Introduction to logic
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Lecture - 32
Characteristics of formal system PM

Welcome back, in the last lecture we presented Hilbert Ackermann axiomatic system and before that we presented one of the important axiomatic system that you will see it in principia mathematic. That is another axiomatic system which is due to Roseland white head; in the Roseland white head axiomatic system what you will see is this. That there is a choice of taking the primitive logical symbols Roseland white head took into consideration the negation and disjunction.

Whereas, Hilbert Ackermann took implication and negation as the primitive symbols, so in the axiomatic prepositional logic our goal was is that, we have some kind of valid formulas and then we are trying to come up with theorem proofs of those theorems. So, how you prove the theorems you state the axioms explicitly. So, they are absolutely true and then we have asset of transformation rules which preserves the truth of a given formula.

Then, you have a simple rule of inference that is the rule of detachment which is also called as principle. And using these 3 things we transform the axioms we trim this axioms in such a way that we derive minimal things such as law of middle and law of contraposition etcetera. So, now in this class what I will be discussing is this that whether system axiomatic system are these consistent in a sense that it is not the case that you derive both x and not x.

In that sense it is consistent or these systems are said to be complete or strongly complete or weakly complete etcetera all these things which will be talking about in some detail in this lecture. So, 1 of the important advantages of knowing this particular kinds of theorems is this that suppose, if your axiomatic system is complete in a sense that whatever what all you prove are valid and all valid formulas are also provable.

Then, instead of checking for example, if you proof is very hard to come by then you can use the completeness theorem and you can say that, you can show that instead of proving the theorem you can simply show that a given well-formed formula is valid. So, that is 1 of the advantages of having this completeness theorem in particular.

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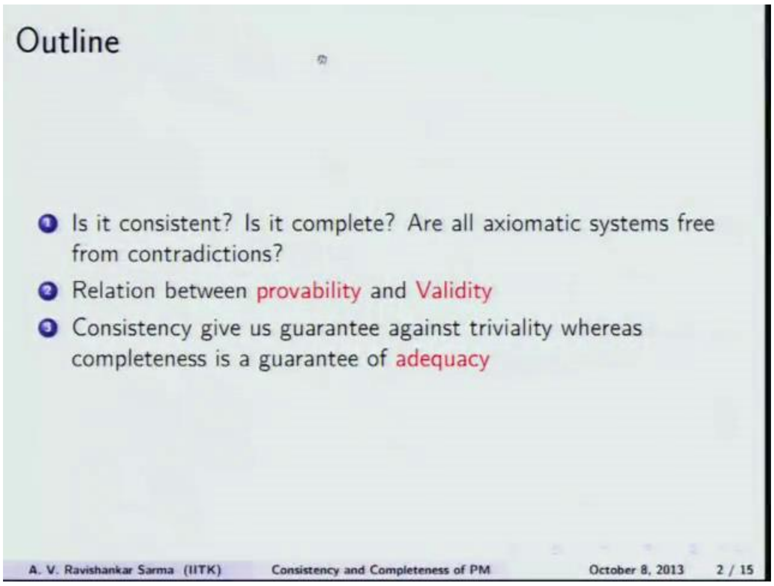

So, the outline of this stock is this that is the 1 which we have presented in the last class, is it consistent? Is it complete? That means, all are all axiomatic systems the 1 which we spoke earlier are they free from contradictions? That means, within your given axiomatic system you should not be in a position to derive contradictions. So, as we have seen earlier if you have contradictions, if you start with a contradictions you can derive anything.

Or from impossibilities you can derive anything. So, that is why we have shown in the last few classes. So, then we will be talking about the relationship between provability and Validity. So, that is what this completeness establishes. So, if all the provable things are also considered to be valid; that means, true tautology, validity are 1 or the same in the prepositional logic.

So, all tautologies are obviously, valid statements; if whatever you have come up with a proof of some theorem and then it so happened that it is also turned out to be valid. But it is the case that all provable theorems in your axiomatic system are considered to be valid. In the case of prepositional logic, this is the case all what all you can prove are obviously, considered to be valid.

Because, these step of your proof is a result of applying either an axiom or theorem which is obviously, true or some kind of transformation rule which preserves the truth. Then, the rule of detachment which also preserves the truth, so each step is considered to be true. So, the final step of your proof which is considered to be a theorem that is also considered to be true; so this consistency gives us guarantee against some kind of triviality results such as x and not x if you derive it at least.

The some kind of trivialities whereas completeness, guarantees, some kind of adequacy and all, so if it makes your systems adequate. So, these are the 2 things which it does consistency guarantees that it ensures that there are no trivialities in your axiomatic system and completeness is you will give us guarantee of your formal guarantees adequacy of your formal system.

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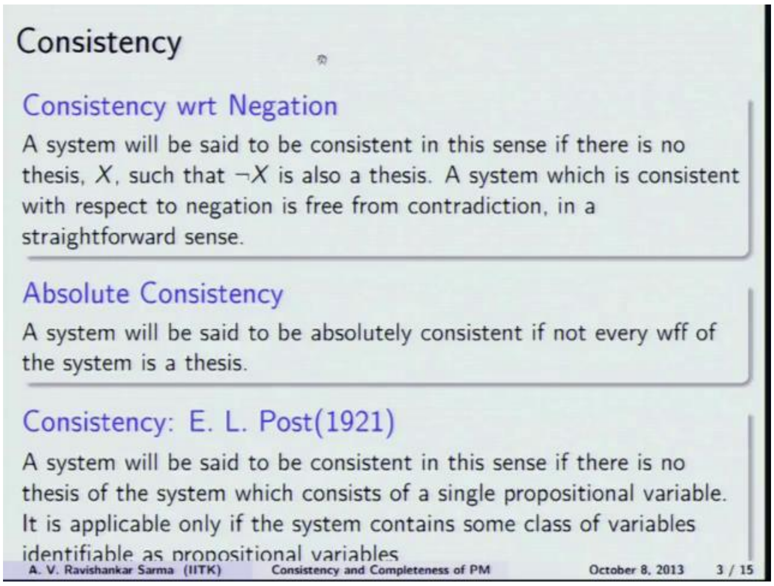

So, there are first to start with will talk with consistency; consistency is the 1 which we have already seen earlier.

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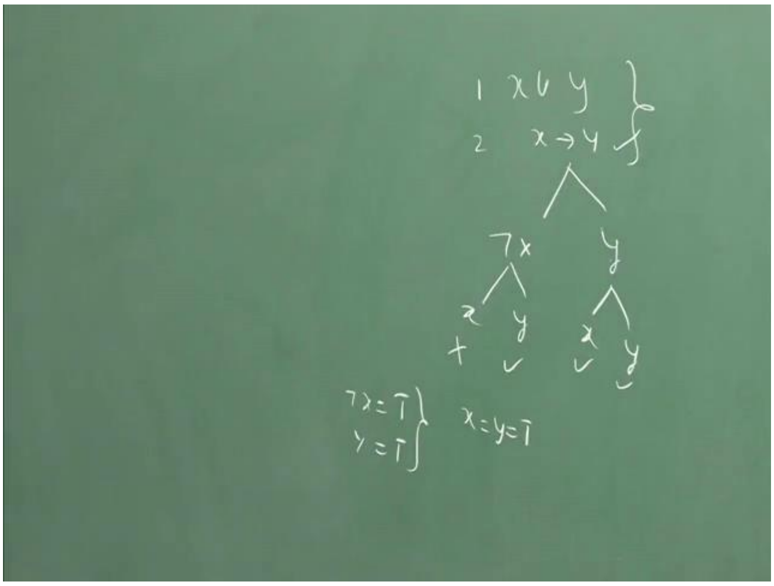

So, whenever we have for example, 2 groups of statement 1 and then suppose if you have another statement like, x implies y etcetera these are the 2 statements that we have. So, these 2 statements are consistent to each other especially when you construct a tree for this 1 using Semantic Tab Locks Method; at least some of the branches open. So, now you construct a tree for this 1 not x and y.

So, this the first we checked it and then this is x y x and y. So, this branch closes, but all the other branches opens; that means, this particular kind of assignment satisfies this particular kind of formula. When not x is t and y is t, then it satisfies this particular kind of formula whereas, in the when both x and y are takes the value t that also satisfies this particular kind of formula.

So, using Semantic Tab Locks Method 1 can find out when a group of formulas are said to be consistent. If at least 1 branch is open then that means, that is set to satisfy this particular kind of formulas and all. So that means, that makes these 2 formulas true. So, it is in that sense we usually call it as consistency.

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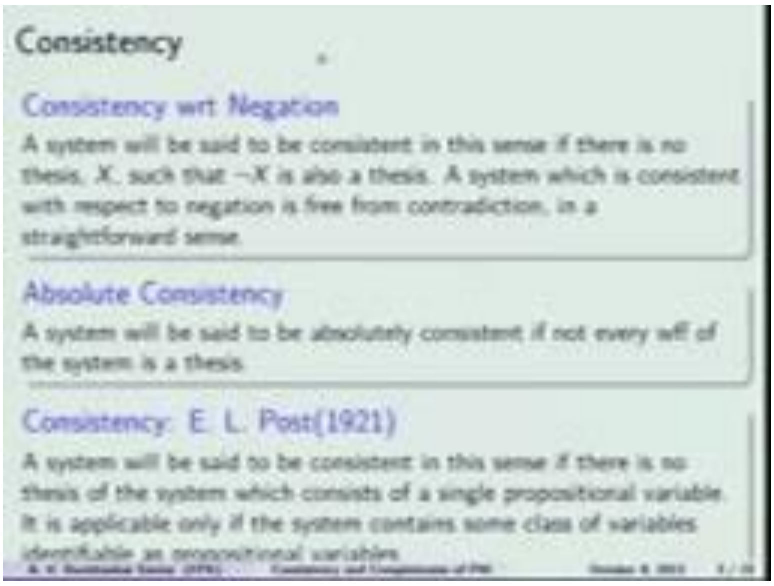

So, now there are 3 kinds of consistency: one can talk about a the first 1 is consistency with respect to the main logical operator negation. So, what do we mean by consistency with respect to negation? If a system any axiomatic system will be set to be consistent in the sense that if there is no thesis; thesis means, it can be an axiomatic can be a theorem. If there is no thesis X such that, not X is also consider to be thesis if that is a case then system is said to be called as consistency with respect to negation.

