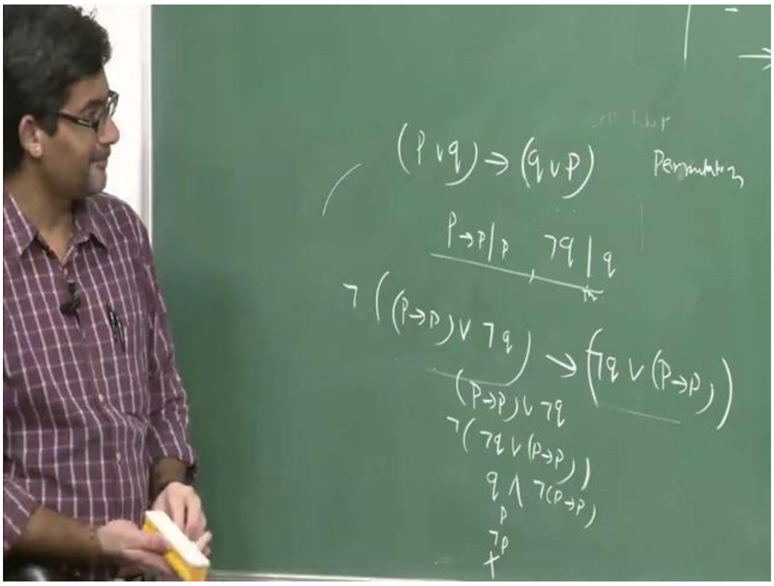
So, there is we have done in the case of natural deduction. So, here we use as many minimal number of rules as per possible; obviously, 1 or 2 at most. And then mostly you will use as many less number of axioms and then the serve things are all derived from.

Of course 1 important rule at we uses, rule of detachment. So, now, what you mean by a rule of substitution. Suppose if x, I will be using word this is this is in the sense, that it can be a theorem or it can be a axiom. An axiom is a self evident kind of truth. A theorem is even which is generated out of transforming this axiom into some other kind of statement. The substitution might in axiom, might lead to another kind of proposition. So, that my lead to theorem.

If x is a this is, this is can be considered as theorem or a an axiom. So, in that since we have some kind of flexibility in using this phrase, that this, is means you can be have the axiom it can even theorem. If there is only 1 for example, in this case let us say considered 2 p a p e implies p. So, that does not need any proof because, that already an axiom. So, the proof of that 1 means simply the same statement. You re trade the same thing p r p implies p, is already an axiom does not require any proofs. Such in that sense p r p implies p can be called as a thesis, in the particular kind of sense, can be called of maximum can be called as a theorem.

So, if x is a, this is and that sense continuing propositional variables p 1 to p n. And y 1 to y n it is considered to be a well form formula, a well form formulas there, in how did you get this x? x is obtain from substituting y 1 with b 1 y 2 with b 2 and y 1 y 1 with t n. Then you will generate some kinds of treatment, that statement is also considered to be a this is this means an already a theorem or it should be an axiom.

(Refer Slide Time: 30:45)



So, it is like this particular kind of. So, this essentially says that, for example, if he take into consideration, 1 particular kind of axiom here p or q or q or p. This is what is in the Russell Whitehead axiomatic system as permutation. So, now this the formula that we have. So, now, we can substitute p implies p for p variable p is there let us say you can substitute p implies p, we will be inform we need to substitute and then the substitute not q for q. It should be write as not q substitute for q means uniformly you are substituting into this particular kind of axiom.

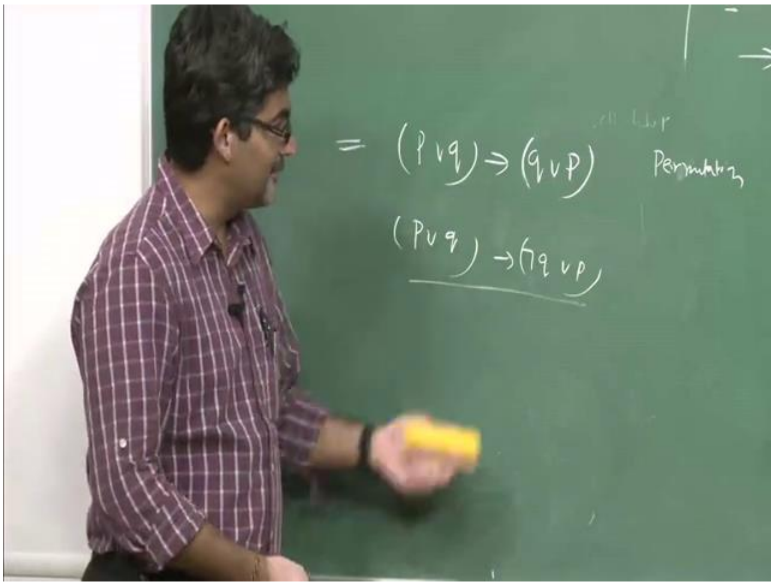
Then the resulting statements that means, this means what is uniform substitution role. So, now, this will become p implies p or q means not q is the first statement implies, now q means a 2 or p means this p implies p. So, this is what we got by substituting listening uniform. So, now if a substituting this way uniformly, the result of statements will; obviously, become true. So, 1 thing check it with the help of ... so far...

