Introduction to Logic
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Lecture -31
Hilbert and Ackermann System

Welcome back in the last class under a section axiomatic propositional logic we presented Russell whitehead axiomatic system that is what we find it in the book principia of mathematica in one of the sections on deductions you will find some interesting proofs such as law of identity law of no contradiction and many other theorems. So, what is our main goal our main goal was this that all the valid formulas in your formal axiomatic system should find a proof. So, that is the reason why we are doing this axiomatic propositional calculus. So, today we will be presenting another axiomatic system which is due to another 2 set of mathematicians.

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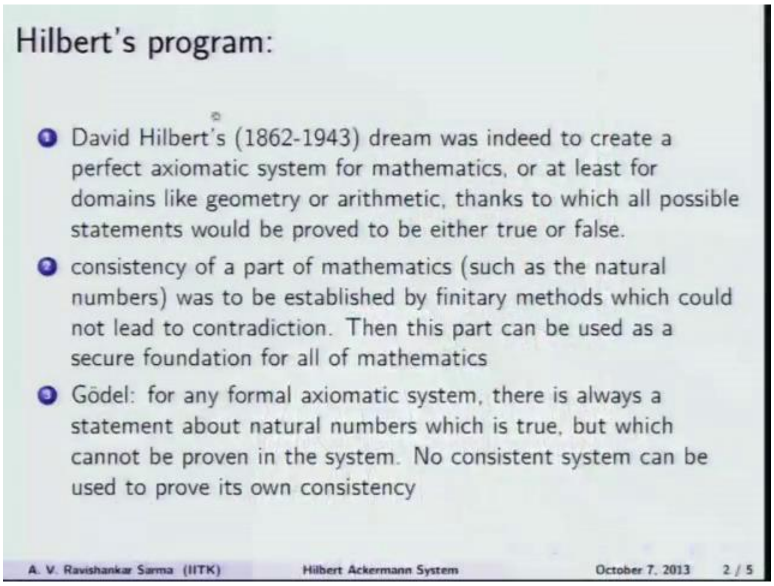

Hilbert and Ackermann Hilbert is famous for his different challenges and all Hilbert there called as Hilbert problems. So, we will not talking about all the problems and all, but. So, we are presenting a kind of axiomatic system which is proposed by Hilbert and Ackermann. So, what Hilbert's interest was mainly axiomatic geometry. So, he live from

9 teen 6 ty 2 to 9 teen 40 3 his dream was in deed to create a perfect axiomatic system for mathematics or at least if if you restrict down self to at least geometry and airthmatic as for us and arithmetic are concerned wants to come up with grand axiomatic system if you can construct this axiomatic system then what will happen is this that all possible 2 statements can be proved or you can show that all the provable thermos can be are automatically true. So, now, .

According to him the consistency is a part of mathematics such as in the case of natural numbers it was established by some kind of finitary method which could not lead to any contradiction i start with tautology you will never end up with a contradiction then this part can be used as a secure formation for the entire mathematics. So, you showed that by using some kind of finitary method you come up with some proofs which are divide of contradictions and that can be used as a pillar for constructing a some other kind of theorems and all. So, it is only till Gödel Gödel has come up with some interesting theorem which is called as incompleteness theorem according to which for any formal axiomatic system as in the case of principia of mathematica are Hilbert Ackermann axiomatic system. So, he is he has come up with.

Radical kind of view that exists there are always some treatments about natural numbers that is in the arithmetic which are; obviously, consider to be true which cannot be proven within the system with using its own axioms and using its own principle etcetera and all so; that means, system leads to incompleteness what is incompleteness anything which is provable it is true or if you can show it to be valid and all.

The valid formulas have to be have to find a proof if that is a case then your system is considered to be complete. So, now, Gödel has come up with an interesting kind of theorem with which you 1 can know 1 can show that no consistent system can be used to prove its own consistency if you think that you know principia mathematica and Hilbert and ackermann system considered to be consistent anyway Hilbert dream has been shattered by this 1 of the important thermos in logic that is Gödel theorems which we will talk about it under the limitations of the first order logic.

So, as for as prepositional logic are consent there are decidable and complete and consistent and sound, but where as in the case of first order logic they are semi decidable and it leads to incompleteness etcetera and all. So, in the context of first order logic when I talk about predicate logic these things will become prominent. So, our goal is to present and this lecture is to present Hilbert Ackermann axiomatic system and then will be proving some important theorems such as p plus pa p or not p etcetera and all and then not only that thing we making use of 1 important theorem prepositional logic axiomatic prepositional logic. So, that is the deduction theorem, if you can use deduction theorem with.

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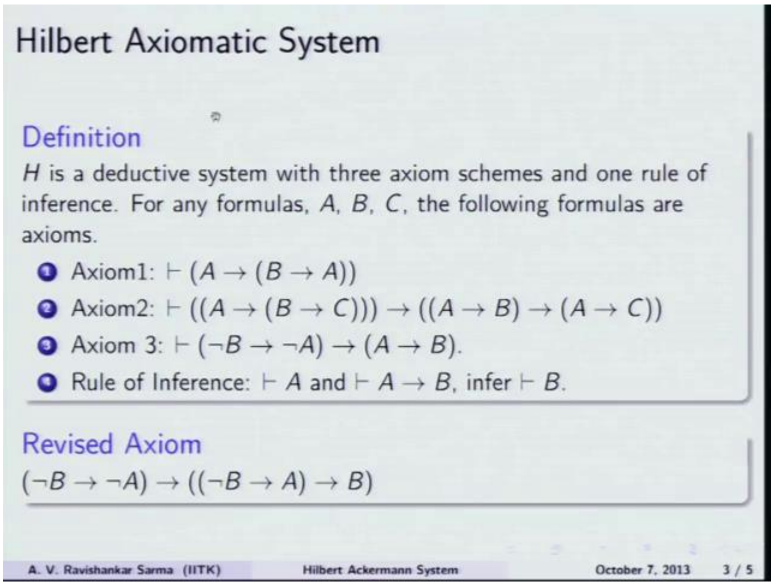

With a set of axioms that you already have then our proves our proves will become simpler. So, now, to start with Hilbert Ackermann axiomatic system it is presented in various ways in various text books in particular in most of the text books these are the 3 axioms that are provided in the text book like Mendelssohn etcetera. So, these are some of the axioms we have to note that Hilbert Ackermann makes use of only 2 primitive logical symbols that is implication and negation.

So, there is only 2 symbols which you commonly find it in you find it in the Hilbert Ackermann axioms. So, to start with axiomatic system consist of assets of axioms and transformation rules and off course the rule of detachment. So, instead of 5 axioms as in the case of we have only 3 axioms here. And all these are in the are expressed in terms of implication and negation the first axiom is a implies B implies a and the second 1 a implies B implies c implies a implies B implies a implies c and now this third axiom stated like this in the beginning not B implies not a is implies b, but in some text books like Mendelssohn inflection to mathematical logic you will find this particular kind of thing not B plus not a implies not B plus a implies b, but when you can show that if you take only in the first 3 axioms that will that will be constitute a formula axiomatic system and then if you take the first 2 axioms 1 and 2 and the 4th 1 which is there at top of the slide then you can formulate a different kind of axiomatic system let say h prime now you can easily show that this h and h prime are more or less.

They are same if you can somehow show that this not B plus not a a plus B will get it as an outcome of this revised kind of axiom that is not B plus not a not B plus a implies B then you can show that these 2 axiomatic systems are similar to each other. So, I will be taking into consideration the first 2 axioms and the revised axioms which was presented by Mendelssohn in in the tradition of Hilbert Ackermann axiomatic system. So, now, using these axioms what essentially we are trying to do is we are trying to prove some important theorems. So, now so far we have seen a some important theorem such as p implies p in Russell axiomatic system.