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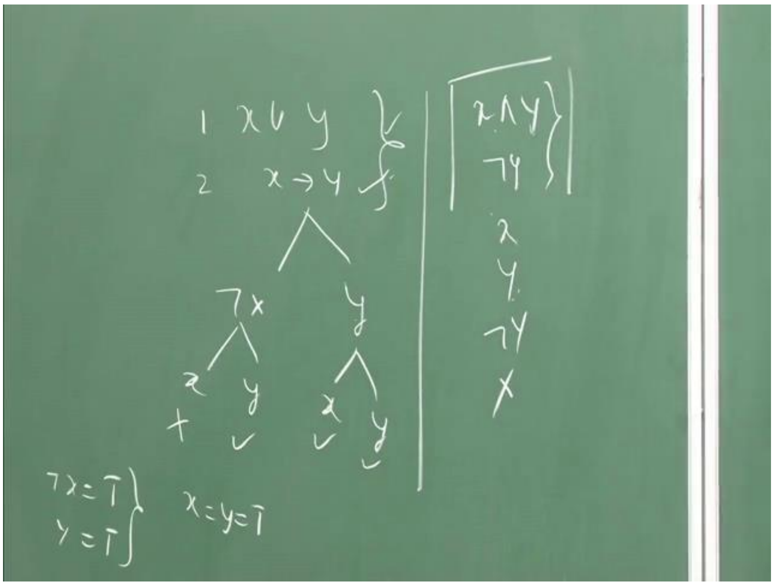

So, that is what we have seen on the board. Suppose if you have a formula like x and y and not y. So for example, this is the 2 groups of statements that we are taking into consideration. So, now again if you construct Semantic Tab Locks Method, then x and y can be written in this sense and not y is like this. This y and not y this a contradiction, so it closes. That means, these 2 statements are inconsistent to each other.

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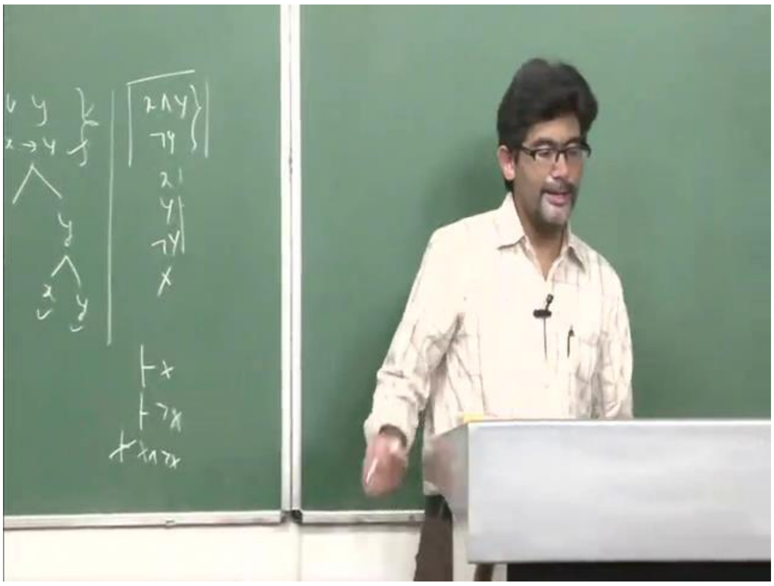

So, this is what we mean by consistency with respect to negation. So, it tells us that either it should be in a position to derive x or it should be in a position to derive not x. But it should be the case that both y and not y should be there in your proof. So that means, either it should be in a position to derive only x or it should be in a position to derive not y.

But it should not x, but it should not be the case that x and not x if both x and not x you derive that system is called as a trivial kind of system. So, it is in that sense there is no thesis if you consider any formula well-formed formula X. If you derive X and if you derive not X also, then your system is considered to be trivial kind of system and that system is consider to be inconsistent; inconsistency with respect to negation.

So, a system which is consistent with respect to negation is usually free from contradictions. As you see here in this case, you constructed you are checking whether these 2 formulas are consistent to each other. In this 1 suppose if you constructed a tree and then all the branches closes and all that means, it leads to unsatisfiability leads to inconsistency. But here it is not the case in the first in the first 1 that is not the case.

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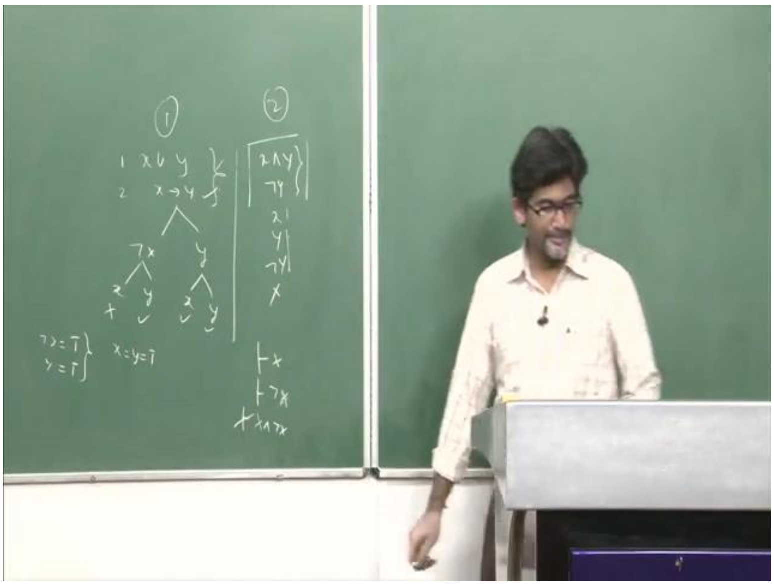

But in the second one you have this contradiction, so all the branches closes. So, consistency guarantees that there are no such kind of trivial things which are present in your axiomatic system; triviality how triviality results in, because you have x and not x. So, it ensures us that your system is free from contradictions.

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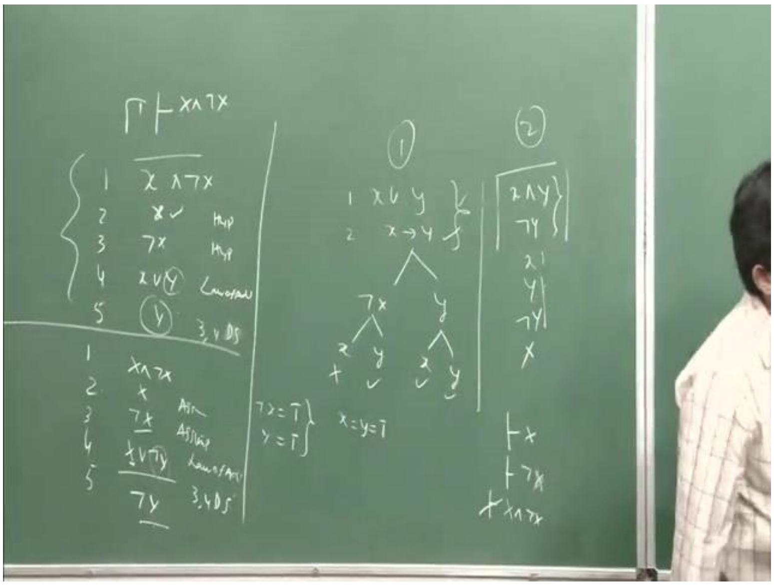

One of the problems with this contradiction is this that, which we have already using classical logics you can derive anything. For example, if you start with this particular kind of thing x and not x which is considered to be inconsistent. Then, this is the first step and then you assume this 1 x and then not x. Now, since x is true even x or y is also true why because so this is Law of Addition.

So, if x is already true our semantics allows us that x or not y is also true. So, this we can add it here now this is Law of Addition. So, this is already hypothesis or assumption or anything which you can take into consideration. So, now 3 and 4 leads to y. So, now from x and not x you can derive y, but in the same way you can produce the same kind of proof and you can derive even not y also.

So, it is like this again you start with x and not x, then 2 you assume the same things x and not x there all assumptions or hypothesis sometimes. So, now in the fourth step since not x is already considered to be true, then you can add any other strange kind of thing even not y also. Because, this is already true this ensures the truth of the whole disjunction; irrespective of whether not y is true or not y is false it does not matter.

Because, it is already true so this makes the whole disjunction true. So, this is the Law of Addition, so here we have added y, here you have added not y. So, then this leads to so now, 3 and 4 disjunctive leads to not y so now, in our system let us say you have set up formulas gamma, and then you have taken 1 inconsistency that is the contradiction x and not x.

Then, you derived y and you have also came up with the same kind of with rules etcetera and all. You have also come up with not y; that means, your system you have x and not x. So this is nothing, but a trivial kind of system; a system in which both x and not x are proved is considered to be a trivial kind of system. So, this what we mean by consistency with respect to negation. So, as far as possible your axiomatic system should be free from this contradictions like this.

So, there is another way of defining consistency which you will find it in the literature; logic literature that is Absolute Consistency. So, what we mean by an Absolute Consistency? A system is said to be absolutely consistent if not every well-formed formula of the system is a thesis. So the this means, that let us say that you have a formula axiomatic system and that there are so many well-formed formulas and all.

So, but not all well-formed formulas are valid kinds of statements; whatever formula that you take into consideration you will not be a kind of valid kind of formula. So; that means, not a true statement so if it so happen that your system is said to be absolutely consistent if and only if, there is not every formula of the system is considered to be a theorem or a tautology. Only selective kinds of things are considered to be either tautologies or axioms.

