Introduction to Logic

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Lecture - 38
Truth, Satisfiability, Validity in Predicate Logic

Welcome back. In the last lecture, we discussed began with a semantics of predicate logic, where we discussed about when do we say that a given formula in the predicate logic is true and its going to be false etcetera. In continuation with last the discussion on the discussion of the last lecture, we will be continuing and we will be talking about some more examples, so that we can get this idea in a better view. So, we will try to talk about the semantics of the predicate logic in greater detail, with some more examples in this particular kind of lecture.

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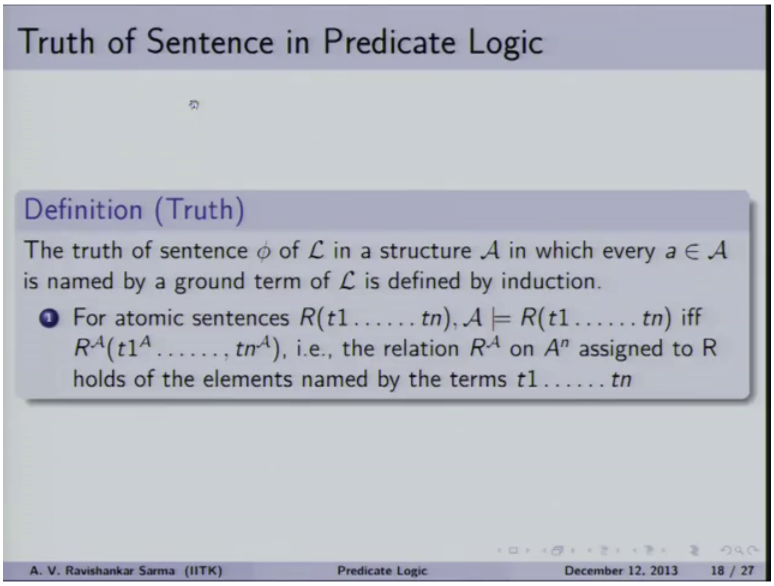

So, what is important for the semantics of a predicate logic is the 2 things; 1 is the domain it does not make any sense to talk about the true value of a given predicate logic formula, without respect to some kind of domain. So, we need to fix a domain, it can be a natural number, it can be a real number, it can be set of peoples it is a rivers etcetera and all. We need to fix the domain and then we need to have a interpretation function i and that constitutes d and i constitutes what we call it as a model structure etcetera.

So, now, in this particular kind of context, we define what we mean by we provide a formal definition of a structure. So, essentially what talked about this is that, in the predicate logic, we have various constants, predicates and functional symbols, each 1 when you assign some kind of values to these things; it has to find some kind of an entity in the domain, where the predicates are mapped to 0 and it means a property whether or not holds it or not is the 1 which we are going to see. And then each individual constant should have a member in the domain d etcetera. Each functional symbol finds another kind of a re functionary symbol in the domain etcetera.

So, now, how can we define truth? The truth of a sentence pi a given formula pi in L which respect to some kind of structure, which consists of domain and the interpretation function structure A in which an element belongs to A is named by a ground term of L and is defined by means of some induction like this. For atomic sentences are t 1 to t n in a given structure A, that is going to hold; that means, R t 1 to t n is going to be true with respect to a structure A if and only if, if u have R to the power of A t 1 raised to the power of A to t n raised to the power of A. So, that means, the relation R A and A n assigned to R holds of the elements named by the terms t 1 to t n, otherwise it is going to be false.

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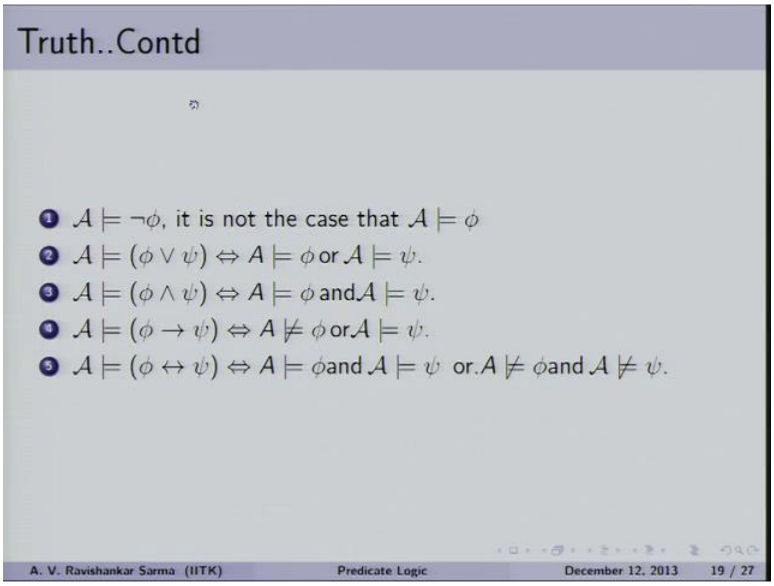

So, we will give some examples to a talk more about definition of truth, with respect to the formulas in the predicate logic. In the context of preposition logic, we have seen what we mean by saying that a particular formula is true or false, with respect to a structure A, a model A. For example, not 5 is going to be true in a structure or model A; obviously, when it is not the cases that, pi follows from A. Actually it should be written in this particular kind of sense.