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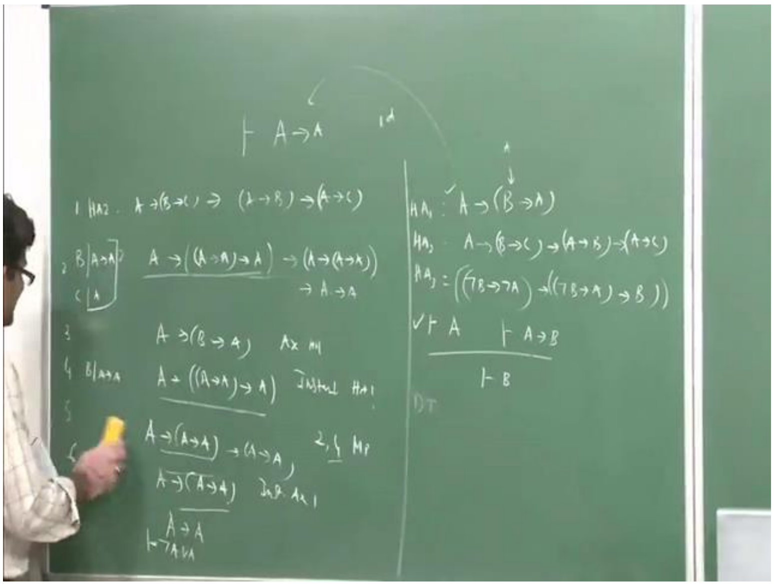

So, now, we trying to show with a help of Hilbert Ackermann axiomatic system you will be proving some of the important theorems such as P implies a. So, now, any axiomatic system this law of identity should come as an outcome. So, now, these are the important things that we need to know these are the axioms you can write h a 1 h stands for Hilbert Ackermann and this is a axiom number 1 a implies B implies a ha 2 a implies B implies c implies a implies B implies a implies c. So, the third 1 is like this not B implies a this is what we are going to take into consideration that the revised axiom this means not B implies a implies B this brackets needs to be closed properly and if you want you can write something and in the rule of detachment is as it is suppose you can assume that a and if you can assume that a implies b.

From these 2 things you will get B as an out comer. So, we will be making these of thsese things on the right hand side of the board in proving some of the important theroms then we later we will use another 1 which is called as deduction theorem. So, which will employ it little bit later. So, now, we are trying to show a implies a by using only these axioms and some kind of transformation rules plus this detachment rule. So, now, what exactly we are trying to do is we are training these axioms in such a way that it leads to another kind of truth these are all; obviously, true statements you crimite in such a way. So, that it transforms into this particular kind of prepositions. So, now, for this you start with Hilbert ackermann axiomatic system to that is a implies b, implies c same thing which we are writing it again a implies B implies a implies c.

So, now, in this what you will do is for variable you find B you substitute it as a implies a and a variable you find c substitute with a a d off course even B also as a so; that means, you are replacing B with a and you are replacing sorry you are replacing B with a implies a c with a and this is not required and all these are the 2 operation that you are trying to do. So, rational behind this thing is is that if you substitute anything to this axiom uniformly that will retain this to tautology hood; that means, it is still it will still act like a tautology it is a tautology.

So, now, this is the first step and the second step is this a is as it is because you are proving a implies a; that means, you have to eliminate this B’s and c's somehow, so that you will find only A’s in your formula. So, that is a reason why we use this uniform substitution root. So, now, what is B here B means a implies a and then c is also considered to be a now second 1 a implies a implies a B is a implies a that is the first 1 and then second 1 is a implies a. So, ultimately we ensure that the last step of your condition last the consequent of your conditional at the occupying the last position is summer closer to what we are trying to prove so; that means, a implies a. So, 1 needs to do is.

Somehow you detach the whole thing and somehow you get this particular kind of thing which that is what we desire. So, now, this is step number 2. So, now, we have an axiom a implies B implies a. So, now, in this you substitute again for B you substitute a implies a. So, now, this becomes what is this is axiom number 1 h a 1 if you can understand 1 proof then we can solve we can proof many other theorems and all. So, so this will become instead of B we write a implies a and then this is as it is. So, this what instance of Hilbert Ackermann axiomatic system 1 i mean you substituted a implies a for B and this is what you get.

In the fifth step you observe this 2 things. So, this is same as this 1. So, now, these 2 what are these things 2 and 4 you have to write justification here more responses; that means, you use this particular kind of rule then this get's ditched and then what ever remains is this 1 a implies a implies a is this 1. So, that is a implies a implies a this part goes and all it is get detached and whatever remains is this portion whatever is there after wards implies a implies a. So, this is how did get this 1 by applying 2 and 4. So, now, in the 6 th step still it is not in the particular kind of form somehow you need to detach this. So, how do we detach this particular kind of thing somehow we need to again fall backen or axiom.

If you can use any 1 of this 2 things you will not get to this particular kind of thing, but if you use this 1 transform it in such a way that for example, you instead of B you put a here then it will become a implies a implies a. So, now, a implies a implies a what is this this is an instance of axiom number 1 because instead of B you have put a here uniformly you substituted a for b. So, this is what you get. So, now, on the 8 h step. So, these 2 are same a implies a implies a and a implies a implies a it is like x and x implies y. So, we will get y. So, now, in the 8 h step you will get a. So, now, what is that we are seeing in this particular kind of proof you might come up with a implies a in may be in less number of steps, but this appears to be the case that at least 7 or 8 steps are involved in proving a implies a.

So, that there are at least 2 things which you need to note I start with a axiom and then you I trim this axiom in such a way that I will form this particular kind of theorems. So, this theorems might come in 4 steps sometimes 7 steps or sometime if you are axiom the choice of your axioms are wrong then you be you will be playing with it and ultimately it might it might take some 6 teen steps sometimes proofs might take even days also. So, the effective proof is considered to be that particular kind of proof which ends in final step in finite intervals of time; that means, if your proof never ends in it goes on and on and that is not considered to be an effective kind of proof. So, should also note that all this proofs you can transform it into some kind of language the programming language and then.

You can talk about this particular kind of thing or you can develop a software in which you give the feed beck of all these axioms and all and then you put this data a implies a whether it is a theorem or not that software will tell us whether this is a theorem or not or there are some software which provide even proofs also that is not we are going into details of it. So, as a first step we are trying to show how you can generate a implies a by producing some kind of rigorous proof. So, with this proof what else we can find out is this that everything is listed here; that means, everything is stated explicitly in terms of axiom which are considered to be; obviously, true and then the rule which is the truth presuming rule and transformation rules also preserve set rule. So, every where we are everywhere you consider as hypothesis are premises.

They are all true and the final step of your proof is called usually called as a theorem that is what we have said in the beginning of discussing this particular kind of axiomatic prepositional logic. So, this is the first theorem which you will get it as an outcome and from this by definition you say is a r a etcetera. So, this is also can be proved and all if this is proved then not here r a it can also come as a thin so. So, now, what we will be doing is ah. So, we will be proving some other kinds of theorem such as not not B implies b. So, let us with this particular kind of axiomatic system. So, what essentially we are trying to do is that. So, we are formulated an axiomatic system then.

In that axiomatic system which consist of only few rules and using this only we have only few axioms and very minimal set of rules and all and with that you generate all kinds of true statements that are theorems. So, you are not suppose to use anything outside these 3 axioms and all. So, if you use anything outside the things and all like in the case of ingredient axiomatic system it also considered to be a formula axiomatic system, but the problem there was is that proofs or not rigorous like these proofs and all in the sense that there are many implicit assumption which are part in parcel of your proof and there are certain things which are not part of the proof also take they also took part in the proof and all. So, in that sense Euclidean axiomatic system is not.