So, we started with some axioms and then we proved some theorems and all. That theorems are set of some kind of well-formed formulas and all. So, out of these well-formed formulas some are tautologies, some are contingent statements, some are also considered to be contradictions and all. So, your sets of well-formed formulas are big enough;, so in that only few formulas are considered to be valid formulas.

So, if you can ensure that not any kind of thing is considered to be a valid kind of statement or a thesis. Then, your system is considered to be absolutely consistent; that means, it should ensure that the only statement that you have are only tautologies and all. If the only statements that you have are only tautologies, what about contradictions and contingent statements and all; if you if you had built your system in such a way that you allowed for only tautologies now this not possible.

But if it so happen that, if your system has only tautologies nothing else then that is not called as Absolute Consistency; but you can build a such kind of system where you can come up with the only there is no way in which you can come up with usually with a system in which your system is considered to be absolutely consistent. Because, it is very difficult for us to construct only it difficult to- visualize a system in which there are only tautologies it is not quite possible.

So, your prepositional logical system is usually considered to be absolutely consistent. In a sense that, not every well-formed formula is considered to be a valid formula or a thesis. So, third this is another kind of consistency which discussed in greater length by E.L Post another important logician. This is also responsible for the truth tables etcetera following, so according to E.L Post a system will be said to be consistent in this sense if there is no thesis of the system which consists of a single prepositional variable.

So, it is applicable only if the system contains some class of variables identifiable as at least prepositional variables. So what happens here is this that let us assume, that you have constructed a grand axiomatic system in that system... if there is no thesis of the system which consists of a single kind of prepositional variable. If that is a case, then also it is considered to be consistent.

So, you should ensure that you do not have the single well-formed formula that is p or q or something like that; which turns out to be a well kind of thesis. Then, so that is not considered to be consistent in the sense of El Post. So, as far as possible you should avoid this particular kind of situation. So, that is a system will be said to be consistent in the sense that if there is no thesis of the system, which consists of a single prepositional variable. So, it is like this particular kind of thing.

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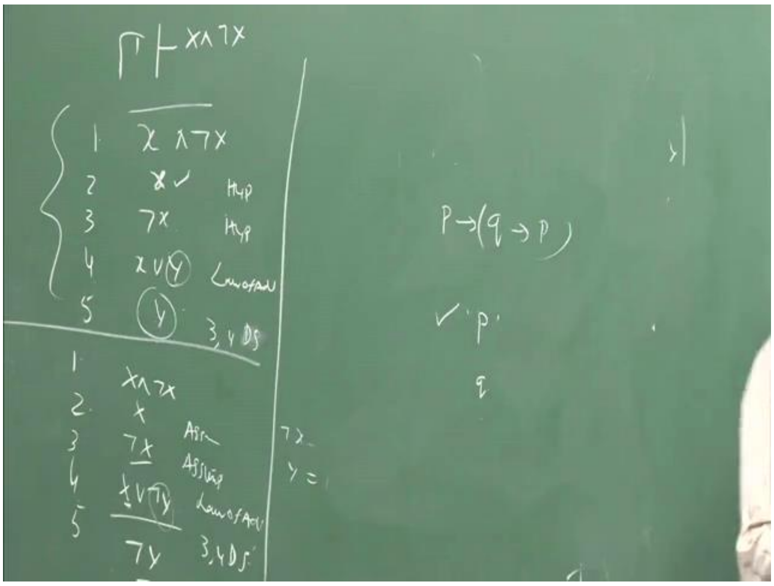

So, we can have these things at thesis and all like we have seen in the Russell Whitehead Axiomatic System etcetera p implies q implies p does not make any problem and all. But if it so happen that only this particular kind of thing is considered to be a thesis in your system, then your system is not considered to be consistent; if this also is viewed as this thing thesis. Then, your system is considered to be inconsistent.

So, according to El. Post we should ensure that these particular kind of formulas like p q symbol single prepositional variables should not be a part of your thesis. What is thesis; thesis is either axiom or a theorem. So, it is in that sense your system is considered to be consistent.

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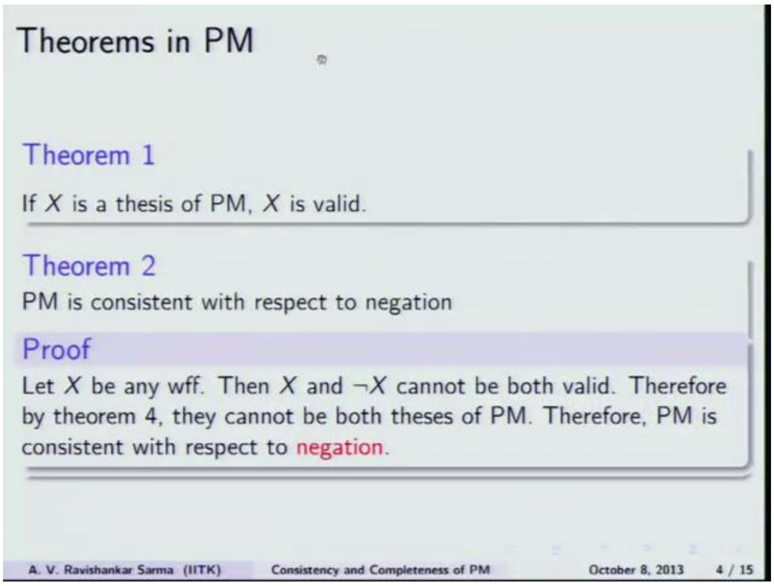

So, now these are some of the important theorems related to the axiomatic system due to Russell white head in principia mathematic. So, the theorem number 1 tells us that if X is a thesis of principia mathematic, that is Russell white head axiomatic system. That means, x is also considered to be valid. So, how do we know that this particular thing is the case X can be either axiom or X can be it is obtained by means of a applying some kind of transformation.

Then, it we could have got this particular kind of thing through and all. So, if X is the thesis of PM that is Principia Mathematic X has to be valid 1 example some examples which you can take into consideration. So, suppose x is considered to be thesis that is it can be either axiom or it can be a theorem. So, now let us consider this particular kind of thing q or p this is permutation axiom in Russell white head axiomatic system.

So, now we have a method with which we can check whether this particular formula is considered to be valid or not. So, that is the Semantic Tab Locks Method suppose, if you take this x as this 1 not x will be not of p or q implies q or p bracket needs to be closed properly. So, now if you construct a tree for this particular kind of thing, then this will become p or q and q or p.

So, now if you elaborate it little bit then it will be not q and not p and then this if you simplify it will get this 1, and then p or q needs to be written here. So, now you have not p here and p here this leads to contradiction and then not q and q is to contradiction. So, what is that we have showed?

So, we showed that not of x is unsatisfiable. Because, if you take the negation of the given well-formed formula which is usually considered to be an axiom in Russell Whitehead Axiomatic System that is considered to be a thesis. So, if you negate the thesis leads to contradiction; that means, not of x is unsatisfiable; that means, x has to be valid.

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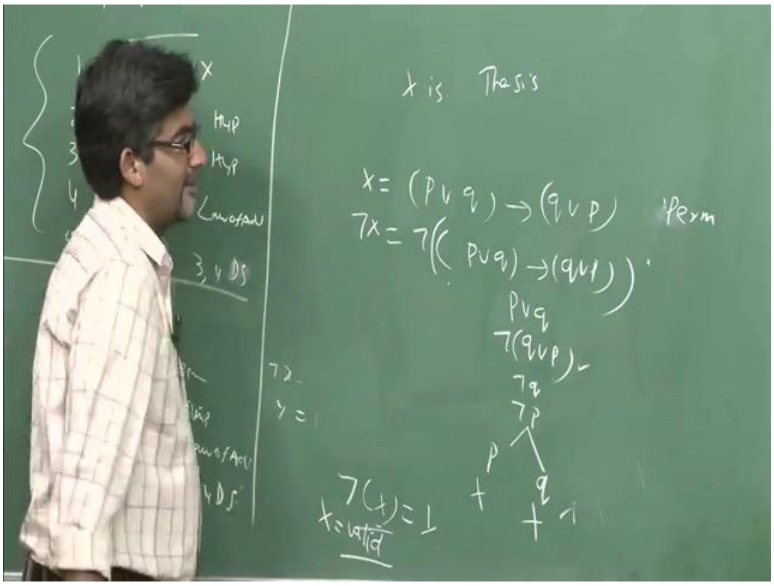

So, now like this any theorem that you can you can take into consideration and you can use Semantic Tab Locks Method and you can establish the validity of those given formulas. So, it will... same thing will hold even for a for the case of theorem if anything is considered to be theorem, in any axiomatic system if you Semantic Tab Locks Method; that means, if you deny the well-formed formula not of x.

And construct a tree by using alpha beta rules which we have seen in the case of Semantic Tab Locks Method. Then, you will see that not of x is going to be unsatisfiable; if not x is unsatisfiable x has to be valid. So that means, you are not able to come up with a counter example in which you're like in the case of argument you are not able to come up with a counter example in which you have true premises and a false conclusion.

That means, a original conclusion holds, so this is what we mean by saying that if x is a thesis of a principia mathematic, that is Russell Whitehead Axiomatic System that has to be valid. Because it is obviously, tautology and then all tautologies are obviously, valid. Now, theorem number 2 tells us that principia mathematic is said to be consistent with respect to negation.

So, what is consistency with respect to negation a system will be said to be a consistent in a sense that if there is no thesis x such that, both x and not x is also part of your not x is also a thesis. Either x has to be thesis or not x has to be thesis; that means, x has to be theorem or not x has to be a theorem.

But not both the things if that is a case then it is called as consistency with respect to negation. So, how do we prove this thing? So let us assume, any kind of well-formed formula X in this axiomatic system. Then, x and not x cannot be both valid it is obviously, the case. Because it is a contradiction so obviously, that x and not x is going to be false; it cannot be true.

So therefore, a theorem 4 which we will be talking about they cannot be both thesis of principia Russell white head axiomatic system. Because, combining both of this things leads to contradiction at least 1 of this things should be a thesis of that 1 either x has to be a thesis of your axiomatic system or not x has to be a thesis of your axiomatic system.