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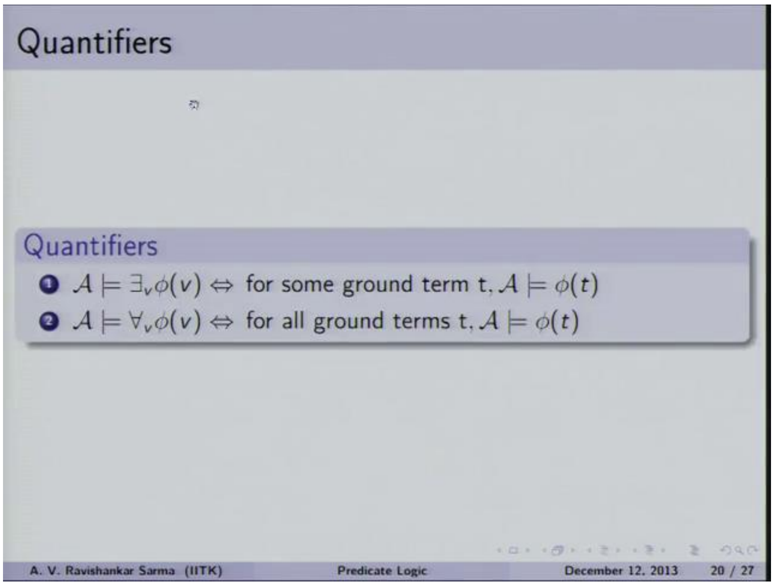
So, A does not belong to pi. If that is the case, then not pi is considered to be true in this particular kind of model A. In the same way pi or psi is going to be true in model A if, either pi is true in the model A, psi is true in a model A, that is the standard definition and the conjunction is going to be true, when both conjuncts are true, that is taken care by the third 1 and implication that is going to be true, only when if you have premises true and conclusion false. In the same way, pi if and only if psi. There are things which are exactly same as the case of preposition logic.



So, the additional things that we have in the case of predicative logic are some of the truth values, with respect to quantifies. What is extra in predicative logic are 2 more operators that are considered to be quantifies. First is there exist some v pi v, that is true with respect to a structure A, if that is going to be true at least for 1 ground term t, that means, if you substitute 1 ground term t, then this there exists some v pi of v holds, then that particular kind of formula is true with respect to structure A. That means, for some ground term t pi of t; that means; x is v is substituted by t and then pi t has to be true in a structure A.

If that is the case for at least one of the ground term t then; obviously, that is called as there exists some v pi of v is true, with respect to structure A. And for all v pi v is going to be true in structure A, if it happens for if pi of t is going to be true for all the values of t, whatever value that we are going to take in consideration for t as in all this cases pi of t has to be true. In that sense we call it as for all v for pi v is going to be true with respect to model A.

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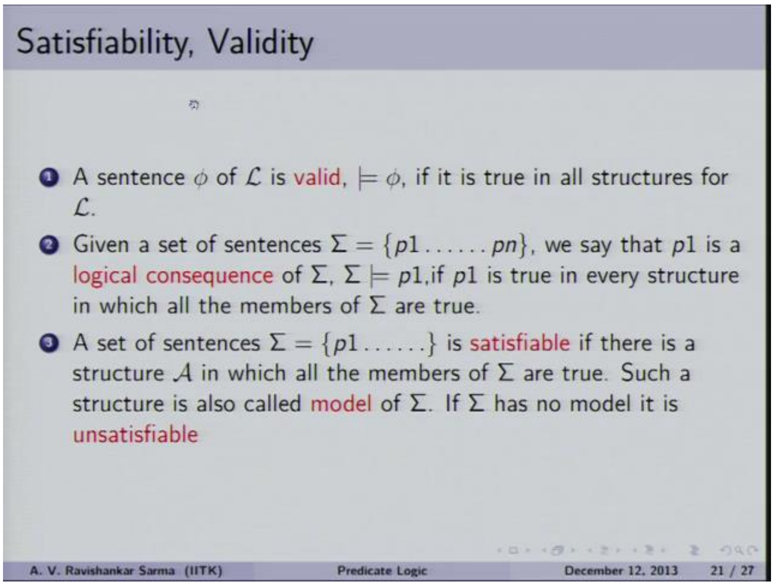