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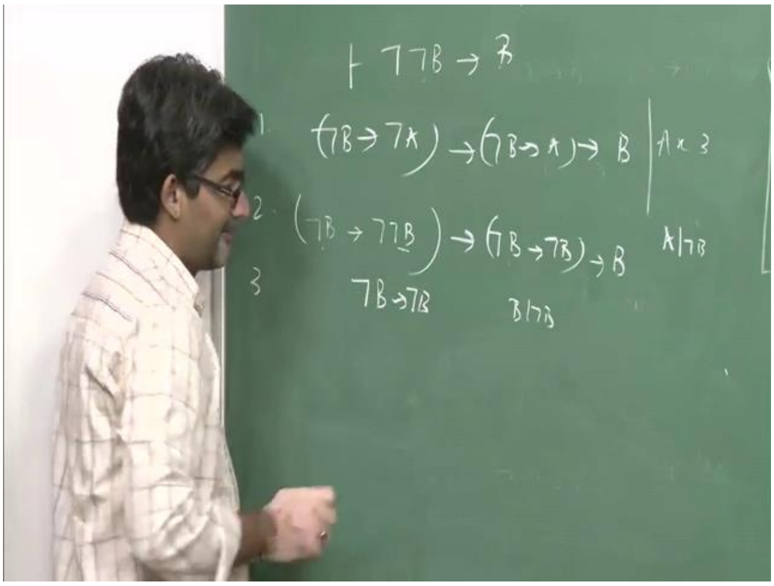

So, rigorous like the axiomatic system that we are trying to present the very purpose of presenting this axiomatic system is to get rid of those, non rigorous kind of proofs. So, this thing everything is stated explicitly there is nothing hidden no hidden assumption are there. So, everything comes through by trimming this axiom you will get your theorems. So, now, let us try to prove another theorem which we have already proved it in the in the axiomatic system due to Russell white head.

So, let us see how we prove this particular kind of theorem. So, depending upon what axiom that you are going to choose ah we can start with any of these axioms and all. So, if you want to show that this is true this is a theorem then 1 needs to start with one of these things because these are the only things which are given to us it seems that the third axiom to take into consideration then somehow you will get into particular kind of formula not b implies a implies b. So, this is axiom number 3 you need to provide justification on the right hand side or may be here left hand side and all. So, now, 1 instance of this particular kind of axiom is this 1. So, now, what you have done here is is that for a you substituted not b. So, wherever you find a you substituted with not B a means not b. So, that is why not is already there that is why it becomes like this. So, now, this is not B and a is not B and then B is as it is. So, this is the first step that we have 1 and 2.

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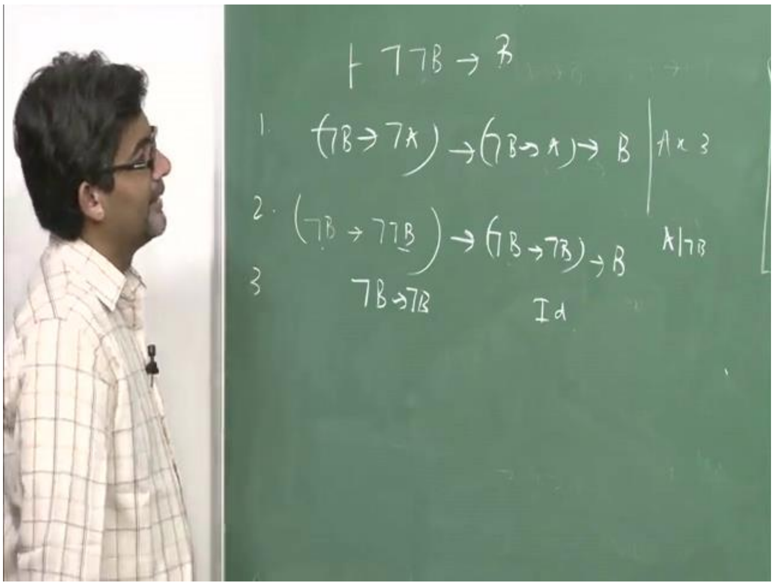

So, now, just now... So, there is law of identity we have showed it just now that is B implies B is the 1 which we have showed just now. So, now, in this if you put B if not B then this will become this. So, this is what law of identity I can say identity this which we have already proved. So, what we will be using is in order to simplify this proof. So, not not B implies B in the case of russell whitehead axiomatic system it involve some some 14 14 steps and all to show that not not B implies B is a is a rule of double negation. So, now, we will pause things for a while here and then we will talk about 1 important theorem in the axiomatic prepositional logic.

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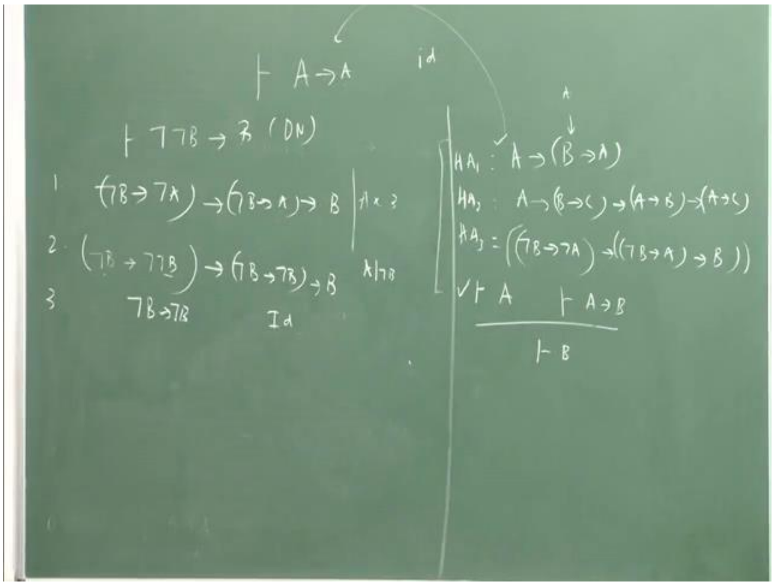

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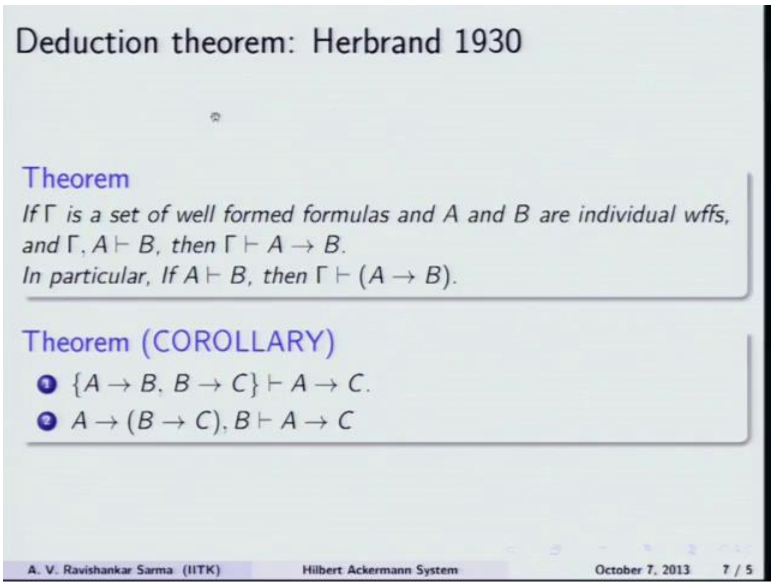

So, that is the deduction theorem and then now then we will make use of this deduction theorem in proving this particular kind of thing. So, this is what is considered to be the deduction theorem. So, I will come back to this particular theorem little bit later. So, now, this deduction theorem is due to herbrand in the year 9 teen thirties and on the same time even natural deduction systems are also come into existence we do not know exactly what kind of relation you will find it between herbrands deduction theorem and natural deduction theorem (( )) proofs using a natural deduction theorem that is due to promise and others. So, this theorem says like this off course every theorem has to find a proof.