So, it is in that sense Principia Mathematic is consistent with respect to negation. So, it is straight forward pretty straight forward that you will not be in a position to derive both x and not x. So, both x has to be part of your thesis and not x has to be part of thesis, but not definitely not both the things x and not x if you have x and not x. If you allow for this particular kind of thing, it leads to trivialities.

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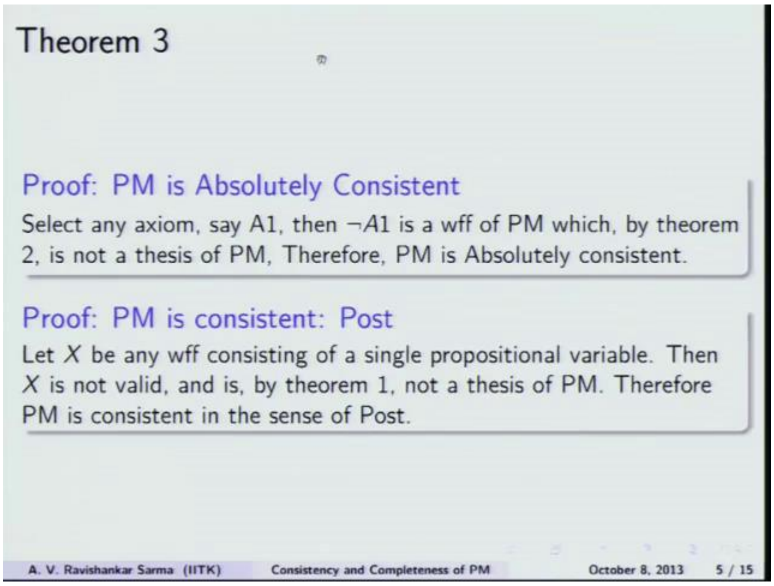

So, 1 can also show that Principia Mathematic that is Russell Whitehead Axiomatic System is considered to be absolutely consistent. So, what is Absolutely Consistent? a system is said to be Absolutely Consistent if not every formula of your system is consider to be a theorem or axiom and all or valid kind of statement. If you ensure that, whatever arbitrary form that you take into consideration is not going to be a valid formula.

Then, system is considered have absolutely consistent; that means, it your system has a other things as well that is contradictions and even contingent statements also. So, when we discussed about group of statements that you commonly occurs in the prepositional logic we have seen that. There are 3 kinds of statement which usually you will see in the prepositional logic that is on the bottom you have contradictions, on the top it tautologies occupies top most position.

So, statement which are always true and in between that there are some contingent kinds of statements. Suppose, if you can ensure that not every kind of formula that you take into consideration. So, how did we construct this every kind of formula? So, by using some kind of formation rules you construct kind of well-formed formula. So, does not mean that whatever formula that you come up with that is going to be theorem and all.

So, that is not the case in that sense it is called as Absolutely Consistent. So, now we are trying to show that Russell Whitehead Axiomatic System that is principia mathematic is absolutely consistent. So, how do we show that that is a case you select any axiom, whether it is A1 or A2 or A3. Suppose if you take A1, then not A1 is a well formed formula of Principia Mathematic which by theorem is not a thesis of Principia Mathematic. So therefore, Principia Mathematic is considered to be absolutely consistent.

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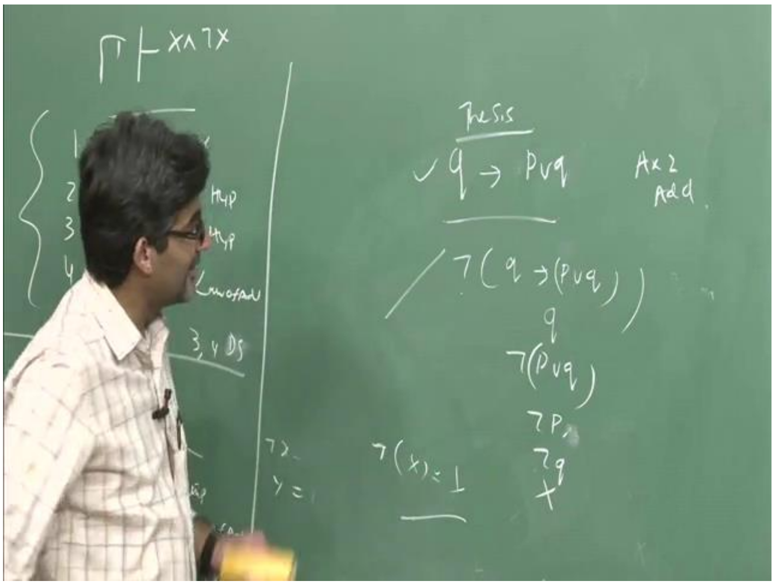

So, what it essentially says is this particular kind of thing. Suppose if you take any particular kind of axiom q implies p or q. So, this is axiom number tool of addition. So, you take any such kind of random axiom and all. So, what we what essentially we are trying to show is this that any formula that you going to take into consideration that is not that should not be considered as a theorem.

Suppose, if this is the formula that we have and I will take the negation of this particular kind of formula. So, this is already thesis of thesis because already in axiom; thesis means, it is an axiom or theorem. So, now I take the negation of that 1 then I will show that this is not part of your axiomatic system. So, now if you take the negation of this 1 again using Semantic Tab Locks Method, you can clearly show that not of x is going to be unsatisfiable.

So that means, is going to be in invalid formula, so how it results you take the you expand this you operate the tree method for this 1. Then, you will have this particular kind of thing then not p and not q since not q and q are there it closes; that means, not of x is unsatisfiable. I mean it is not a thesis of your axiomatic system what we establish? Any kind of thing which you pick it randomly that is not going to be a thesis of your axiomatic system.

That itself will be the for us to show say that, your system is considered to be Absolutely Consistent. Ensures that, not any kind of formula is going to be a theorem if that is a case then system is obviously, considered to be Absolutely Consistent. So, now Principia Mathematic is also consistent in a sense of El. Post.

So, what is considered to be consistency according to the famous logician El. Post it is like this a system will be said to be consistent in the sense of El. Post if there is no thesis of the system, which consist of only a single prepositional variable. So, you say that you know suppose if you have a q and you say that that is a thesis of your axiomatic system if; that means, that system is not considered to be consistent.

So, if prepositional variable simple prepositional variables also serve as a thesis and all, then the system is not considered to be consistent. So, we can show that even this kind of consistency also holds for this famous axiomatic system. So, how do we show that? Let x be any well-formed formula consisting of only single prepositional variable that is let say x.

Then, a single prepositional formula cannot be valid or invalid and all. So, if you if your statement is true, you can only talk about truth of a prepositional variable. If suppose, if you say that this is a duster and the negation of that 1 is this is not a duster and all. But only you can talk about validity only; when you combine with another kind of variable that is it is a duster or it is not a duster that is x or not x that is going to be valid and all.

It’s a tautology, but not a single prepositional variable can be taken as a valid kind of statement it 1 only be true or false. So, in that sense x is not considered to be valid and all. In that sense, anything which is not a well valid formula should not be a thesis of your axiomatic system. So, in that sense Principia Mathematic is said to be consistent even with respect to the consistency that El. Post talks about.

That means, in your axiomatic system there is no way in which you can have a single prepositional variable as your thesis. That is the axiom or you only have that particular kind of thing and all. So, that is not permitted and all. So, if that is there then it is not consistent with respect to El. Post. So, then now we just discussed 3 kinds of consistencies and all. So, mostly we will be using consistency in the with respect to negation; that means, any axiomatic system you should not be in a position to derive both x and not x.

If you derive it then it is considered to be a trivial kind of axiomatic system. So, consistency ensures that there are no contradictions in your system. Once you have contradictions, you can prove anything you can prove x and you can prove not x and you can prove any other strange kind of prepositions and all.

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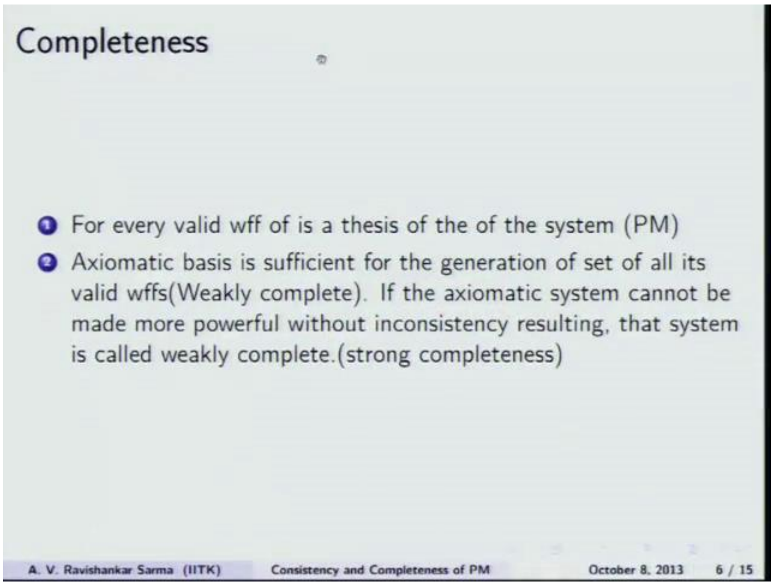

So, now let us move on to Completeness so far we discussed about consistency and we showed that principia mathematic is consistency with consistent, with respect to negation consistent, with respect to absolute consistency and even consistent with respect to whatever El. Post talks about. So, what do you mean by Completeness? For every valid well-formed formula of given axiomatic system is considered to be a thesis of your axiomatic system.

So, now you have all the well-formed formulas and all. So, there all considered to be thesis of your axiomatic system; that means, either it should be a theorem or if it is not a theorem, it has to be an axiom. So, that is a thing then it shows that all the true formulas are can be shown to be provable so; that means, all valid formulas should find a proof if that is a case then it is usually it is called as completeness. Suppose if you say that all provable things are true then it is sound. And then all the true formulas are also find proof if not today or tomorrow then it is considered to be complete.