So, let us considered this as x. So, now, in we use as semantic tableaux method for this not x is this. So, you put negation will add this and will see whether negation of this formula reaches to branch closer or not. So, what essentially we are trying to show, this is the in a given axiom whatever substitute you uniformly and the result and statement is also considered as theorem; that means, it has to be a valid formula.

So, how can you show that this whole formula a valid formula, you denial this for problem denial this formula and then you constructed tree. And if all the branches closes; that means, not x is unsatisfiable; that means, x has to be valid. So, now, this will become p implies p or not q and then not off not q r p implies p. So, now, is this will be at q not not q is q, negation of this junction is convention and then negation of p implies p. So, now, negation of p implies p is nothing, but p and not p. So, this thing q forward by the not of p implies p. Since we have not p this branch closes and you not have worry much about statement.

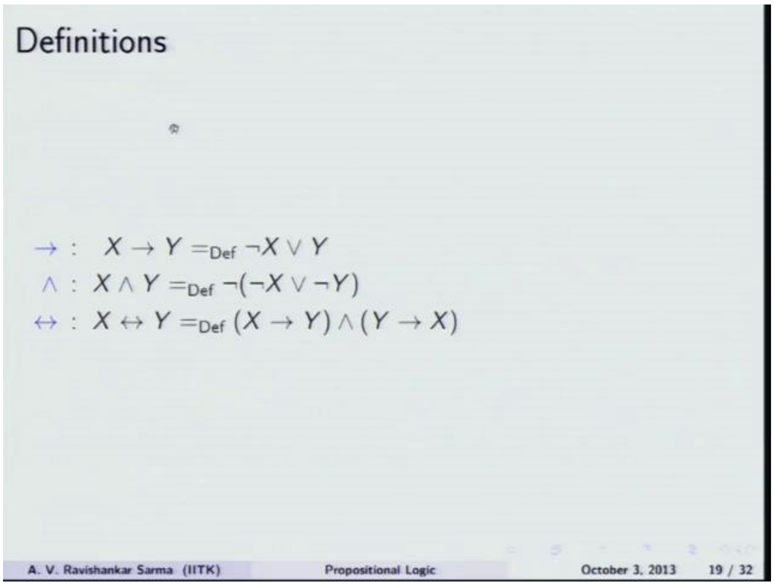
So, what essentially we showed is simply is that, we need substitute anything uniformly for q, anything uniformly for p it tautology put. So, that is about we mean that uniform substitution. So, you can substitute some complex kind of thing into these 1, but still it will turn out to the theorem.

(Refer Slide Time: 33:59)



For example: let say, p substituted p are p p r q r r for p. And then let us say not p implies q not p implies r for q. So, if you uniform sub uniformed substitute with any kind of proposition, then the result and formula is also going to be a theorem. But we will to ensure that, the substitution should be uniform. For example: if a substitute p or q here and then you substituted not q or p and all, this not a uniform substitution because, for q has substitute q here only, but here a change can your use not q. Then, you yourself will see that, this is not going to be tautology. That means, not going to be a theorem.

(Refer Slide Time: 35:39)



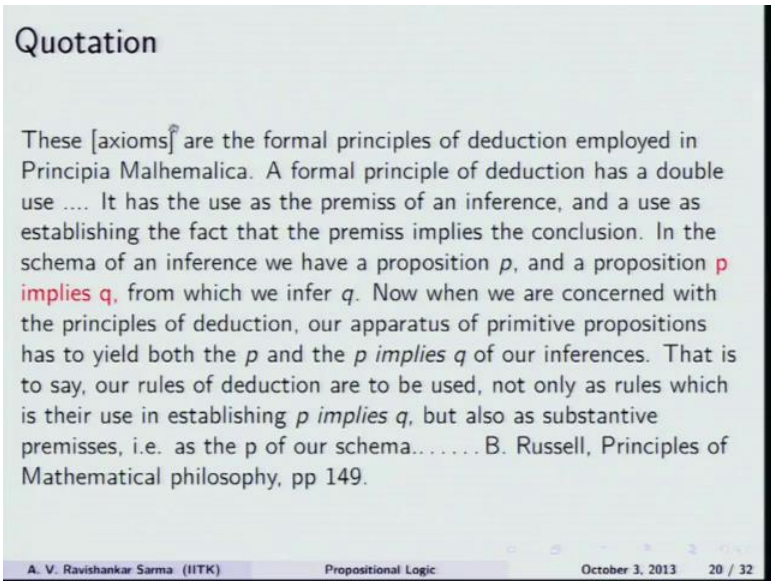
So, substitution has to be uniform. So, then only your exam will turned to another thing which is considered to be true. So, that we mean by rule of substitution. And the second rule is simple rule, which is called as a modes rule, are can be also be called as a rule of detachment etcetera. If x and x implies y are this is, there already assume to be true or at least this is then y the result and 1 this also called as a this is, there is also considered to be true.