We can talk about some other important logical properties, which are satisfiable value t in this context. So, a sentence pi in a predicate logic, first of all what is considered a sentence in predicate logic, it does not have any few variables then; obviously, it is called as a sentence, otherwise it is going to be a formula in the predicate logic. So, that sentence pi of l. So, the language of predicate level is considered to be valid, which is usually represented as a models and pi, especially if it is true in all structures for l. If that is going to be true in all kinds of structures, whatever interpretation that you are in all interpretations that is going to be true, then is considered to be tautology.

So, that is what we mean by validity truth in all structures is considered to be what we mean by validity. And given a set of sentences sigma which consists of p 1 to p n, we say that p 1 is considered to be logical consequence of sigma, if and only if p 1 is true in every structure, in which in that particular kind of structure, all the members of sigma are also going to be true and all the members of sigma are true, p 1 also has to be true. If that is the case, then we say that p 1 is logical consequence of sigma And third important thing is that, a set of sentences let us say p 1 to p n is going to be satisfiable, if there is 1 structure A in which all the members of sigma are true, where p 1 is also true and such a structure is also called as model of the given set of formulas sigma. If sigma has no model then; obviously; that means, there is no interpretation in which x is true. That means, you are not able to find out at least 1 interpretation, in which you are x is true, then it is called as unsatisfiable.

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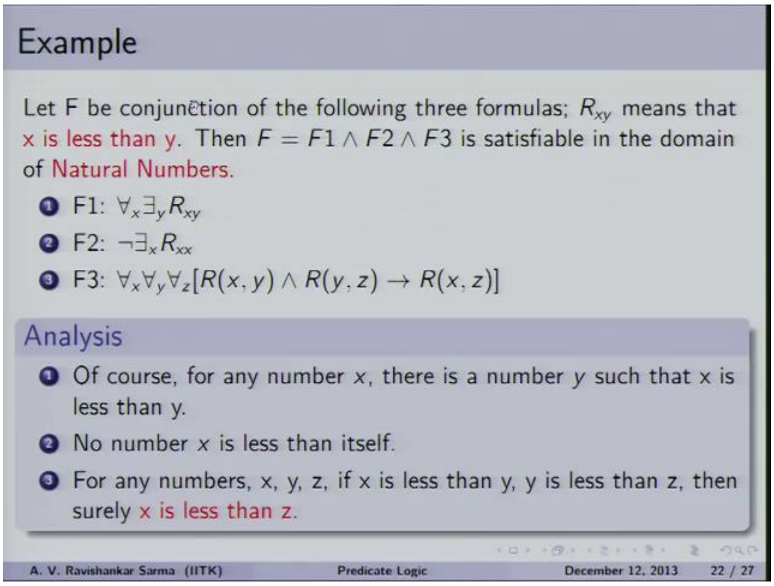


Let us consider some examples, let us take this into consideration F, we conjunction of the following 3 formulas. There are F 1 F 2 F 3. The first 1 is represented as t read as for all x there exists some i r x y. Now R x y means here, that x is less than y. Suppose if we take 1; obviously, it is less than 2, if we take the natural numbers into consideration, if we pick up x as 1 and y as 2 then; obviously, 1 is less than 2. That is what we mean by R x y and then you have F which is a conjunction of all these things; F 1 F 2 F 3.

Now we are going to show that it is going to be satisfiable in the domain of natural numbers. It might be false with respect to real numbers, some other numbers etcetera, but we are going to say that, it is going to be satisfiable. When you say that this conjunction of formulas are going to be satisfiable? For at least in 1 interpretation, in which this particular kind of property of F 1 and F 2 F 3 is going to be true, then it is called as satisfiable, otherwise it is going to be unsatisfiable.