So, axiomatic basis is sufficient for the generation of set of all its well-formed formulas. So, we know that if we have some solid foundations based on axioms what are these axioms? They are considered to be self-evident which are obviously, considered to be absolutely true. So, there itself is sufficient enough for us to say that, since axioms are absolutely true they are also considered to be well formed formulas. You can use Semantic Tab Locks Method, any other decision procedure method and you can check this particular kind of thing.

So, usually in general axioms does not require any proof. Suppose, if your axiomatic basis is sufficient for the generation of set of all its well-formed formulas which is usually called as weakly complete. If the axiomatic system cannot be made more powerful without inconsistency resulting, then the system is called as weakly complete though in the first sense it is called as strong completeness and all written in a wrong way here.

So, an axiomatic just your axiomatic system is itself is sufficient for the generation of all the well-formed formulas and all. That means, what essentially it means is that, you have an axiomatic system which consists of some set of axioms and transformation rules and etcetera. That is all you need to generate all kinds of well-formed formulas that exist either, in any given field and all either you are talking about arithmetic or geometry and anything.

All the truths of arithmetic and geometry should be should come else an outcome of just these axioms and all. It is in that sense mathematics can reduce to logic or you talk about all the mathematical concepts in terms of the concepts of logic using only conjunction, disjunction and some set of axioms etcetera. So, either it should be in that sense or your axiomatic system cannot be made more powerful like in the case of the first case; which is considered to be strong completeness.

Without some kind of inconsistency resulting in the given system, then that system is considered to be weakly complete. So, this is what we mean by the difference between what we mean by strong completeness and weak completeness. So, the first case is considered to be a strong completeness and the second 1 is considered to be weakly complete.

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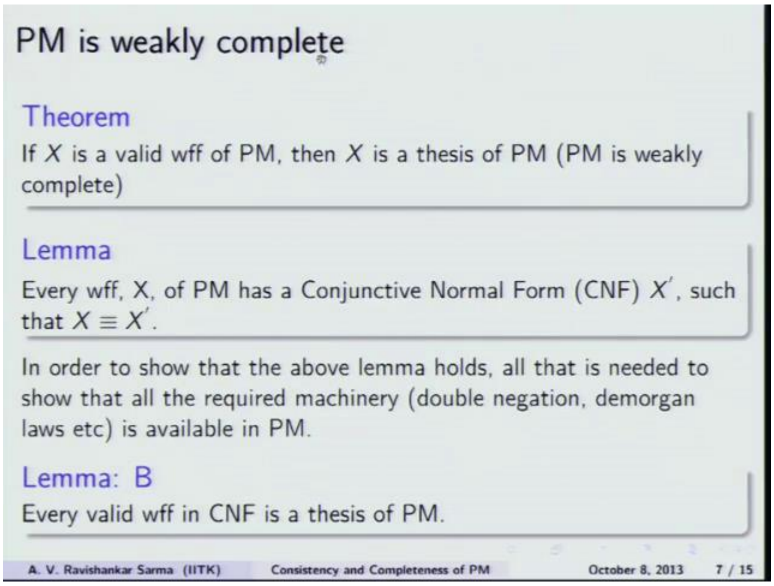

So, now we need to show that Principia Mathematic is weakly complete. Weakly complete in a sense that the second thing if the axiomatic system cannot be made more powerful without some kind of inconsistency resulting then the system is called as weakly complete. So, these are some of the theorems which I will just go into the I will just give you a brief idea of this particular kinds of this theorems.

But all the proofs are already there in any either in the book any important book that you read it in the formal logic. So, there are some differences given at the end of this slides and all. So, in those books you will find proofs of all this theorems, but what we need to get is the central idea of central idea behind this theorems. So this theorem tells us that, if X is a valid well-formed formula of a Principia Mathematic it is Russell white head axiomatic system.

Then, x has to be a thesis of Russell Whitehead Axiomatic System or Principia Mathematic. So, it is in that sense pm is weakly complete. So, now 1 corresponding lemma based on this thing is this is this particular kind of thing. Every well-formed formula X of Principia Mathematic has its corresponding conjunctive normal form let us say x prime such that this formula is equivalent to its x prime.

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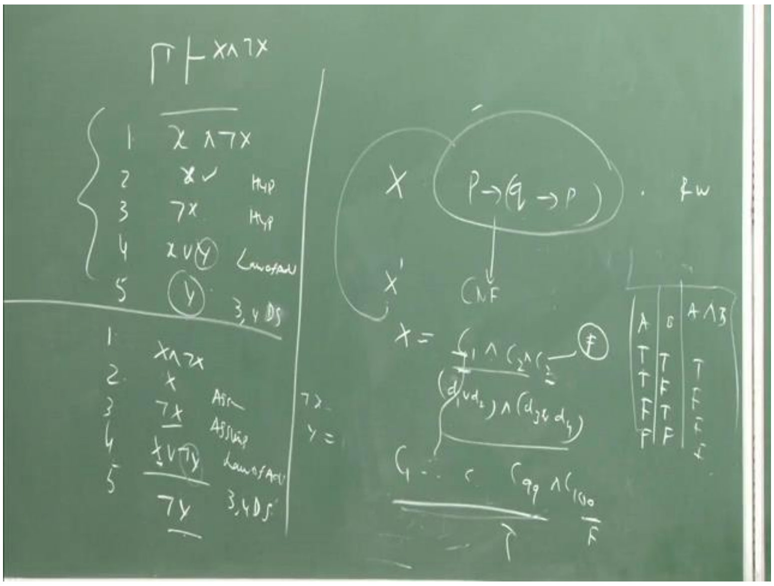

So, what essentially it says is this that you have a thesis like this particular kind of thing let us say, p implies q implies p this is considered to be thesis in Russell white head axiomatic system. So, now this is the formula x so now, if something x is considers to be a thesis, then it has its corresponding CNF. So, what is CN? It is conjunctions of disjunction. So, where it is it is conjunction what is a conjunction of disjunctions d1 or d2 and d3 d4 etcetera.

So, each conjunct is each conjunct consist of disjunctions set of disjunctions. So, any given formula can be appropriately transformed into its corresponding conjunctive normal form. So, that is what this particular kind of theorem tells us, in the same way 1 can transform a given thesis into disjunctive normal formals. So, there is a advantage of converting a given formula into conjunctive and disjunctive normal forms.

It is quite simple suppose if you have a formula x is c1 and c2 c3 and all. If anyone of this conjunct is false irrespective of whether for example, c1 c2 c3 c99 even if they are all true if the c hundredth 1. So, that particular kind of disjoints I mean in that what are the elements you have only disjunctions. If all disjoints are false, that makes the whole disjoints false and hence sees hundredth 1 is false.

So, that makes even though 99 are true, the hundredth 1 is false. So, we have the semantics like this for and this is A and B. So, you have A and B T T F and F T F T F and it’s going to be true only in this case when both the conjuncts are true, then only it becomes true in all other cases it becomes false. So, that is why even if you have in your conjunctive normal form.

Let say, there are 100 kinds of conjuncts like this if 99 conjuncts are true, but 100th conjunct is false that is c100 is false; that makes the whole thing unsatisfiable and all. That means, this makes the whole formula false the conjunctive normal formula becomes false means it’s unsatisfiable; unsatisfiable means its invalid. So, this is the this particular kind of Lemma any given formula you can transformed into its corresponding conjunctive normal form.

So, in order to show that the above lemma holds all that is needed to show is that we have sufficient kind of machinery; that means, we have all the rules of such as double negation De Morgan laws etcetera and all then you can transform any given formula into its corresponding conjunctive normal form. Now Lemma B tells us this, every well-formed formula, every valid well-formed formula which appears in the conjunctive normal form normal form is also considered to be a thesis of Principia Mathematic.

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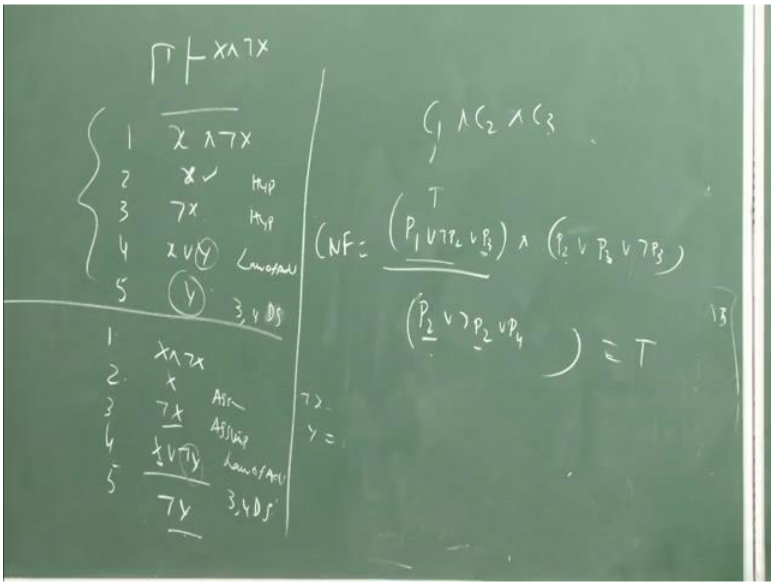

So that means, so when a given formula is going to be valid formula let us assume that you have a formula like this. So, this c1 c2 c3 and all then only it will said to be in conjunctive normal form. Where each c1 let us say, it is like this p1 or not p2 or p3 and p2 I am selecting in a clever way such that, you know each disjoints will automatically be true p3 and not p3 and p2 or not p2 or p4 or p5 etcetera.

So, let us try to talk about only this thing, so now this is the CNF. So, this is a conjunctive normal form conjunction of disjunctions that is why it is in CNF. So what this Lemma tells us that, any such kind of conjunctive normal form which is which holds and all; that means, you can clearly see here that a literal and its negation which appears in a given formula.