So, these are the minimal transformation rule 1 requires. So, now, in any axiomatic systems, we need to have some kind of definitions. So, since Russell and Whitehead has used only disjunction and negation. So, now we need to talk about either important connect is that is, implication, by implication and conjunction. And since only negation and disjunction, all the other things should come either outcome of that. How do you come out with this particular kind of definition?

So, he made use of the concept of material implication. So, implication is depending in since x implies y means, by definition it is not x or y are you can even write it in the form of a conjunction, that is, it is not the case of x and not y. So, now, the conjunction is written in the sense. So, using some kind of a Demorgan's laws which are already there, us; x and y is equal to it is not in the case in not x not y. Transform the same thing it will become x and.

So, now, conjunction is define in the since of disjunction, by using only negation and disjunction. The 1 which is there in the right hand side, you will find only disjunction. So, and x in an only if y is same as x implies y and y implies x, where x implies y is find as the first 1 that is not x or y. So, these are the definitions that are already there.

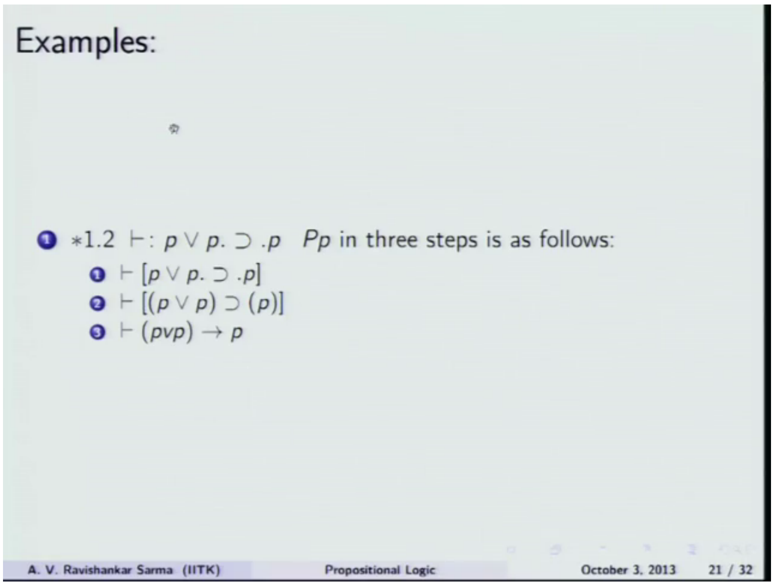
(Refer Slide Time: 37:06)



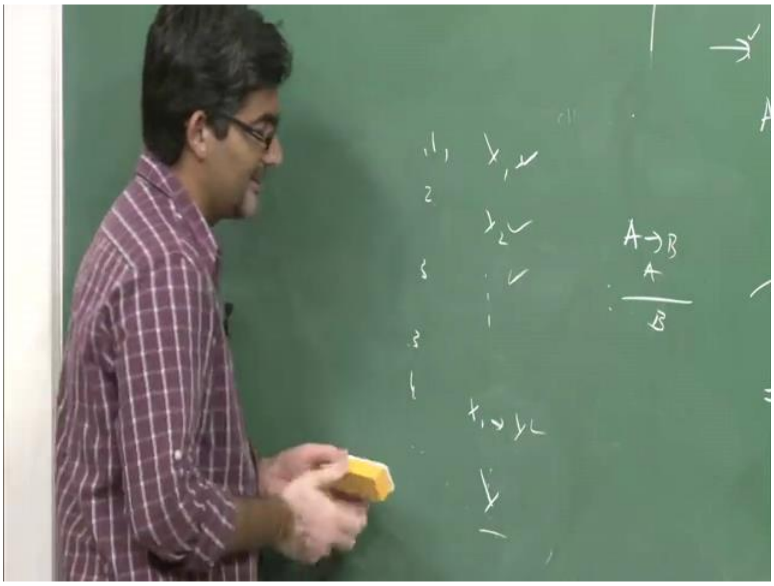
And according to Russell and Whitehead, this is one of the important quotations, which we has used it in the book; principals of mathematical philosophy, page number 149. So, the axioms that we have presented here, the 5 axioms or considered to be as the formal principles of deduction, employed in the principia mathematic. The formal principle of deduction has kind of double role, what is the double role? It has the use of premises of an and use as establishing the fact that, this premises leads to some kind of conclusion. In the schema of ...; that means, 1 propositionally stands from to another propositions, we have a proposition p and the proposition p implies q and from this 2 you will generate q; that means, you are already using the material implication.

So, now when we are concern with the principle of deduction, our primitive propositions has to will both p and p implies q as our influences. So, now, what you are going to see in the proof that follows there from now, is that, when you are moving from 1 proposition to another proposition, somehow in some stage we need to have a propositions is in the form of p and another proposition in the form of p implies q. That allows us to by using the definition of material implication, are there are rule of detachment. So, that is a saying our rules of deduction are to be used not only has rules, which is their use in establishing p implies q, but also as substance to premises, that is, as p of our particular kind of scheme. So, this is what he tells us in his book; principia mathematica.