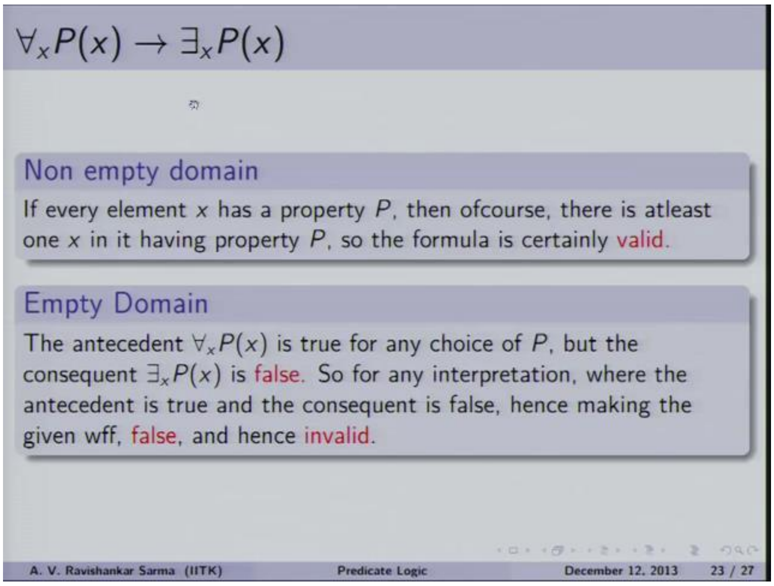
So, now, F 1 is the formula which is depending on this sense, for all this it is some kind of y R x y. In the context of natural numbers for all x, whatever number that you have taken into consideration the domain of natural numbers, where all this exists some kind of y, where that particular kind of x is always x is less than y. So, for example, if you take into consideration, 1 to be the particular kind of thing, there always exists there exists some kind of y to which is less than, if you take x to be greater than 1, then all the elements all the elements greater than 1 2 3 4 5 6 etcetera and all. For all those numbers; obviously, 1 is less than those particular kind of numbers.

The second 1 is this x such that, R x relation is x is less than x and for all x for all y for all z the third 1 is stating that, if x is less than y y is less than z than; obviously, x has to be less than z 3 numbers 1 2 3 etcetera and all, 1 is less than 2, 2 is less than 3 and obviously, 1 has to be less than 3. In the same way, 2 3 4 you take into consideration in order, 2 is less than 3 3 is less than 4; that means, 2; obviously, has to be less than 4. So, now, analysis is like this.

So, now, in at least 1 particular kind of case where, it this property holds in particular, then F 1 is going to be true, F 2 is going to be true, F 3 is satisfiable. So, then each F 1 F 2 F 3 is satisfiable then F is; obviously, considered to be satisfied. So, the first 1; for any natural number x will always enable y number y, there is number y such that, x is less than y. You always find some kind of arrangement like this or any number you take into consideration x, there always exists some kind of y, where x is less than y. Natural numbers it consists of 1 2 infinity and all. For example, you take any number such as let us say 25 you take into consideration, then always there exists some kind of number which is greater than that 1, which is less than other number say let us say 6 29 or 30 35 etcetera and all.

So, at least 1 kind of situation it happens. So, that is why for all x, whatever number that you are taking into consideration in the natural numbers, there always consists some kind of y where x is always less than y. So, that holds that satisfies. Now the second thing is that does not exist x R x x. So, it is written as for all x it is not the case of R x x, there is no number of x is less than itself which is; obviously, the case in terms of natural number. Suppose if you add 0 to it, then this will change or minus if you add integers to it this may not hold. But in the case of natural numbers, if you take 2 3 4 anything into consideration, 2 cannot maybe less than its own number that is, 2, it has to be equal to 2, it is definitely not less than 2. So, that also holds.

The third 1 for any numbers you take any actual numbers into consideration in some kind of order, if x is less than y this holds and y is less than z, then obviously, x is; obviously, considered to be less than z. If you take 2 3 4 etcetera and all 2 is less than 3, 3 is less than 4; obviously, 2 is less than 4.



So, now, let us consider some interesting formula, that is stated as stated in this way. For all x P x implies there exists x P x. This formula is going to be valid in all non empty domains. When do you say that domain is non empty at least it has if the domain is domain has some kind of object, otherwise the domain is considered to be empty. For example if you talk about set of people, at least some kind of people have to be there at domain. Otherwise if there are no people at all, only animals, non living beings etcetera and all, that domain is considered to be empty.

So, if every element x has a property P, then of course, there is at least 1 x in it, having that particular kind of property. For example, if you say that all human beings die in some day or other, for at least 1 human being has that particular kind of property, we mean every1 has to die some day or other. So, then it means by saying that some x y z, if you take it arbitrarily from the domain of people, that also there also have they also satisfy that particular kind of property P. So, the; obviously, the formula seems to be certainly valid in case of non empty domain. That means, the domain consists a set of people, in that if it happens for all the things, for example if you take into consideration; set of birds for example, birds crows in particular.

If all crows are black, most of the crows are black and all. Then it holds for all the crows and all, then you take any 2 or 3 birds into consideration which are taken to be crows which are; obviously, considered to be black; obviously. So, for all x P x if it holds, then there exists some x P x also holds. So, this happens only with respect to non empty domain. But what happens if you take into consideration an empty domain like; for example, we make cons devil, demons etcetera and all are empty domains, which does not exist. So, in this case what happens is that, for all x P x is going to be true, for any choice of P because, empty set of all the sets. In that sense, for all x P x is going to be true of any choice of P, but the consequent in this condition that is, there exists some x P x, that is going to be false because, that leads to the existence of x and all.