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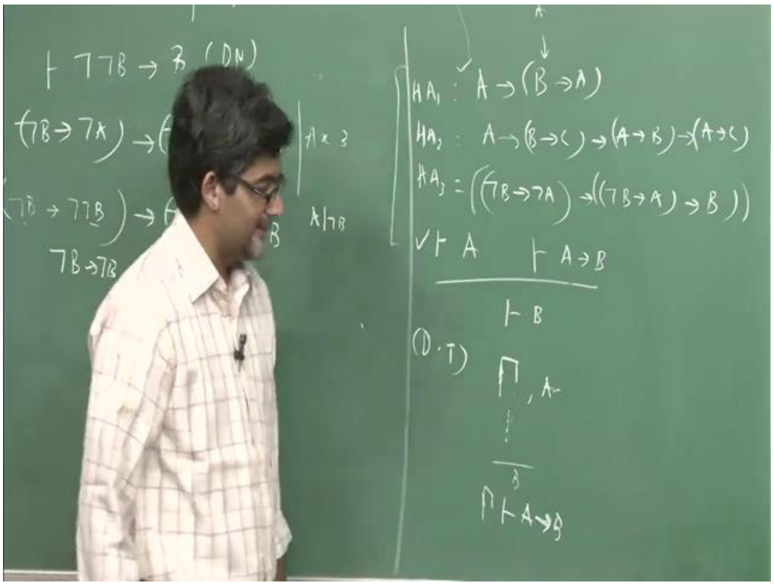

Suppose if you if you take gamma as set of well form formulas in the sense that it has all the well form formulas which you can think of and you single out true formulas a B they are considered to be individual formulas and if it is. So, happens that B is reduced from gamma and a then a implies B can be reduced from gamma. So, that is like this. So, this is what we discuss about it. So, now, you started with particular kind of set of well form formulas now when from that you also have a and from this you reduce b. So, if you can reduce B from this in; that means, d has come after some kinds of steps some finite number of steps you got b.

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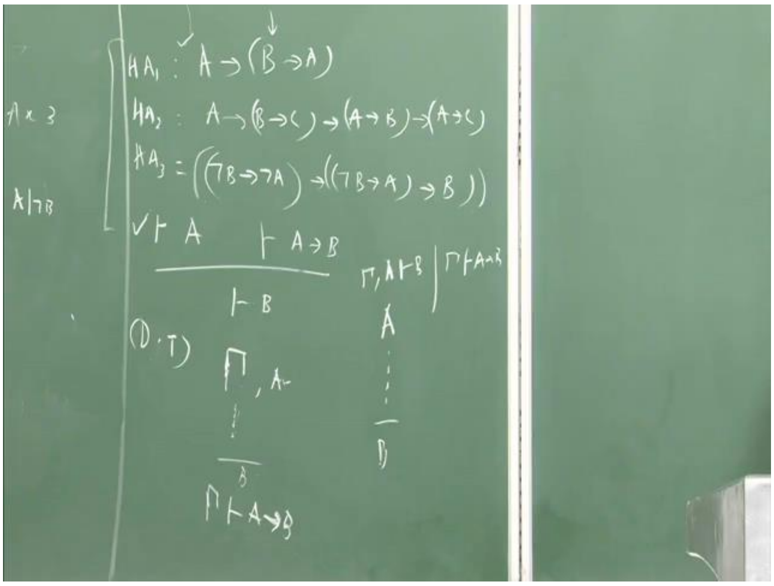

If that is a case then you discharge the these assumptions and then you talk about this thing some gamma you can even derive a implies b. So, that is what is the case. So, this is what is called as deduction theorem so; that means, if I given set of formulas gamma and taking an assumption a you reduce b; that means, you all you already set to reduce a implies B from a given set of formulas gamma particularly if you have this particular kind of thing a and from that you generated B this is what you write it in this way then off course gamma is already there here then you say that it is gamma a implies B this is the same thing which we have said already.

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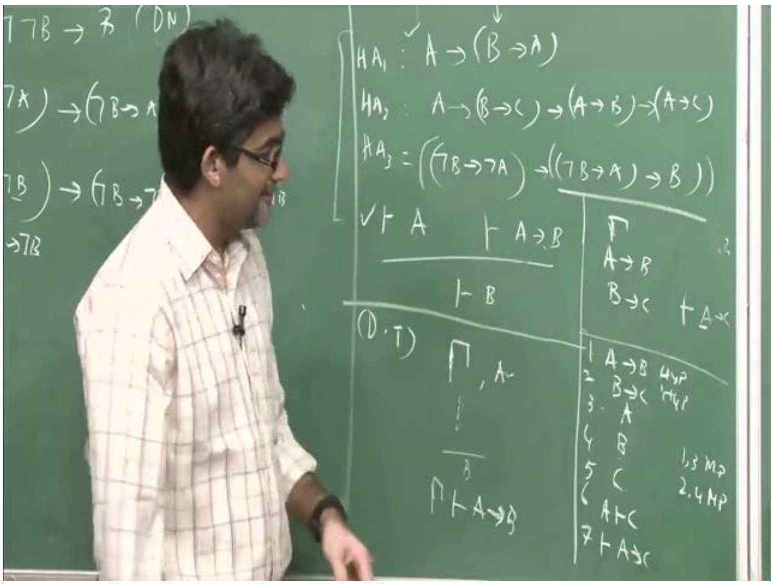

So, this is what is considered to be a deduction theorem actually in mathematics every theorem has to find a proof and all at this moment we are not trying to produce proof for this particular kind of thing otherwise it has to end with suppose if you say that it is a theorem, and if you do not have a proof then it is not considered to be a theorem. So, every theorem has to find a proof, but due to the limitations of time and all we are not going to the details of proof of this particular kind of theorem, but we make use of the idea behind this particular kind of theorem. So, there are 2 important corollaries for this particular kind of theorem.

So, they are like this suppose if a implies B and B implies c are there already there and one of the outcome of this 1 is that you can reduce a implies c. So, you have you have reduced let us assume that these are the 2 hypothesis and all and now let us try to prove this thing you have a set of formulas gamma and then you have a implies B and you have B implies c and from that you will get a implies c. So, how do we prove this thing to take a implies B as the hypothesis and B implies c as another hypothesis now you assume the antecedent of this conclusion let us say this is the conclusion a implies c. So, now, you take the antecedent of your conclusion which have placed in the form of a condition. So, these are the steps that we have. So, now, 1 and 3 modus ponens 1 and 3 modus ponens you will get b. So, now, in the fifth step 2 and 4 modus ponens that is this principle we have used from a a implies B then you can reduce b.

So, now these 2 modus ponens you will get c. So, now, in the natural deduction proof and all you can use deduction theorem now since from some a you got c it is; that means, it is like this. So, from c c is obtained from a with some kind steps and all 1 or 2 steps are there involved in this thing; that means, you can write here in the 6 th step you can write like this now in the 7 th step you can simply write like this this is a left hand side goes to the right hand side and you will get c. So, a implies c is the 1 which we were trying to reduce. So, 1 of the important corollaries of this particular kind of theorem is is that if a implies B is reduced and a implies B b implies c is hypothesis then from that you can reduce a implies c.

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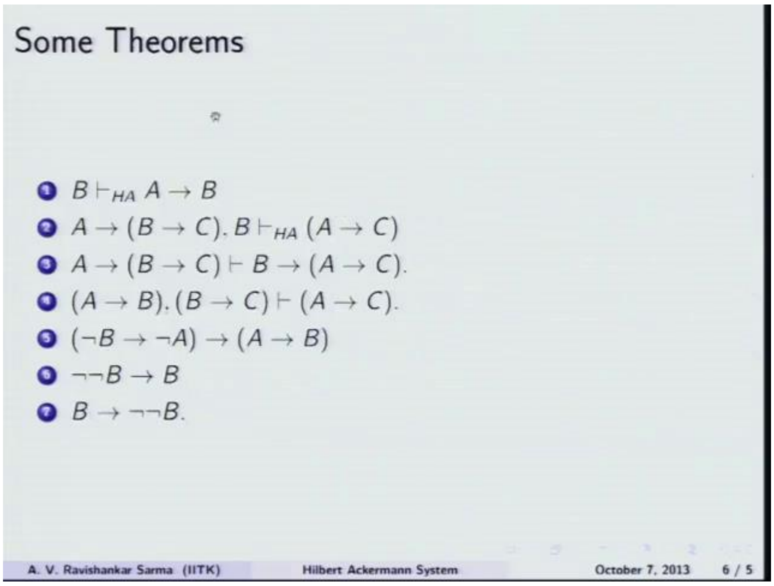