(Refer Slide Time: 39:02)



(Refer Slide Time: 39:08)

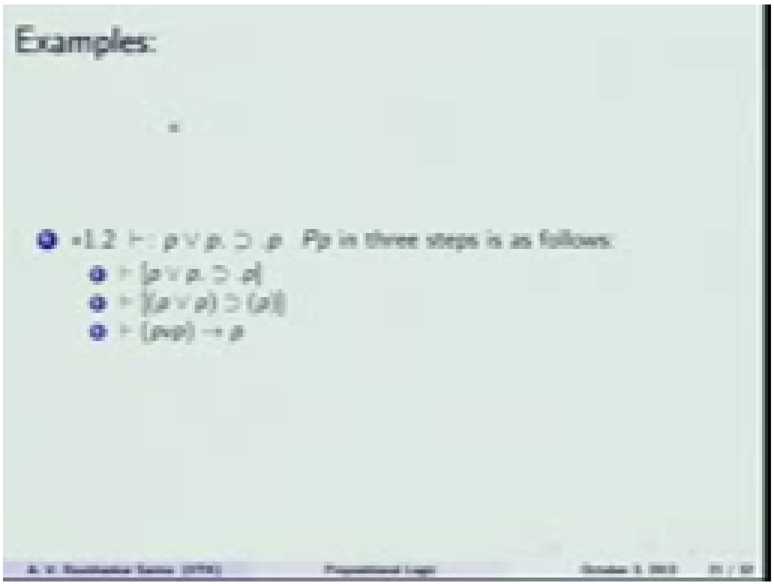


So, what is essentially he says this is the. So, a proof is considered to be a fined sequence of steps etcetera and all, 3 etcetera. So, now, we have reduced y from this 1 let us say. So, now, that means, x 1 implies y. Now deduce this thing x 1 implies y and all. So, now, from 1 and 4 there is a way in which can move, to the x 1 detaches and then you will generate y. So, these are all considered to be premises etcetera and all. In addition to the au we can generate some kind of statements like this, by using the material implication, I mean by using the definition of material implication, as well as rule of detachment. So, this is what is; the rule of detachment.

So, now, what is the Russell and Whitehead have stand to show. So, now, you are formulated and axiomatic system, which consist of disjunction and negation. And we have some sort of axioms and then we also know that, at 2 statement will first kind of proposition is talking about. And then you have transitions which precise the tautology in a accents. If your accent are trending in such a way that, but the will lead to only tautology is only. So, now, we end ensure ourselves that, what we can generate tautologies.

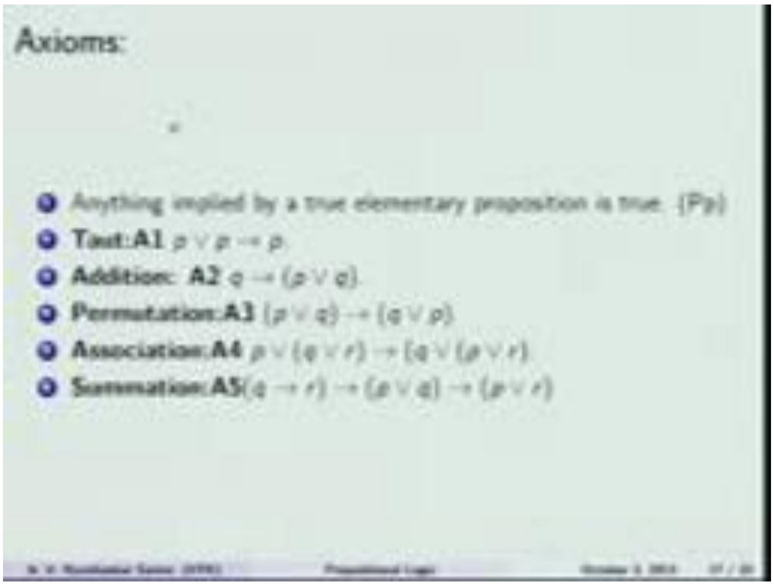
So, now, all the valid formulas which you can think of, should come as a theorem of theorem by using only this rules and you only by using this rules and the absent that have given.

(Refer Slide Time: 40:48)



So, now, here is 1 of the examples which is produce it as it is, which is used in the context of principia matematica, but we use there is different kind of notation. So, start means and then chapter 1 the second kind of proposition something like that. So, now, that single followed by colon and that should be written as p r p, the whole thing in brackets, p r p implies p. So, now, this you will obtain it with a help of 3 steps. So, now, how do you generate this particular kind of thing? So, we have you need to generate this p r p implies p.