So, for all x P x does not need not have any commitment that, particular x has to be exist in the universe. With respect to the empty domain for all x P x is going to be true and with respect to empty domain, there exists some x P x is going to be false. So, for any interpretation; that means, any structure that you take into consideration which has domain and interpretation function and etcetera, where the antecedent is true here; that means, the power of x P x is going to be true, whereas, the consequent is going to be false here. There exists some x P x is false. Hence the given well formed formula is going to be false; hence this formula is going to be invalid with respect to empty domain.

So, in general, when we try to evaluate the well formed formulas; that means, in the evaluate in the truth conditions of given well formed formula in the predicate logic, we usually take into consideration that, the domain is non empty. It can also take into consideration empty domain, then in that case only universal quantifies, the formulas which begin with the universal quantifies are going to be true and others the property P x with the universal quantifies is going to be true and existence quantify, there exists some x P x is going to be false. So, these are some of the things which we need to talk about context of semantics of predicate logic.

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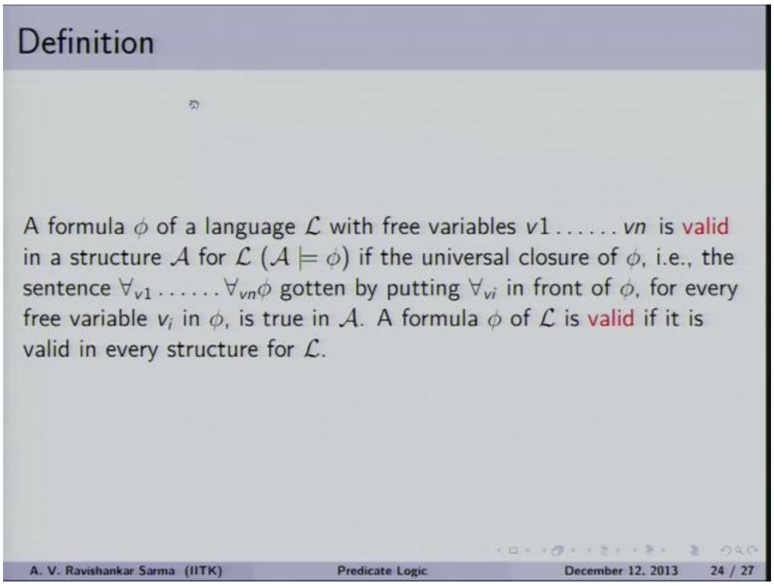

So, now a formula pi of a language L, which consists of free variables v 1 to v n, is considered to be valid in a structure A for L, which is represented as pi models A, pi is a semantic consequence of A in structure A. If the universal closure of pi that is the sentences for all v 1 to v n to pi, which you got it by putting for all v i in front of pi, for every free variable v i that exist in pi. And that happens to be true in A for all for all v i, but the formula is going to be true in that structure A and; obviously, pi is true in that particular kind of form structure A. So, formula pi of L is considered to be valid, if it is valid in every structure for, otherwise it is considered to be an invalid formula.

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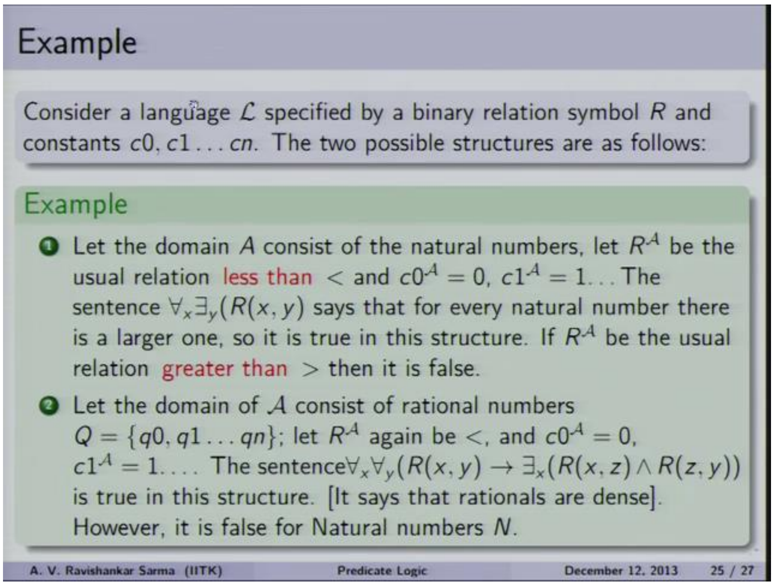