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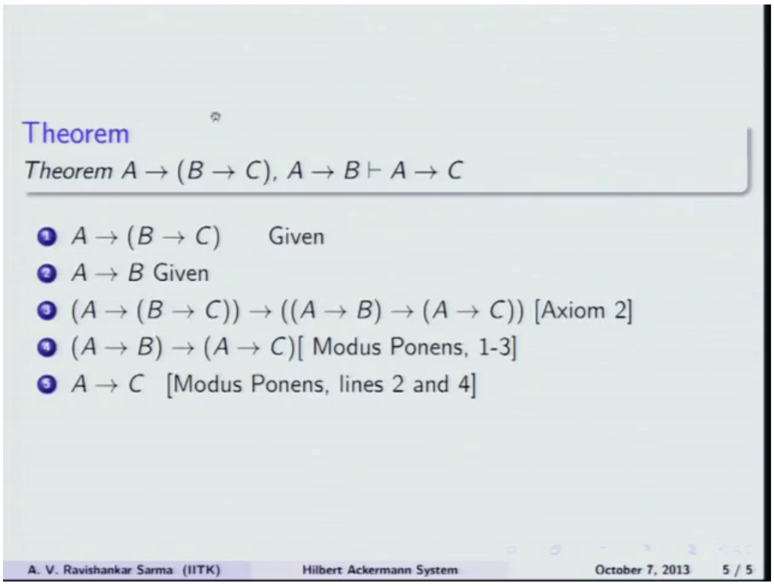

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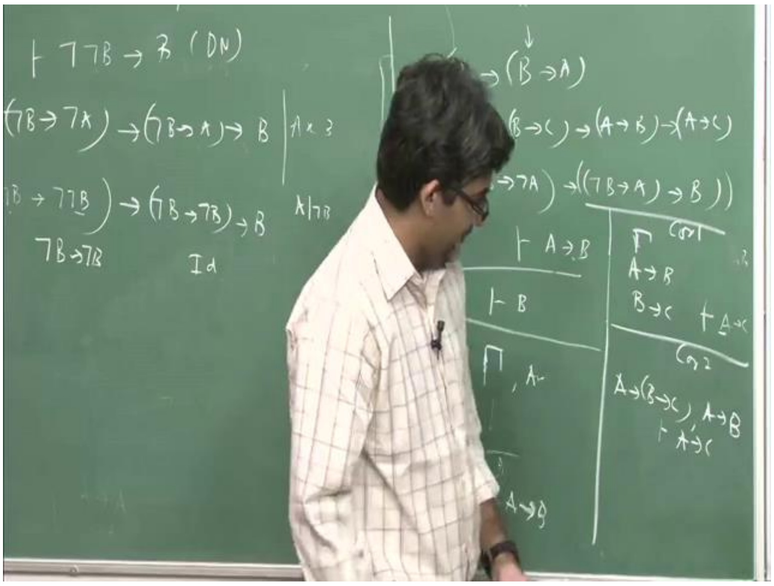

So, this we can make use of it and the other important corollary is this thing from a implies B implies c and you have B then you can reduce a implies c. So, this is in this kind of thing. So, this is 1 of the important corollaries of deduction theorem. So, what is that will write it down on here a implies B implies c a implies B and then a plus c. So, this is another 4teen corollary and then I will going to the proof of this thing. So, these are the 2 corollaries corollary 1 and then is corollary 2. So, now, you have a implies B implies c and a implies c sorry a implies B from that you can reduce a implies c.

So, now let us see how we can do it. So, now, first thing which you will find it in the slide is is that a implies B implies c what is given to us. So, now, a implies B is already given. So, there is a 2 things which you find it in the hypothesis and there was the given kinds of things. So, now, using axiom number 2 that is a implies B implies c is a implies B implies c implies c. So, that is a axiom that we will make use of it and then you apply a modus ponens on 1 and 3 because you have the same thing a implies B implies c then what you get is a implies B implies a implies c. So, now, we already have a implies B so; that means, you will get a implies c if you want show it carely and all I think we do not have space here.

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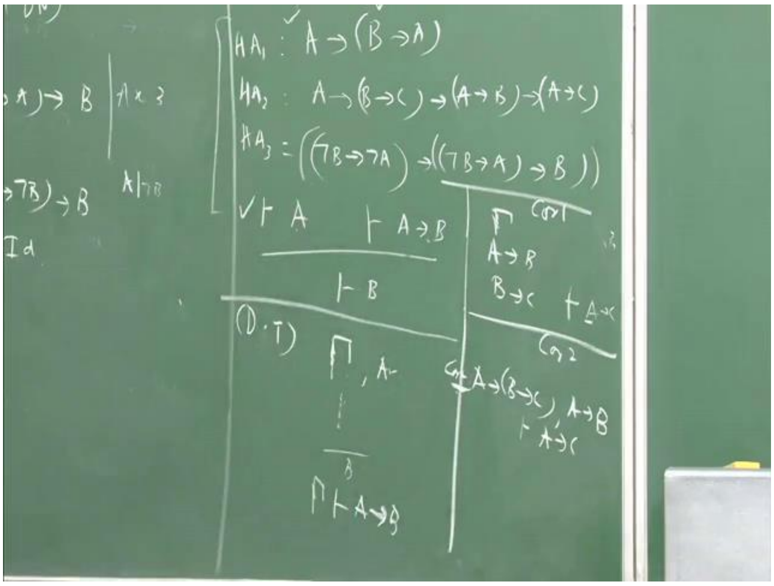

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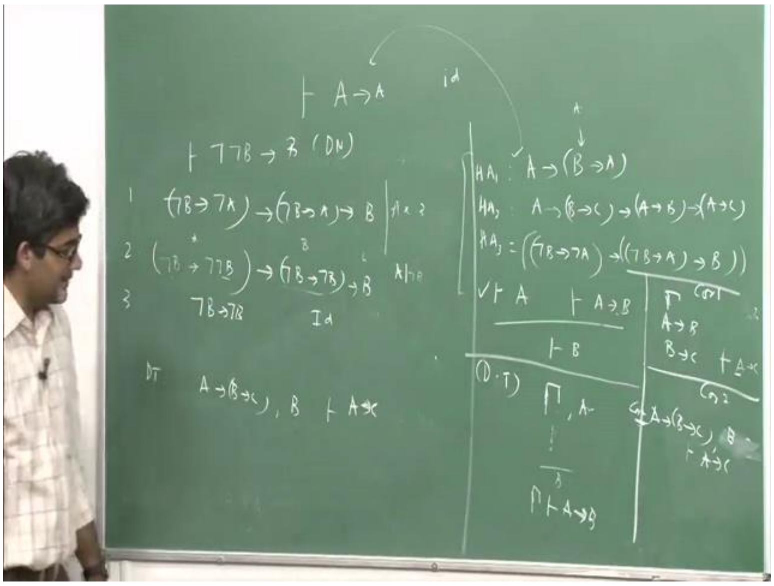

So, we will not going to the details of that 1 this proof is already here. So, now, this is corollary 2 now we make use of these theorems in proving that is not not B implies b. So, now, this is in this particular kind of format. So, for example, if you take into consideration this as a and this as B the whole thing and this as c it is like this thing a implies B implies c a implies B implies c. So, now, the second statement actually the the important corollary of that is 1 is like this suppose if you have a formula like this a implies B implies c and then B then you will get a implies a implies c. So, this is also 1 of the important corollaries of deduction theorem. So, that is the theorem which e have just simply B from that you will get a implies c. So, now, in this sense now to take this a as this 1 the whole thing and B has this 1 c has b. So, now, we have formula like this a implies B implies c the first formula and then B is same as this 1 this particular kind of portion and from that you should be able to get a implies c. So, that is. So, what we should get here if you apply this particular kind of thing.