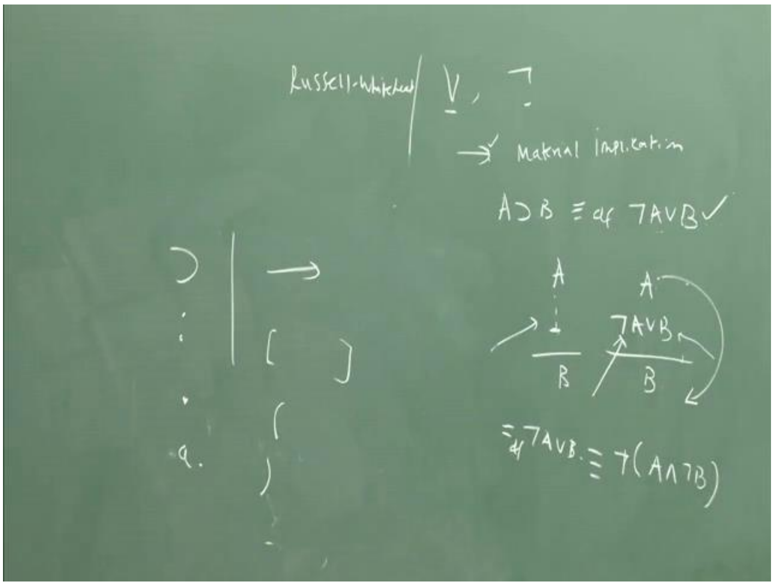
(Refer Slide Time: 41:35)



So, now, these are some of the axioms that we have. So, now, let us say that, we are generating p implies p r p. So, now, this is what is the axiom 1 that is p r p implies p, I mean we are what essentially we are trying to say this is the these the notation which is used by Russell and Whitehead is not a theorem of anything is consider to be an axiom. Axiom 1 is this 1. So, this can be obtained in our mode and notation as follows. So, first we will eliminate this colon and you will put some kind of square bracket. And then this will become p r p and you will use the dot symbol a dot a and we do not disturb this dot.

So, now, in the second step, we assuming we are assuming the dot means is their brackets. So, now, p r p is in brackets and then p is also in brackets. And then we are removing the excessive kind of brackets and all, unnecessary kind of things and all out of brackets we removed, but still we can retain the same thing. Even in the second statements p in brackets does not make any difference and all same as p only. So, now, the formula becomes p r p implies p. So, like this we are you are trying to translate the ones which we you are we which you see in the principia methamatica into our modern notation.

(Refer Slide Time: 43:32)

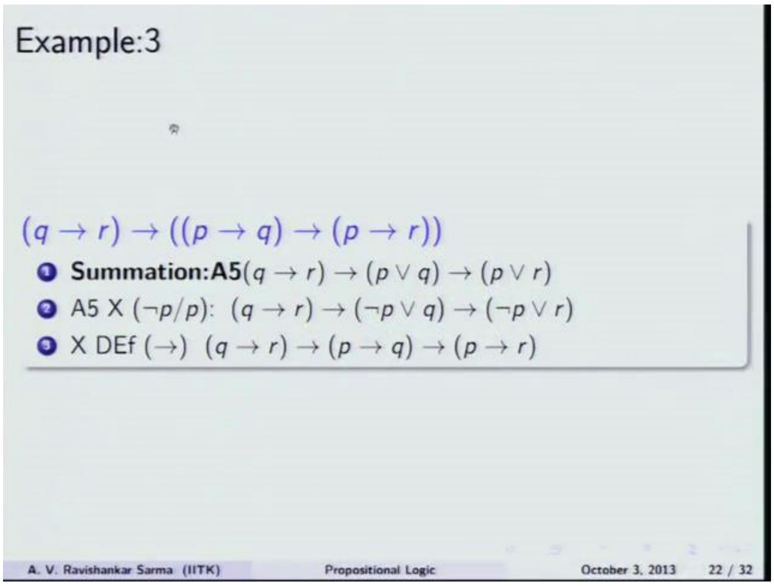


Some steps are involved in it. First we need to you can eliminate this colon etcetera and all, put it into the brackets etcetera and all. This will become a modern notation. But Russell has used as in the a heading that is forward by that this colon p r p and dot stop there, we have to give a pause here and the symbol, but we are using the straight arrow. So, making Russell has used this 1. And then this means some kind of square brackets and then dot means the kind of bracket. Example a dot is before this 1 means is a left bracket a dot is after this 1 and the right hand side, this considered to be right bracket and all.

So, like this we are used and then we have translated into some kind of convenient kind of notation. So, now what you will be doing now, will simply reach that. Any axiomatic system what you are trying to talk about, we have 3 laws of logic, that is, law of p implies p and law that is p or not p and of non contradiction. It is the not the k z p and not p at the same time.