So; that means, this formula is; obviously, going to be true irrespective of whether p3 is true or not, this is going to be true now you have p3 and you're not p3 is absolutely true. And whether p2 is false or p2 is true does not matter, it is going to make this true. And the same way here, you have p2 and you have not p2 here, so that makes this whole formula true.

So; that means, you have shown that each conjunct is true c1 is true, c2 is true, c3 is true. So, that is why the whole formula is also going to be true according to the semantics of conjunction. So, it is in that sense any CNF which is considered to be valid should also be a thesis of your axiomatic system that is we talking about Principia Mathematic. So, that is why should be part of your axiomatic system.

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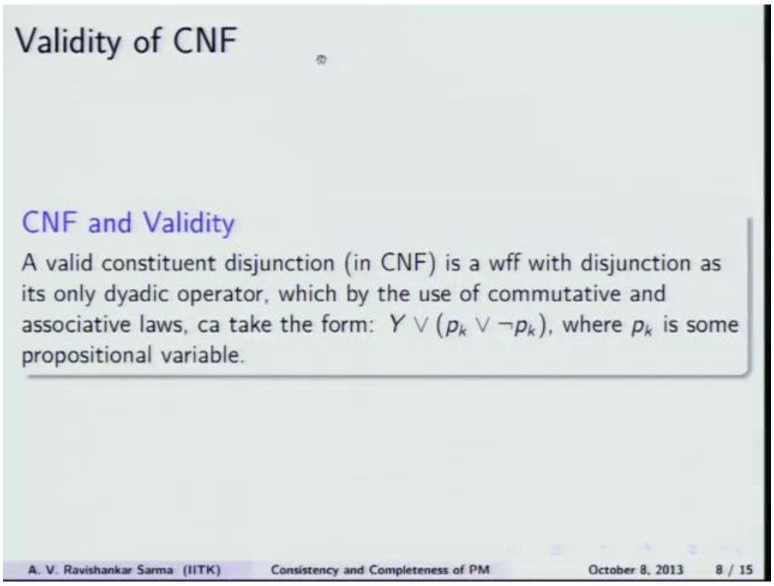

So, we have this particular kind of the thing which is related to the validity of any given CNF formula. So, this is like this a valid constituent disjunction in CNF; in CNF what we have each it is a conjunct which consist of disjunctions. So, now if you observe the interior part of it that is disjunctions of each conjunct we have only dyadic operators that are or are the usual sign which you will find. Which by use of Commutative and Associative Laws can take this particular kind of form Y or pk are it's a literal and its negation is there in a given formula.

So, pk or not pk is always going to be true. So, some kind of prepositional variable pk and its corresponding negation is there in that. Then obviously, it makes the disjoints true and c1 also true. So, if each c1 c2 c3 all are true then your conjunctive normal form is also going to be true somehow your formula should be like Y or pk or not pk.

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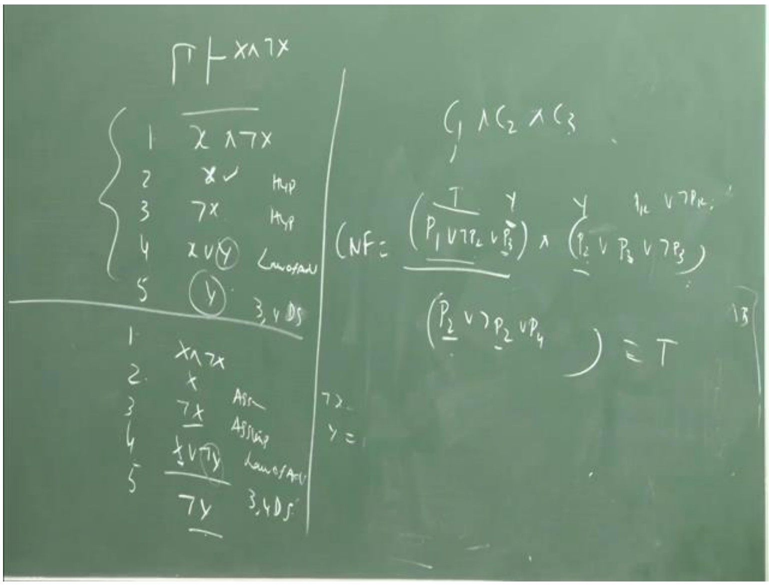

So, now in this case Y is considered to be this 1 and then all the other things are pk or not pk. And here in this case Y is considered to be this formula, and then this is considered to be pk or not pk etcetera. So, like this each and every conjunct will have this particular kind of things. So, that is why a given CNF is considered to be a valid kind of formula.

So the idea here is this that, in any given CNF you should ensure that you have a literals needs negation present in a disjunctions of your each conjunct.

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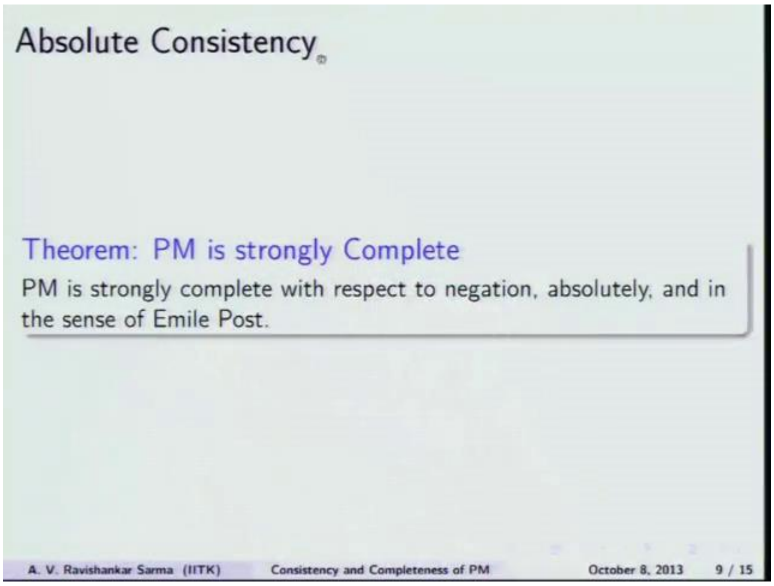

So, now 1 of the another thing is this that PM is a there is a Principia Mathematic also considered to be complete with respect to negation that we have talked about and absolutely and in the even in the sense of El. Post also. So, these are some of the theorems which we can talk about with respect to Principia Mathematic.

And similar kind of things can be we can do it with respect to even Hilbert Ackermann axiomatic system as well Just I will quickly go into the details of whether or not Hilbert Ackermann axiomatic system is consistent, complete and sound etcetera and all.

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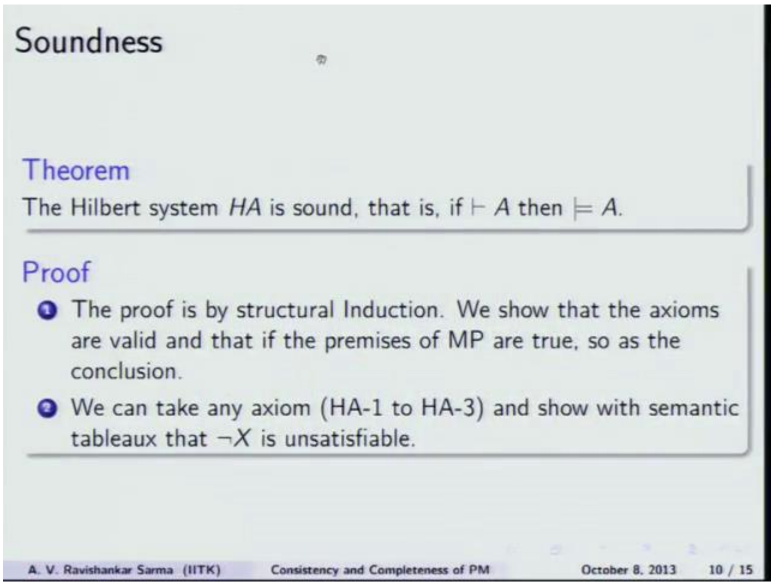

So, now we have we presented our axiomatic system earlier in the last class. So we are now saying that, as a Hilbert Axiomatic system is considered to be sound. So, a system is said to be sound especially when your proved something that is A is provable and that means, A has to be valid. Something you proved, that whatever you have proved is considered to be a valid statement.

So, this can be done by using I do not want to go in the details of the proof and all, but this proof can be done by means of some structural induction. So, what we show here is this that the axioms are considered to be obviously, valid and all. Because, you can check with Semantic Tab Locks Method and you can check that all the axioms are going to be obviously valid.

Because, there is no way in which you deny the axiom and then you will it leads to unsatisfiable it leads to satisfiability and all. So, all axioms are obviously, considered to be valid and that if the premises of another thing important thing is this that, the other rule that you have used is that is p p implies q and q. So that means, that also it should be that rule also should be truth preserving.

So, now if the premises of that is p p implies q are going to be true and obviously, the conclusion also have to be true. There is no way in which p p implies q is absolutely true and then q is false q is false. So, that makes the argument invalid, but that is we cannot come up with a counter example which can establish that is wrong we can establish.

So, we can take in the same way we can take any axiom into consideration in Hilbert Ackermann axiomatic system like p implies q implies p. And then take the negation of that 1 obviously, negation of this particular kind of thesis that is axiom number 1 leads to unsatisfiability. Unsatisfiability means not exist invalid that means, x is considered to be valid or x has to be true.

So, like this in away can check all the things that you have proven to be absolutely to be true. So, it is in that sense whatever you proved that is single A there is whatever is provable is also turn out to be true. At the end of the day it also turn out to be true, as you can see clearly you can use your we can see from the proof itself.

What is considered to be a proof? Each step of your proof is obviously, considered to be true. So, that is why the final step of your proof that is the theorem which obviously, considered to be true. So, in that sense you can establish that Hilbert Ackermann axiomatic system. if something is provable in the axiomatic system that, has be to true statement that is it has to be valid formula.