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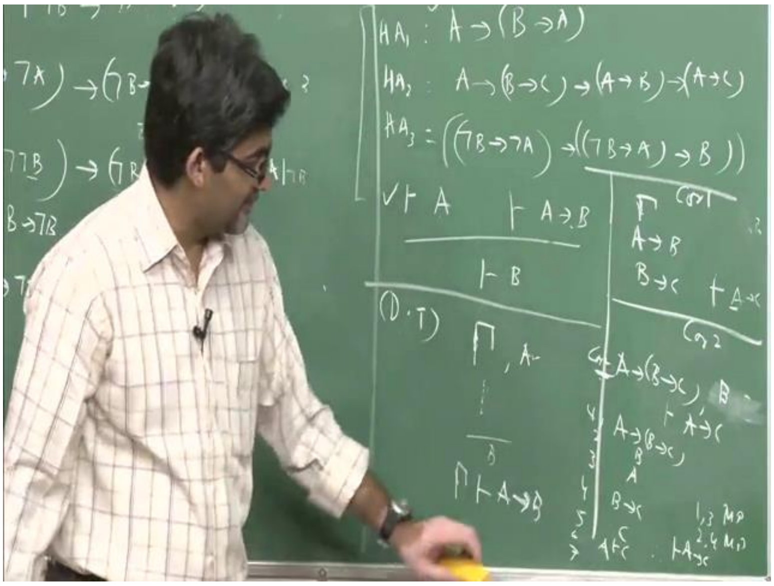

So, this a implies c is the 1, which you need to get what is a here this is not B implies not not B implies c is b. So, that is what you will get by using corollary 2 actually this needs to be modify in this particular kind of sense a implies B implies c and B from this you will get a implies c 1 can prove it by 1 can show it by using a set of things which we already know 1 of these axioms you can take into consideration and may be modus ponens and etcetera you apply on this 1 you will get this a implies c or this you start with a implies B implies c there is a first step and then you assume this second thing and then third thing is you assume the anti strength of your condition that is a.

So, now as the 4th step 1 and 3. One and 3 modus ponens you will get B implies c. So, now, fifth step B and 2 and 4 again modus ponens; that means, this 1 B and B implies c you will get c. So, now, you have reduced c from a so; that means, you apply deduction theorem again and this will become a implies c means this is a implies c. So, that way we can prove this particular kind of theorem corollary will come has an outcome in this way. So, you not have to apply any axiom here you just use modus ponens rule then ultimately you got this particular kind of thing the same role is employed here and this is now what a we get. So, now, till now it is not in this particular kind of format now.

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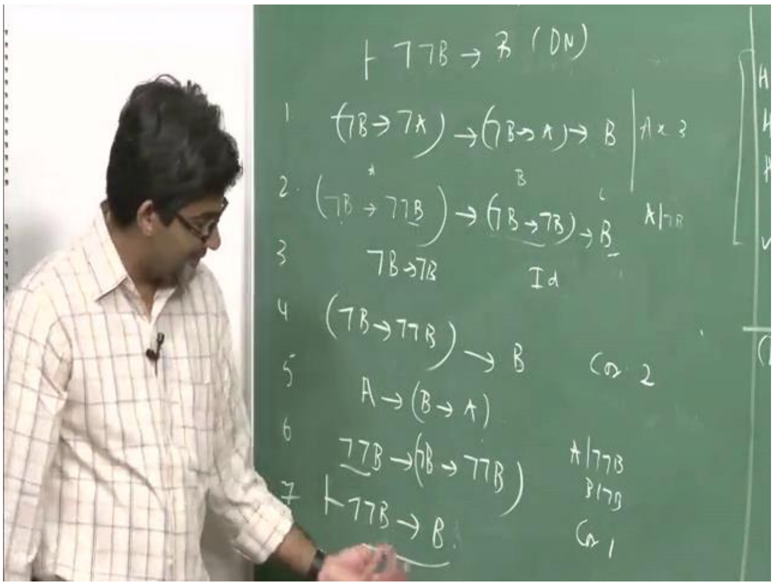

Some of we need to use axiom number 1 a implies B implies a if you can substitute a for not not a fore not not B and B remains as it is and then for a it is not not B implies B is as it is then a is not not b. So, now, observe this particular kind of thing not not B implies B implies this 1 and the same thing not 1 second and B has substitute B has not not. So, then it will remain the same thing. So, now, .

Not not B implies this 1 some x and this x implies this this thing now use corollary 1. So, that is a implies B b implies c a implies c is the case. So, you have to read it in this way from 6 to 7 you need to not not B implies this 1 and the same thing implies this 1; that means, this will become not not 3 implies B this is what we are trying to show. So, now, using there is deduction theorem and it is important corollaries we might simply our proofs. So, again what is this deduction theorem again it preserves the truth. So, every step of your proof is a kind of proof preserving kind of thing which we are employing here. So, that is why the final final step of your proof is also considered to be a theorem. So, final step of your proof is usually consider to be a theorem. So, that is why this is proved in this particular sense.

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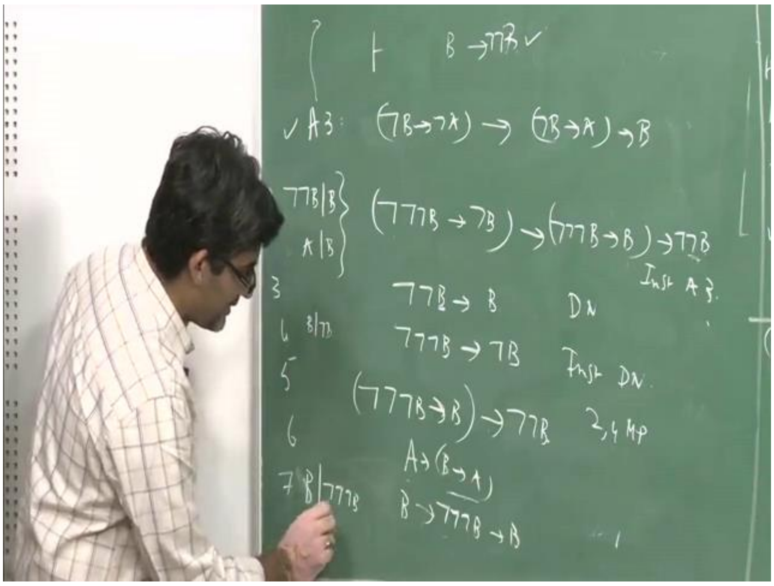

So, there are some other important and interesting proofs 1 can do it this is this is only for our practice. So, more and more we practice more and efficient we will become in deriving this theorems. So, now, let us say we are trying to prove instead of not not B implies B we are trying to prove B implies not not b. So, this is the other not not B implies B is a double negation, but we are trying to show this. So, how do we a g go about this thing. So, instead for proving this particular kind of thing you need to choose some of this axioms. So, they are like this first you start with axiom number 3 what is this axiom number 3 not B implies not a not B implies a implies b.

Off course you can do the same thing by a using it is corresponding axiom and all which is there on that is not B implies not a implies B implies a implies B 1 can use this 1 also, but we are making use of revised version of Hilbert ackermann system. So, now, you start with this particular kind of thing now you substitute not not B per where ever you B occurs you substitute with this thing and a where ever you find a you substitute with B then this will become what is B now not not b. So, not of what is a here this b. So, B means not not B this a means B and now B means not not B what is this instant of instance of axiom 3. So, this is what is concerned to be an instance of axiom number 3. So, what is that we are trying to derive we are trying to derive B implies not not b. So, you might ask you might ask ourselves that that. So, why need to follow all these steps you know I can jump to take 1 axiom and jump to this particular kind of thing and all usually would not get it like that. So, it is a path which leads to this particular kind of truth 1 truth is leading to another kind of truth. So, this is not at over somehow we trim this axiom in such a way that the last in the suppose if you take .

In this as a whole well form formula the last part of this conditional is somehow standing out to be this 1 that is coming closure to this 1. So, this is the second step now just now we showed law of double negation this is what we already proved this is what is double negation. So, now, in this 1 this substitute for d not where ever B is there you substitute with not B then it will become not B and B is this 1. So, now, for this is what instance of double negation. So, now, fifth 1 2 and 4 modus ponens because this is same as this 1 this 2 modus ponens you get not not B implies B that implies whatever you say not not b.

So, now, till now we did not get this thing and all somehow we need to use some other kind of axiom and we need to converted into appropriate form. So, now, we have this axiom a implies B implies a. So, now, here in this 1 suppose if you can somehow you convert this B implies a as the same thing then you can say this particular kind of thing. So, in this 1 what you do is you substitute variable a is there you substitute with B and variable B is there to substitute it with not not b. So, now, this will become instead of a you have B here.