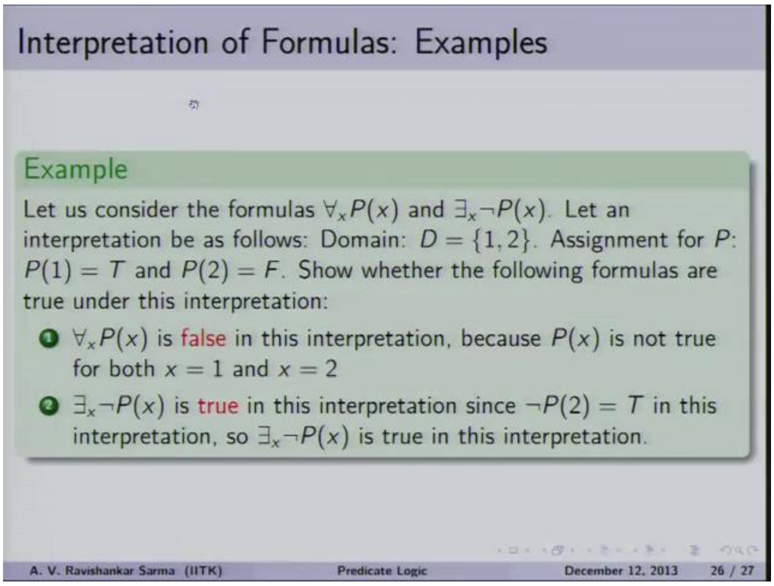
So, now let us consider some more examples. Consider a language which is specified by some kind of binary relation symbol R, which relates to objects in some way. It can be plus, it can be greater than minus etcetera and all, there are all binary operations and we have some constant c 0 c 1 to c n. And we can talk about 2 possible structures, in the context of the formal definition of validity a structure that we are given earlier. Now let us talk about a domain, which consists of a or d, sometimes you write it as a d etcetera. It consists of natural numbers and let R A be usually the relation R with respect to structure A, that is the usual relation, we take into consideration less than. And then there are some constants which find some kind of members in the domain, you write it as c 0 raised to the power of A, that is when it is the case it is 0 and if you take 1 into consideration c 1 it takes the value 1.

Now in that context, the sentence for all x there exists some y R x y says that, in the context of natural numbers. You have taken the domain as D and now we assign some kind of values to the constants and our relation which is a function between these things. And now sentence for all x there exists y R x y states that. So, it is like this; for every natural number there is a larger 1 that R x y transfer this thing; for all x there exist some y means, for every natural number that is for all x, there exists some y means, it is a larger 1 y. So, obviously, that formula that all x there exists some y x y, where x is less than that particular kind of a. For example, if you take a number as 25 25 etcetera and all 41, there will always be number 42 which is this 41 is always less than 42, they always come across the number which is greater than 41.

So, if R A is considered to be usual relation; with greater than, then this particular kind of sentence is going to be false. For example, if you take 1 and 2 into consideration, 2 and 1 into consideration, then for all x there exist some y, that is there exist some case 1 y there's 1 which is less than greater than that 1 1 is not greater than 2. So, that is why this sentence is going to be false. So, depending upon how you define your function that R x y and the domain that matters to us. So, interpretation also changes.

So, now let us consider domain A to be rational numbers, Q to be q 0 to q n and R A to be just taken as a relation less than, any constants represented in this sense c to the power of c 0 a 0 and c 1 a 1 we are taking into consideration 2 constants 0 and 1. Now the sentence for all x for all y R x y implies R there exists some x R x z then R z y, it is going to be true in this structure, it says that usually rationales are dense; however, the same thing is going to be false with respect to R is going to be true with respect to Q, but same thing is going to be false with respect to natural numbers. So, what essentially I am trying to say is this that, same formula is going to be true with respect to some kind of domain of natural numbers, same thing when we take real numbers into consideration in the same formula, here in this case for all x for all y R x y lies so and so. That formula is going to be false.

So, now let us consider some more examples. True formulas for all x P x and there exists some x not P x. Now let an interpretation be as follows; you have a domain d which consists of 2 numbers, usually natural numbers 1 and 2. And you have an assignment for P.



So, whenever you have P to the power of 1 1 is going to be t when it is 2 that formula is going to be false. And we have to show whether the following formulas are true under this particular kind of interpretation. So, now, the first formula for all x P x, this kind of property P x is going to false when it takes the value 2. So, it is not true for all the values of x. So, that is why for all x P x is going to be false because, P x is not true with it not true for both, even if it is true for x is equal to 1, but definitely it is not true for x is equal to 2 because, we said that P of 2 is false. It is not true for all the things, it is true for only 1 particular kind of thing, only there exists some x P x holds, rather than for all x P x.