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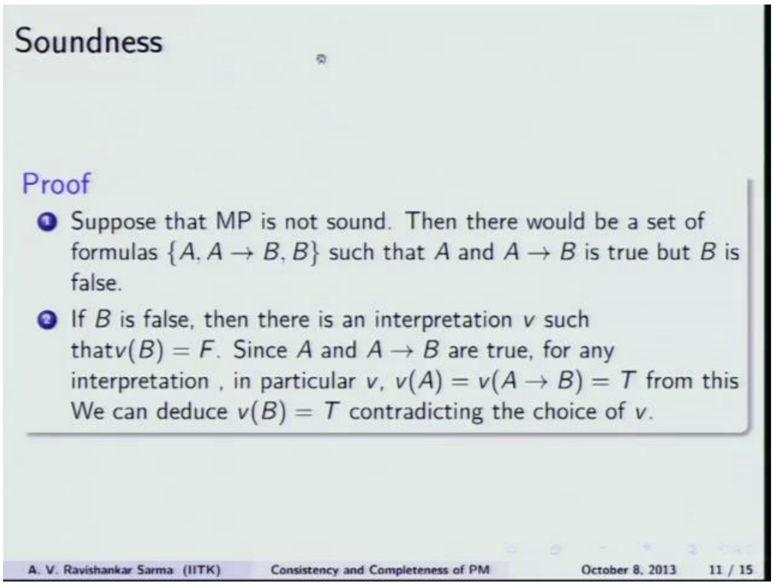

So, now we can as per as axioms are concerned there is no way in which you can show that they are wrong and all. They are absolutely true, you can use Semantic Tab Locks Method you can check all the axioms to be true. There are 3 axioms which you can check them to be true using Semantic Tab Locks Method which we have already or you can use any decision procedure method like truth table or anything. And you can check the validity of a given formula or whatever you have.

Now, the next thing which is important is this that we have used also has 1 of the important things in our axiomatic system. So, how do we know that is true. So, now suppose that is not sound that means, let us say p p and q it does not lead to q. Then, there would be set of formulas like this particular kind of thing A implies B and B such that the first 2 A A implies B are true, but B is false.

So, now since B is false then there is an interpretation in which v such that v of B is going to be false that is what we mean by B is false. This is the way we write this particular kind of thing. Now since A and A implies B are obvious already true for any interpretation in particular v that same interpretation we have v A and v A implies B that is to be true.

So, from this we can deduce that whenever you have v A equivalent to v A implies B the valuation of A implies B is true, then valuation of B also have to be true. There is no way in which valuation of B can be false because, we know that we valuation of A implies B is also true. If it false then it valuation A implies B will become may become false. So, there is no way in which you can get a valuation of B to be false.

So, we get only valuation of B to be true, but we started with valuation of B to be false. So, valuation of B is equal to T is in contradictory with valuation of B that is false that is that is what we began with. So, it is contradicting our choice our choice; what was our choice? In the beginning valuation of B is false; that means, valuation of B should not be false, but it should be T.

So, there is no way in which you can question the in this way that is also considered to be that is also truth preserving rule, which is also considered to be sound. So that means, you can prove the rule, but that also turned out to be a valid kind of formula it is truth preserving kind of formula.

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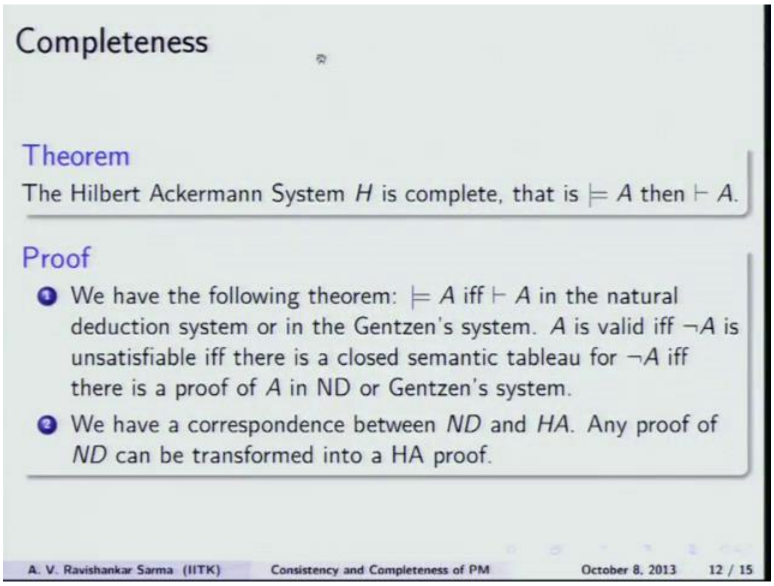

As per as completeness with respect to Hilbert Ackermann Axiomatic System is concerned this s what we mean by, completeness which we already discussed in the case of Principia Mathematic. If you can discuss with 1 system and all, then it’s same as other systems as well. So, Hilbert Ackermann System I think with this I will end it. Hilbert

Ackermann System H is also considered to be complete, in a sense that a valid formula is also provable.

So, this is the beautiful thing about prepositional logic that is all the provable things are obviously, true I mean soundness and choose that they are all true and all the valid formulas are true prepositions are also provable. If that is a case then whenever we can use this theorems in proving, in checking whether or not a given system is complete etcetera and all.

Suppose, if you are asked to prove a complex kind of a statement and all preposition in the complex well-formed formula and all. Then, instead of proving that thing using axiomatic system etcetera and all, you can invoke the completeness property assuming that in a preposition logic is considered to be complete. If it is complete, then it is as good as checking the validity of a given formula rather than finding a proof sometimes proofs might be very difficult to combine.

So 1 can use, 1 can employee Semantic Tab Locks Method and you can check the validity of a given formula. How do we check the validity of a given formula? You negate the formula and look for the unsatisfiability. If you can establish the unsatisfiability that means, in a if you construct a tree and all the branches closes, then that is considered to be not x is going to be unsatisfiable that means, x has to be valid.

So, we have the following theorem that is in the in the case of Semantic Tab Locks Method if something is a valid statement if and only if it is provable in the natural deduction system or another system which we actually did not discuss. But is more or less similar to natural deduction Gentzen's natural deduction system. So, if something is valid that as that is also provable we know that that is a case in the case of natural deduction system.

So, A is considered to be valid if not a is unsatisfiable. So, if and only if there is some kind of closed semantic tableau for not A and if only if there is a proof of A either in natural deduction system or in the Gentzen's system and all this is what we have already discussed. So, that is indeed the case, so we have a correspondence between natural deduction system and of course, this Hilbert Ackermann System or even the Principia Mathematic.

So, any proof of natural deduction system can be appropriately transformed into a proof in the Hilbert Ackermann System. So, if you can do that thing a since whatever all the valid formulas are obviously, provable in the case of natural deduction. In the same way, we have corresponding kind of proof in the Hilbert Ackermann System corresponding to the natural deduction proof. Even that in that case also all the valid formulas are also provable even in this case that means, Hilbert Ackermann System.

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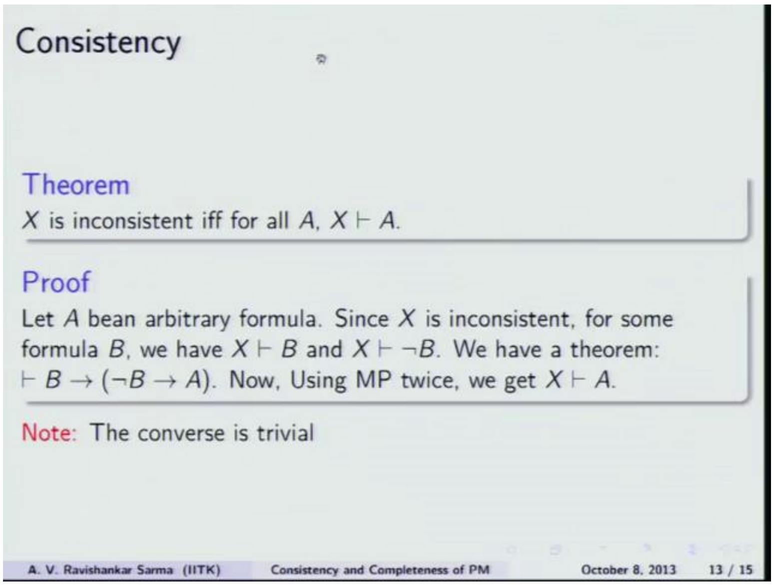

So, finally we can talk about consistency with respect Hilbert Ackermann System that is, X is considered to be inconsistent if and only if for all A if A is deduced from X whatever be the case A is deduce from that particular kind of thing. Then it is said to be inconsistent. So, it is we are talking about in this case Absolute Consistency.

So, proof can be like this you take any arbitrary formula A B and arbitrary formula and since X is inconsistent for some kind of formula B we have both the things B is derived from X and not B is also derived from X. So, we have another theorem such as this is a thing we have A implies B implies A, but instead of A we substituted we have already this particular kind of rule B implies not B implies A.