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B means not not not B implies B is as it is B as once second. So, what is that we are trying to true somehow this needs to be converted into this particular kind of thing 1 second. So, you use only B is same as not not not b. So, this will become this 1 and a has 1 second a implies B implies a not not a as not not b. So, now, this axiom will become.

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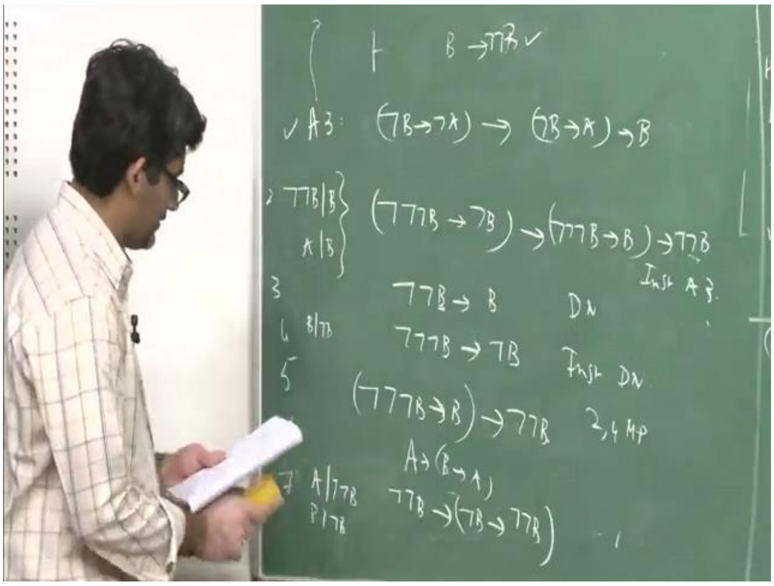

This 1 1 second. So, what you have done here is this will become not not B implies not B implies a means not not b. So, what you have done here is is that for a you substitute it not not B and for B you substitute with not b. So, now, this is what it becomes. So, now, what we have here is this thing I am sorry here this very sorry for this. So, it is a implies B implies a, this is axiom number 1.

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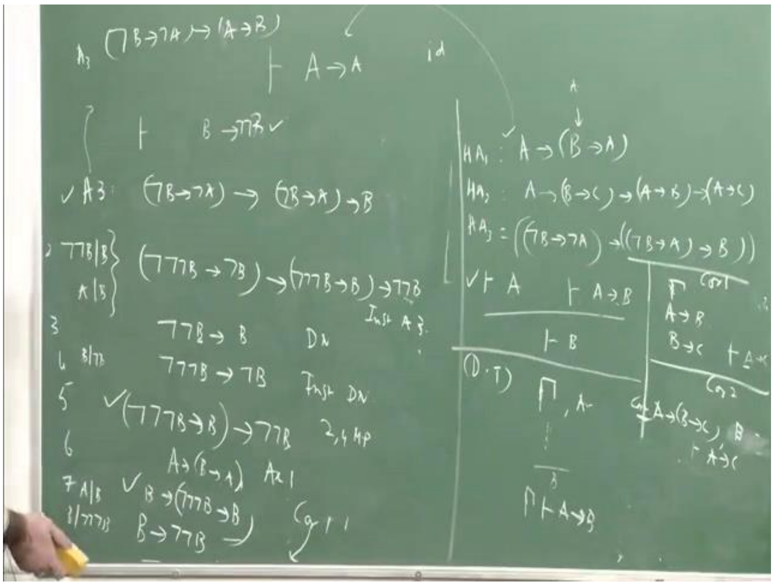

And somehow this should con should be converted into this particular kind of format. So, now, for a if you can take as B and then B as not not B and then a will become this thing. So, this simple kind of translation and all. So, I am just. So, what you have done here for a you have substituted it B where ever a is there you substitute with B and for B you substituted not not B this is what happens. So, now, this is the 7 th step. So, now, observe this 2 things this 1 and this 1.

So, now, this is B implies some x and this x implies not not B so; that means, using corollary 1 of course this follows here, you can say that it is B implies not not B why it is a case because B implies not not not B implies B whether same thing not not not B implies not not b. So, then this goes to this particular kind of thing. So, that is what we are trying t p prove. So, in this way we can prove B implies not not B how did we do this thing you started with a axiom and then again we use 1 important corollary of deduction theorem then our proof has become simplified here. So, let us consider 1 or 2 more proofs and we will end this lecture is only for our practice we are trying to talk about more number of proofs I mean proofs of theorem. So, now, this is what is famous kind of instance material implication? So, now, this is what you are trying to show. So, from not a a implies B follows.

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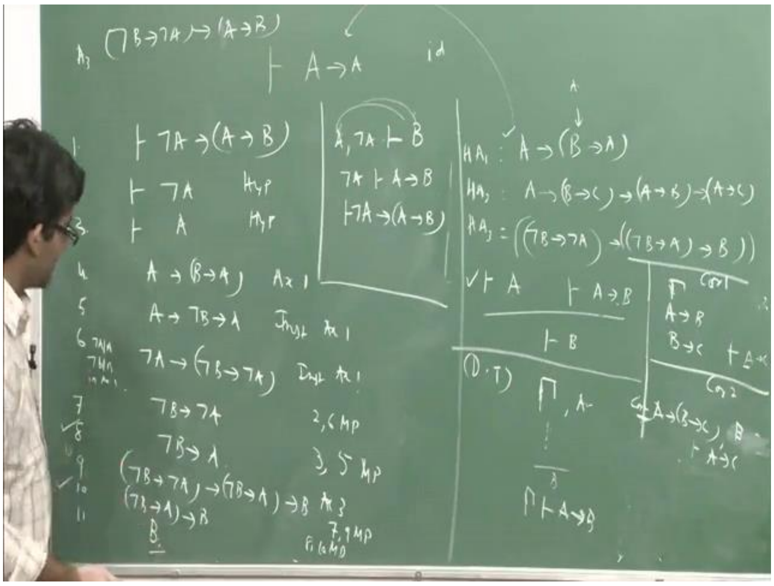

So, now, how do you prove this thing first you list out this, this is considered to be One hypothesis that you write it like this and then you take a also as hypothesis then we have. So, we have an axiom that is a implies B implies a. So, this is what is called as axiom number 1 and if you transform this into certain way if you substitute not B for a this is also considered to be an instance of axiom number 1 this is fifth step. So, now, you apply modus ponens on this things you will get this particular kinds of things now another instance of this particular kind of axiom is this. So, a implies suppose if you substitute not a For a and not B for B in axiom number 1. So, you will get not B implies not a is also instance of axiom number 1. So, ultimately we need to show the B should come as an outcome of this particular kind of thing. So, this is what is also an instance of axiom number 1? So, now, 2 and 6 modus ponens what is 2 here not a and not a plus this 1 this modus ponens you will get this thing 2 and 6 modus ponens you will get this. So, now, 2 and 4 and 3 and what else is thing here 3 and 5 3 and 5.

Modus ponens you will get not B implies a. So, we list out the hypothesis in this conditional not a, and we also assume that a is a case. So, now, we are trying to show let B follows in this particular kind of thing if that is a case then you can show that not a implies a implies B is the case. So, now, till now the proof is not at over. So, we have generated not B implies a and we have not B implies a not a. So, now, the axiom number 3 is is like this not B implies not a is same as not B implies a implies b, but if this which is axiom number 3. So, now, observe this 7 and 9 the 7 and 9 modus ponens again you will get this particular kind of portion.

Not B implies a implies B and 10 th step. So, now, observe 8 and 10. So, 8 and 10 again modus ponens 8 and 10 is here not B implies a and not B implies a less from these 2 you will get B same as this 1 this gets detached and what you get is this 1 b. So, these are the end of the proof and all. So, what we essentially show is this thing is I am writing it here. So, now, what is that we got from a and not a what you got here b. So, now, this is what we are showed. So, now we need to apply deduction theorem twice. So, that these 2 things will come and at the right hand side. So, now, first time when you apply deduction theorem this goes to the right hand side.