So, now if we take the second thing into consideration, there exists some not P x which is going to be true in the interpretation because, not of P 2; obviously, is going to be true in this particular kind of interpretation. So, if we have satisfied at least 1 particular kind of interpretation, then there exists some x not P x is going to be satisfiable, otherwise it is going to be unsatisfiable. If it is true in all the interpretation, then it is unsatisfiable, if it is false in all the interpretations, it is considered to be unsatisfied. So, here there exist some not P x is true in this particular kind of. So, at least 1 interpretation in which the formula is going to be true and that will serve our purpose.

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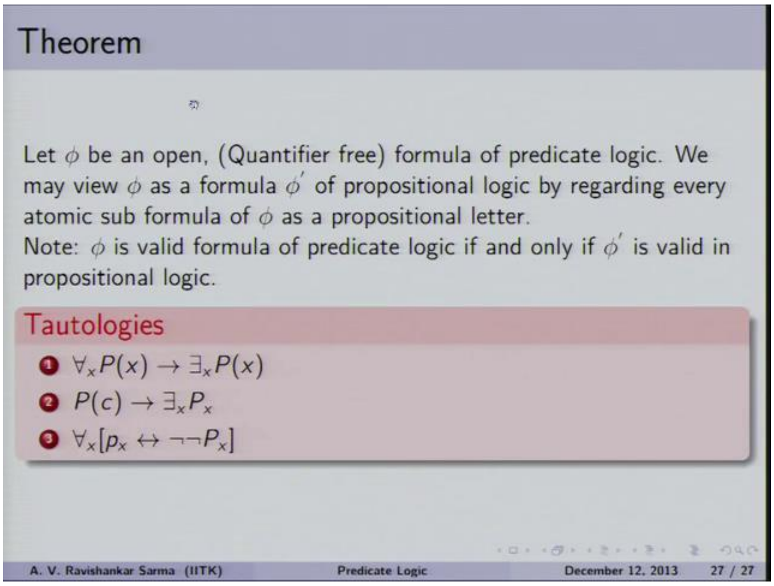

So, now let us consider important theorem which is stated in this sense is stated as follows. Let pi be an open formula of a predicate logic; that means, quantifier free, it has at least free variables in that kind of things. So, it is a formula and we may view pi as a formula pi pen of prepositional logic, regarding every atomic sub formula of pi as the prepositional letter. So, what is this theorem essentially says this is that, we have some kind of tautologies in prepositional logic and if you substitute with some kind of instances, when you have a substitution instance the, which are formulas and the predicate logic, that are also going to be tautologies. For example, in this case P implies P is considered to be a tautology or p x implies p x.

So, now, you substitute it like this thing x P x implies for all x P x for example, if you say that thing it is; obviously, going to be a tautology. So, in this case for all x P x implies, there exists some x P x, that is going to be true in a non empty domain, but definitely it is going to be false, it is what we have seen earlier, that is the formulas going to be false. So, now, if you have a formula P c, if something holds some particular kind of a entity, then you can say that it is at least 1 kind of entity, which has this particular kind of properties. If at least 1 chalk piece is white in color, then you can say that there exists some chalk piece such that, this chalk piece is white in color. So, they always holds. So, that is why it is considered to be a tautology.

So, in the same way for all x here, we know that P implies if and only not not P is true, then you replace it with P x in this particular kind of formula, an instance of preposition logic is the substitutions of tautology in the preposition logic and that id also considered to be a tautology. So, now let us consider some more examples so that, you will understand this particular kind of the semantics of the predicate logic in a better way. So, let us consider 1 single example and we will stop here.

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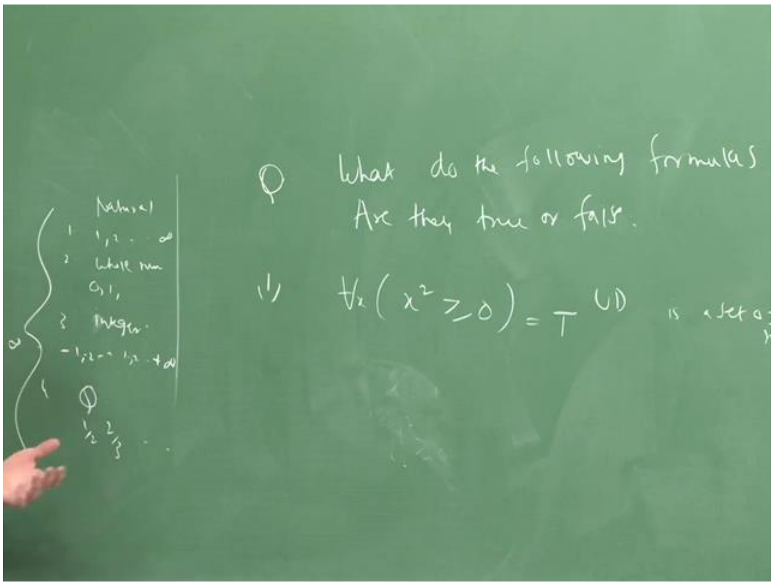