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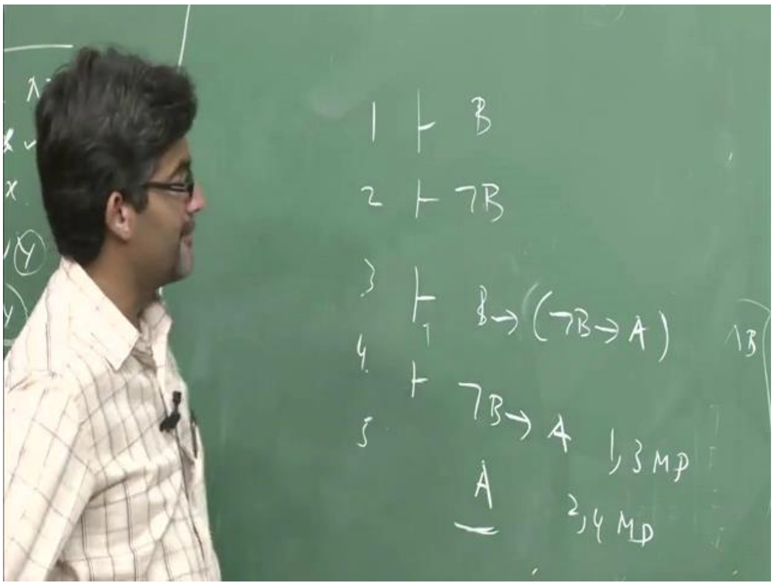

So, now this is like this thing, so what it essentially says is this that. So, let us assume that your system is consistent not B and we already have a thesis which is like this B implies not B implies A. So, this already a thesis although you can check whether it is valid or invalid. So, now first time when you apply on these things 1 and 3 you will get not B implies A. So, now you apply again then you will get not B and not B here, so you will get this thing.

So, now how did you get this 1 1 and 3 and 2 and 4 you will get this A. So, using you will get a as a single prepositional variable as a thesis. So, the if you can come across this particular kind of thing it is synchronisation in the sense of El. Post, so what we have established it here. The converse of this 1 is this that if A is deduced from X then that x is inconsistent that seems to be little bit trivial.

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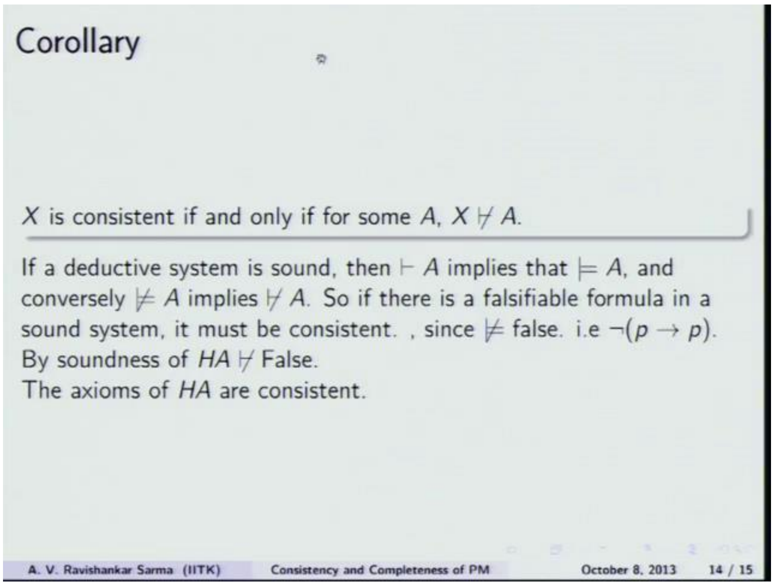

So, this is 1 of the important corollaries of this 1 is this if X is consistent if and only if for some A is not a consequence of that particular kind of X.

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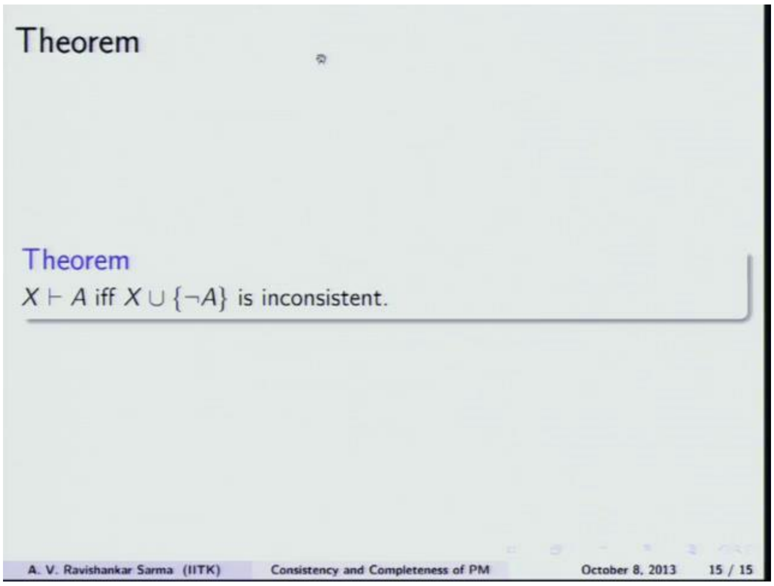

So, another important theorem is this that if A is deduced from X if and only if X union if you add not A to it, so that system is that will become inconsistent.

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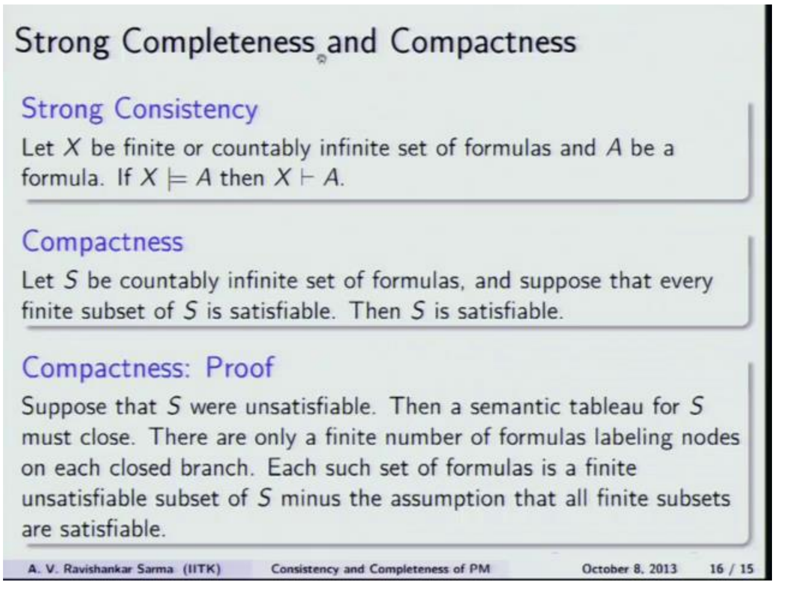

So, with this I think we have discussed all the important theorems and all. There is another important theorem which I will discuss it in the context of when I discuss about predicate logic it is also considered to be 1 of the important theorems that is what is called as compactness.

So, the Compactness tells us that is roughly I will talk about this thing. So, if let us say s be a countable infinite set of formulas x1 x2 x3 like that, which are some kind of formulas. And suppose that every finite subset of S is satisfiable, then S is going to be satisfiable.

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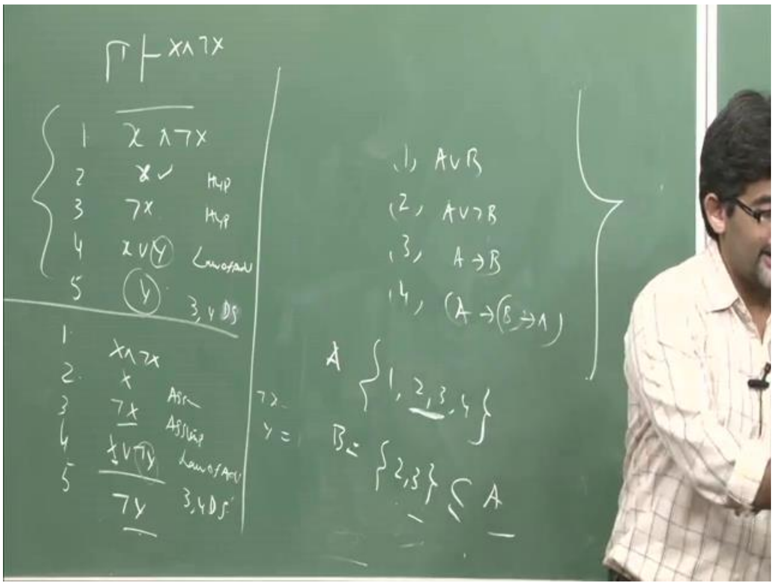

So that means, for example, we have some kind of statements like quickly I will end this 1 A or B A or not B and then A implies B A implies B B implies A etcetera. So, now these are the 4 statements that we have. So, now the compactness property is 1 of the wonderful properties that will happen in the case of classical logics, it is the prepositional logic. So, instead of checking all the statements to be consistent to each other.

So, what you do here is this that suppose if you take into consideration this is the set which consist of these 3 prepositions 1, 2, 3, 4 etcetera. So, now the compactness property ensures that you take any 2 statements and all, if you can establish that these 2 are consistent to each other that is good enough to show that your whole set is considered to be consistent.

So, this is the finite set suppose if you take single out only these 2 things 2 and 3 only. So, this is a subset of let us say A and this is B B is A subset of a if you can show by taking only 2 and 3 to be consistent, then that is good enough to show that the whole set a is also considered to be consistent; this is what we mean by Compactness. So, with this I think I will stop here.

So, what we discussed in this lecture is simply like this that we presented Principia Mathematic and Hilbert Ackermann System in the last few classes. Now we questioned couple of interesting questions they are like this is Principia Mathematic complete or principia mathematic consistent etcetera or is this sound etcetera. So, now we showed that principia mathematic are Russell Whitehead Axiomatic System or you take any axiomatic system into consideration Hilbert Ackermann.

Then, some other axiomatic system which follows, so they are considered to be complete, consistent and considered to be sound. So, 1 of the advantages of having your system complete is this that, instead of checking a formula to be instead of checking instead of providing a proof for a given formula you can check whether a given formula is considered to be valid.

Because, all the valid formulas according to the completeness theorem should find a proof. So, that simplifies our tasks in particular in a sense that you know if your proofs are very hard to combine. Then, you can check the validity of a given formula and say that, so that that will have a particular kind of proof.

So, in the next class we will be talking about we will be entering into the third module of this course that is, we will be talking about the predicate logic. So, I will be focusing my attention on the predicate logics which, whatever preposition logics could not achieve. So, we try to fix some of the problems related to prepositional logic by using the predicate logics.