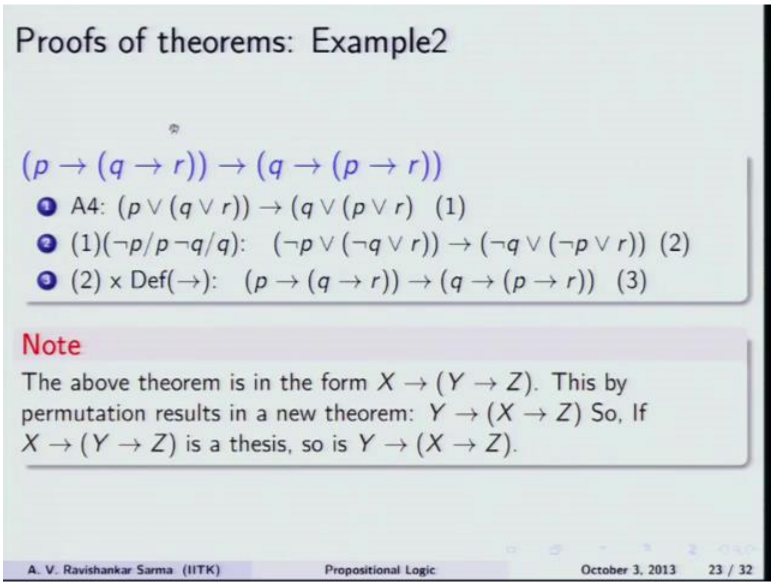
So, now, at least 3 laws of logic should come as an outcome. Of course, there are many, it is also expected that all the valid formula should find a proof and all. But the this 3 thing should come as an outcome. He formulated a grand formal axiomatic system and should now we need to ensure that at least this 3 laws of logic becomes and outcome because, all the other things are constructions of these 3 fundamental laws of logic.

(Refer Slide Time: 44:51)



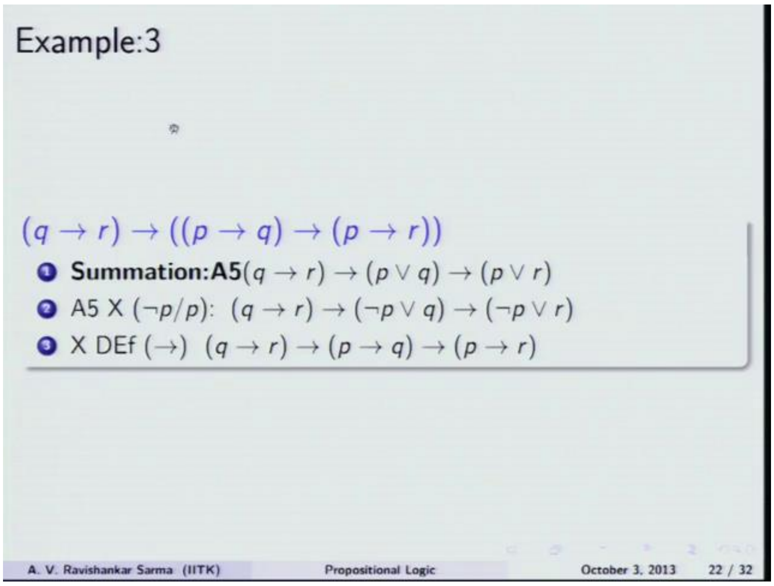
So, now let us try to prove some kind of theorems.

(Refer Slide Time: 44:55)

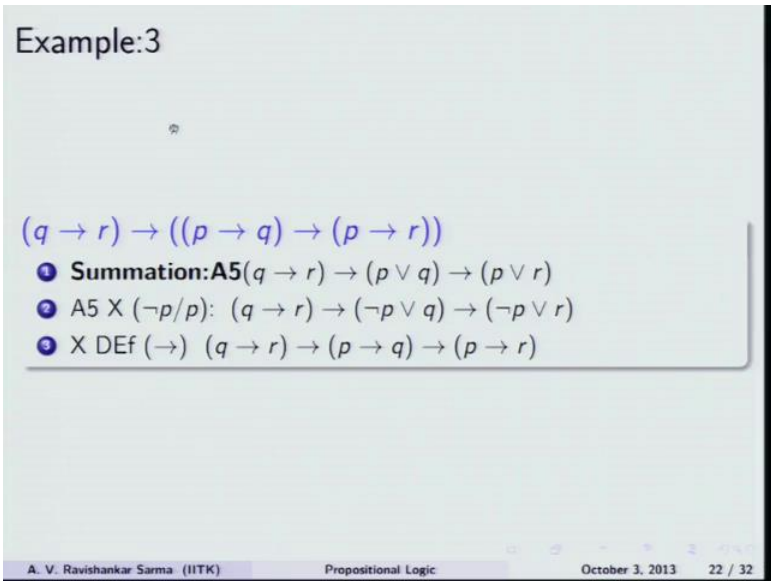


So, that is like p plus p etcetera.

(Refer Slide Time: 45:03)



(Refer Slide Time: 45:42)