Here is an order which we need to follow, suppose if you have 2 formulas a not a and B first time when you applying modus ponens this goes to the right hand side and it will become not modus ponens deduction theorem it goes to the other hand and then it will become a implies b. So, now, next time when you apply ah the same deduction theorem it will become not a implies this goes to the right hand side it will become a implies b. So, like this 1 can use deduction theorem 2 or 3 times and all we can move all the left hand side things to the right hand side. So, this is the way to prove this famous of material implication this is one of the instances of paradox of material implication. So, this is the way to show you.

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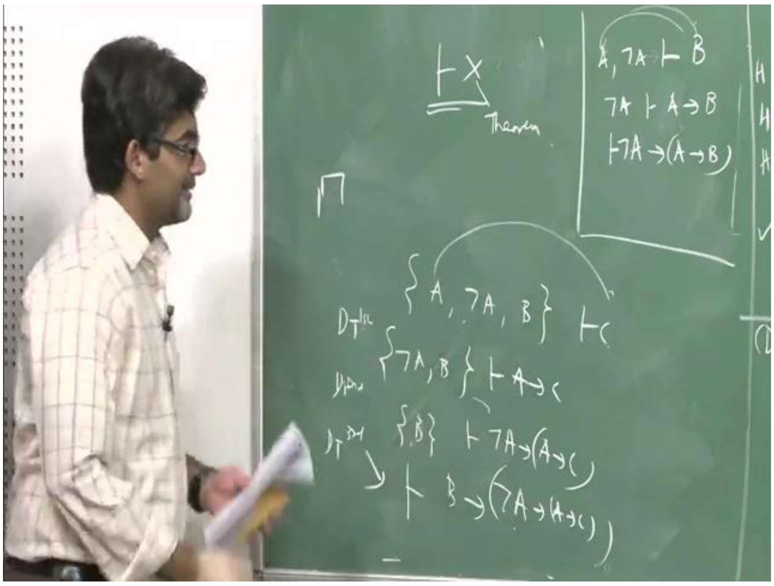

So, now which will get with this deduction theorem for example, if you have a set of formulas like a not a and B and yet you get c. So, this is what you obtain from let us say this c is obtain from this 3 things. So, now, you keep on applying deduction theorem for the first time when you apply it this goes to this particular kind of thing, then it will become a implies c. So, now, the next time when you apply this particular kind of thing off course gamma is already there set of well form formulas taken together with these things leads to c next time when you apply the deduction theorem this goes to the other side then it will become not a implies a implies c. So, now, if you want to eliminate this also then you need to apply this is second time you apply first time and deduction theorem applied third time these 2 this 1. So, this goes to the right hand side this will become B implies ah.

Ah. So, what is that B implies not a implies a implies c. So, whole thing is in brackets. So, like this 1 can use deduction theorem n number of times and ultimately this this 1 can show it as a tautology whatever there in the right hand side suppose if you write it like this there is nothing at the left hand side; that means, whatever follows up to this 1 is considered to be theorem. So, that is what we can 1 can show. So, now, let us consider the last kind of theorem which is called as law of contraposition when using same kind of Hilbert ackermann axiomatic system. So, with this I will end this lecture. So, what is this law of contraposition that is not B implies not a implies a implies B.

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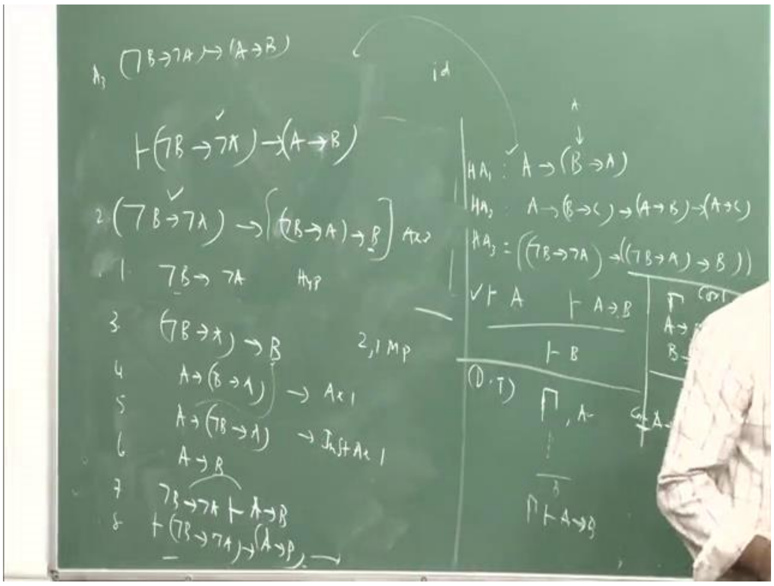

In each and everything which you are trying to prove what essentially 1 requires is is that what kind of axiom 1 needs to take into consideration. So, that that goes as far as possible closer to this particular kind of thing. So, again we make use of this thing and the last axiom a not B implies a implies B this is axiom number 3. So, now, we are proving it with a help of this thing now take into consideration some of the hypothesis. So, that is not B implies not a as your hypothesis. So, this is the if you suppose if you assume that this is the whole conditional and all. So, now, the first part is considered with a antecedent and this is the consequent.

So, now in this antecedent part is assumed. So, that is not being implies a and you also assume this particular kind of thing. So, that is not B implies a this also considered to be hypothesis or assumption etcetera now from this you need to prove b. So, so not B implies not a of this particular kind of thing which we assumed. So, what essentially we have done here is is that first we started with this hypothesis that is antecedent part of your conditional that is considered to be assumption are hypothesis now we have used this particular kind of axiom. So, now, ah this needs to be stated below that, but it does not matter let us say this is the second step and the third step not B implies not a not B implies not a same thing is by Modus ponens 2 and 1 modus ponens you will get not B implies a implies B this is still not in this particular kind of format a implies B we need to do little bit of this thing now we make use of axiom number 1 that is a implies B implies a this is axiom number 1. So, now, 1 instance of this particular kind of axiom is like this a implies for B to substitute with not B and this will become like this. So, now, this is instance of axiom number 1. So, now, we have a implies not B implies a and not B implies a implies b; that means, in the 6 th step you will get a implies b. So, how do we get this 1 this x implies y and y implies z; that means, x implies z that is a implies b.

So, now what we have shown here is is that from assumption not B implies not a a implies b. So, now, you apply deduction theorem here then this goes to the right hand side and this will become not B implies not a implies a implies b. So, this is what is considered to be law of non law of contraposition. So, in this lecture what we did is simply like this that we presented Hilbert Ackermann axiomatic system both in the unrevised format and the revised form to take into consideration in the revised form h prime which consist of the third axiom this 1 otherwise it is it was like this that not B implies not a is nothing but a implies b. So, that also you can take it has 1 of the important axiomatic system. So, we presented the axiomatic system which involves only implication.

And negation signs and then we use transformation rules and modus ponens then we derive some of the theorems and we also made use of one of the important theorems of axiomatic prepositional logic. So, that is deduction theorem deduction theorem in an action tells us that if you have set of formulas gamma and you have a formula a and from that if you reduce B then you also set you have reduced a implies B a by using the same set of formulas gamma and there are 2 important corollaries that we have discussed in greater detail that is you also showed you also proved this things, suppose if a implies B and B implies a has hypothesis and then from this a implies c will come as an out comer. So, that is a kind of rule of syllogisms and all. So, another important property is that if a implies B implies c in the case and B is the case then continue and reduce a implies c we made use of this corollaries and deduction theorem and then we have simplified the proofs that are there in the there in the given axiomatic system. So, so far we have studied principia mathematica the Russell axiomatic system due to Russell whitehead and another axiomatic system due to Hilbert and Ackermann. So, in the next class what we are going to see is that are these system complete; that means, in a sense that all the provable things that are there in the this axiomatic system are true a valid or all the valid formulas are considered to be true whether or not the system is complete etcetera. We will establish these things by using by making use of some kind of theorems such as theory of consistency theory of soundness etcetera and all. In the next class we deal with whether or not the principia mathematica is complete etcetera, all these important courses we will be dealing with in the next class.