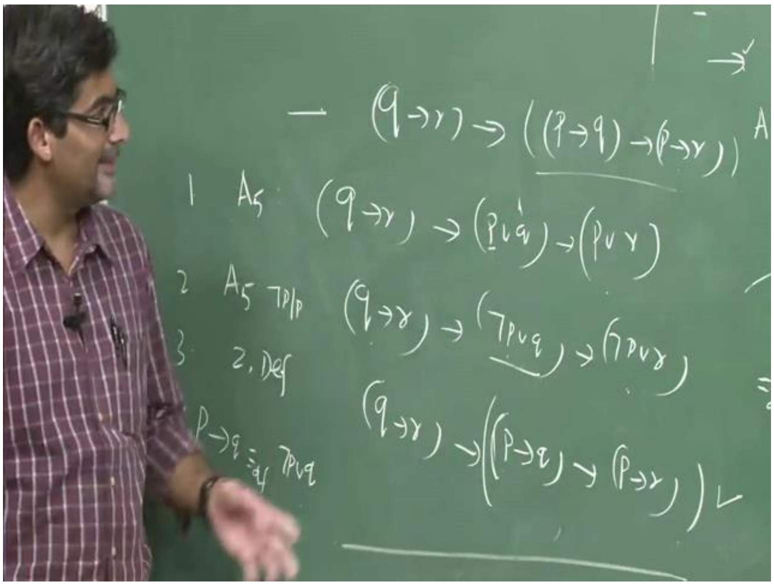
So, now let us try to prove this particular kind of thing, which is considered to be kind of transitive property, that is, q plus r implies p implies q and p implies r. So, now, how do you prove this particular kind of thing? So, now, we use to only the axioms that are listed there, that is, the 5 axioms that we have. And you have to use transformation rules and we have to use only more respondence rule. And ultimately that propositions which is up appearing in the blue color and the first 1 q implies r and p implies q and p implies r. So, you are trimming this in such a way that, it will lead us to this particular kind of theorem, that is, q plus r plus p plus r p plus r.

So, now, in these cases you must note that, what kind of axiom that we will be choosing. That depends upon our kind of some kind of creativity and choosing these axioms; we might choose any one of these axioms, but if a take some selective kind of axioms and all, then your proof might be consisting of less number of steps. So, that depends upon only our creativity etcetera.

So, in a way it is proving some kind of theory proving theorems is also kind of an art just like programming there is a kind of art. So, this proving this theorems is also considered to be kind of art because, in somebody's proof, it will have only 4 steps or in some 1 might struggle and then you will come off with 15 page proof. So, you might ask what is the term we will be getting from this things is because, so in the reason why we are working on this proof is because of this particular kind of thing, that is, in the Euclidian geometry is also considered to be an axiomatic system.

So, there are many things which are not part of the proof are also outside the proof are also taking part in the proof. So, that we should avoid on few, if a proof has to be rigorous, then everything needs to be stated explicitly, that is what we mean by axioms. And then from the axioms, we transform it in by using transformation rules and the more respondence etcetera and will transformed into some other theorem. So, now, how do we prove q implies r implies p implies q and p implies r? So, now, to start with, we used axiom number file, that is this 1 q implies r implies p r q implies p r. So, this is what we have began with.

(Refer Slide Time: 47:52)



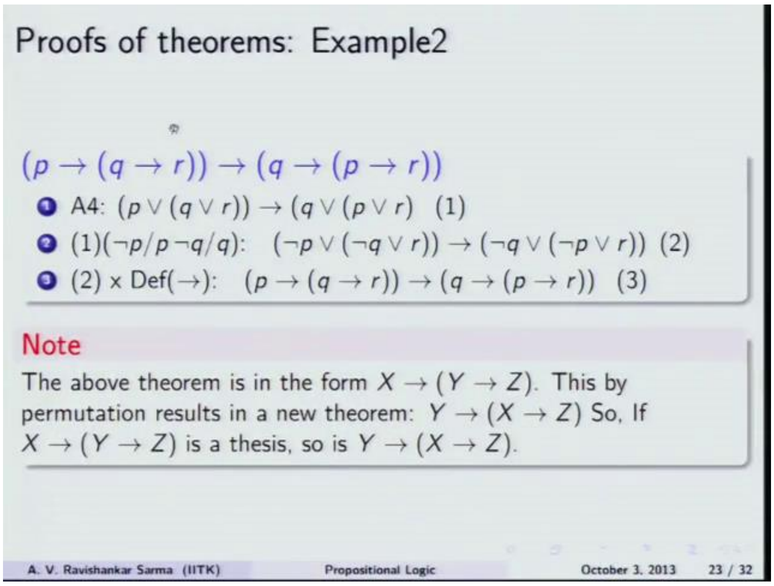
So, let us considered to this particular kind of thing by using this 1 which is what we are trying to improve q implies r implies p implies q not this q implies r is q implies r means p implies q that p implies r this what we are trying to prove. So, now, you have began with this axiom and dot 5, see that is p r q implies is that axiom au p r. So, now, we begin with this particular kind of axiom. So, now, it appears to be a more or less, somehow we need to change these things in such a way, this axiom is to such a way that you will generate these particular kind of thing.

So, now they are many ways such you can think so that, this will transform to a particular kind of thing. Example what kind of substitution 1 is to make so that, to transform this things into a particular kind of thing, q implies r is same as this 1 and some kind of substitution we need to make so that, it will be this 1. So, now, in axiom number 5, suppose if you substitute not a for p and this variable p occurs is substituted with not p then what will happened, axiom 5 is transformed into this thing; q implies r implies. So, now, p will become not p r q and this will become not p r. So, now, this 1 let us say this is 1 and 2 and 3 2 you have to use definition here. What is the definition you will using?

So, p implies q is nothing but by definition not p r q. So, now, wherever not p or q is means this q implies r reply this means this. So, now, justification of this 1 is use is that, this by definition is nothing, but this 1 p implies q. So, we need to put that it here. So, now, not p or r implies p implies. So, this is what we are trying to generate. So, now, in this proof, you have only 2 steps you have 1 substitution, you can transform this axiom, you have trimming this axiom into another kind of thing statement, which is usually considered as theorem.

So, why it is called as a theorem? Each step of your proof is considered to be true then; obviously, the final step of your proof is what we mean by the theorem, that is what we have defined idea and the beginning of the axiomatic systems. The last step of your proof is considered to a theorem. So, this is the way to proof this particular kind of proposition.

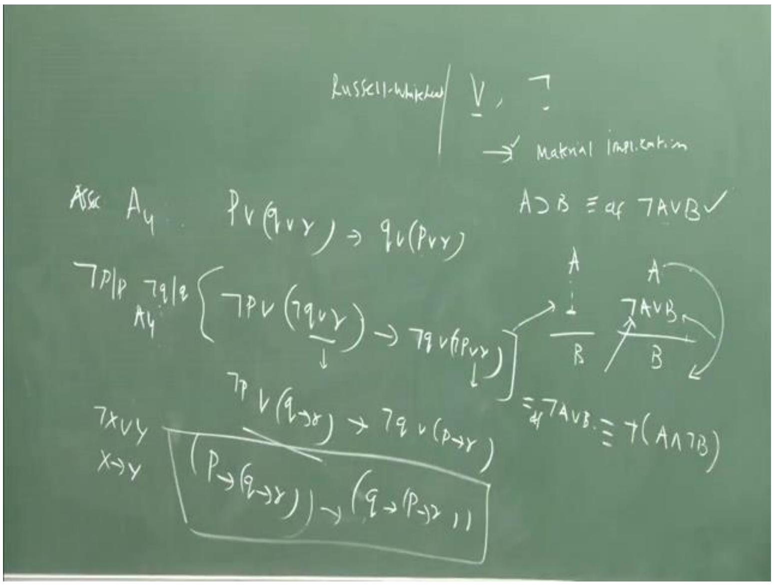
(Refer Slide Time: 51:19)



So, there are other theorems, which we can take a and we can prove this things and this things. So, let us say you are trying to prove this particular kind of thing; p implies q implies r implies q implies p implies r. So, now, for this, again we need to think second way. What kind of axiom which you can use, so that you can come closer to this particular kind of theorem. It is not in 1 step at least by transforming into some other kind of steps using transformation rules and more respondence, etcetera.

So, now, you are started with axiom number 4, that is, law of association in this case. You started with p or q r implies q r p or r. So, now, in this axiom 4, you have substituted q for not q, wherever q is there not q. So, then this will become this is what we have done in the second step, that is, what you find it here not p for p and wherever you find q, you substituted with not. So, then this association kind of principle, you will get this thing.

(Refer Slide Time: 52:35)



So, now let us consider this thing p implies q r r implies q r p r r. So, this is what is called as axiom 4 and we will also called as association. So, now, what you have done here, this is the wherever is there you are substituted with not p another p and wherever q is then you will substituted with not q in if. So, now, this will become not p r not q r r implies this is q means not q or not p r r, you will put in the square brackets so that, we will avoid the confusion.

So, now, see this by definition use this thing, second step you will write it q implies r. So, now, again this by definition, we will get q implies r. So, now, again we will hook definition on this 1. So, this is not x or y; that means, it is x implies y where x is here p and p u not p au x is p and then q implies r so; that means, it is p implies q implies r implies q implies p implies r. So, this is the way to prove this particular kind of theorem.

So, now, 1 might ask many questions here. So, how do we generate an affective kind of proof? So, now, it depends upon what kind of axiom that you are going to take into consideration. So, in principle, you can take any axiom into consideration and then you can generate proof for this 1. But if I chosen the a 4 axiom, then my proof would be simpler. I can generate a proof in 2 steps. In the same way, you can generate proof in even 6 or 7 steps also, by using may be you can start with a 1, ultimately if we did not do not work out and then we will move to some other axiom, working such a way that, somehow the or training this axioms in such a way that, you will generate whatever is considered to be a theorem.

So, in this lecture what we have done is then we have presented Bertrand Russell Whitehead axiomatic system. And then we are seeing that, any axiomatic system should consists of sort of axioms transformation rules and the rule of inference. In the case of principia mathematica, you will find only disjunction and negation as primitive symbol and the transformation rules and the rule of detachment. And making use of these things, definition of material implication, that is, A implies B is nothing, but not A or B. Many you could talk about all the other kinetics, based on this 2 primitive kinetics by using the definitions.

So, now, we have seen some simple kind of proofs, in which we have transformed the given axiom, is starting with an axiom. And then we applied some kind of transformation rules; that means, in a way we are used uniform substitution etcetera. And then we trim these axioms in such a way that, we generated whatever he decide to prove.

So, in the next class we will be talking about some more proofs in principia mathematica like, at least this law of entity law of excluded middle etcetera. And then we are going to see whether or not is principia mathematica is going to be consistent or whether this system formal arithmetic system going to be complete etcetera, by the things, which you will be talking about in the next class.