So, these are the problem states like this; what do the following formulas mean. Meaning of a formula means in the truth conditions, that is what we mean by that. Are they are true or false. So, now, we are taking into consideration few examples, simple examples. So, for predicate logical formula to be true false; we need to have domain first of all and then you need to have an interpretation. So, now this is the formula which we have x square greater than 0.

So, now, where the universal that I have discourse are a domain is like this. A set of real numbers, this universal discourse is considered to be set of real numbers, which are represented as R. So, what are real numbers; we have all this things natural numbers 1 2 infinity and then we have whole numbers that is 0 1 2, all the natural numbers together with 0 whole numbers and then we have integers like minus 1 minus 2 or minus 3 etcetera, this is minus infinity and then plus infinity 1 2 3 etcetera and then you have rational numbers 1 by 2, 2 by 3 etcetera and all. So, all these things are considered to be a real numbers.

So, now if you take this particular kind of a formula into consideration with respect to real numbers, now we want to see whether this particular kind of formula is going to be true or not. So, now if you take universal discourse to only the natural numbers. So, now, for example, if you take natural numbers into consideration, you take x into consideration x as 1, then it says that 1 square is less than 0; obviously, it is less than 0. So, for natural numbers it seems to be the case that, whatever value that you substitute for x, this is going to hold x square which is for is; obviously, greater than or equivalent to 0 which is greater than 0.

So, now this particular kind of formula; for all x there x square is greater than or equivalent to 0, for every real number x, we have this particular kind of thing x square is greater than 0 is the case. So, that is why this is going to be t; that means, in all this situation, even if you take into consideration minus 2 or minus 1 etcetera and all, minus 1 whole square is equal to 1; obviously, 1 is greater than 0. So, this formula holds for the real numbers. So, hence that is that formula is going to be true.

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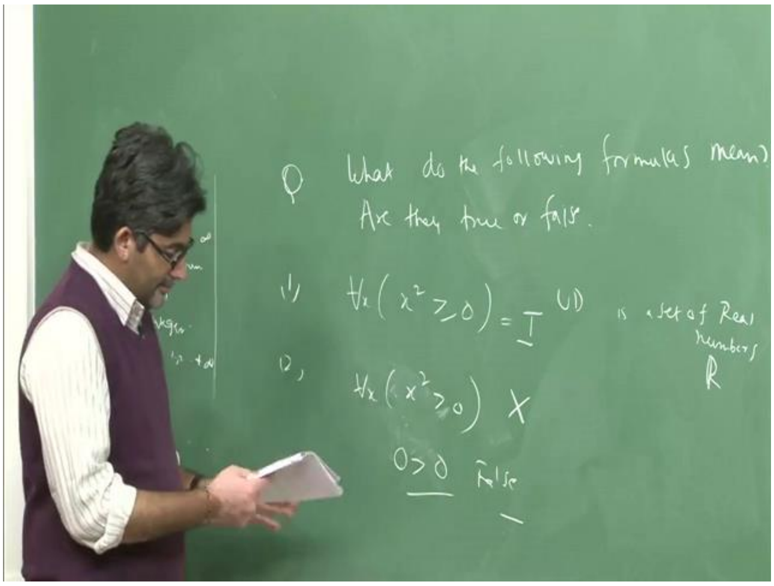

So, now let us consider another example. For all x, x square is greater than 0, you remove this thing; x square greater than 0. But here real numbers also consist of this whole numbers also, example if you substitute 0 square; then definitely 0 is not greater than 0 and all, but 0 is greater than or equivalent to 0. So, now, if we take x square greater than 0, now if we take this into consideration and that is going to be false. In at least 1 instance, this formula is going to be false; then this does not hold. For all x x square is greater than 0 does not hold.

So, that is why this formula is going to be false. Whereas, this particular kind of formula are going to hold because, if we take 0 into consideration, this formula is telling us that at least 1 x for all x for example, if we take 0 into consideration, 0 square is 0 only that is greater than or equivalent to 0. The second condition holds and all 0 is equivalent to 0. But in this case, it is strictly stating that, 0 is greater than 0, which is considered to be false. So, this formula does not hold in particular for the real numbers. So, now if we take another kind of formula, now let us consider the domain to be real numbers only.

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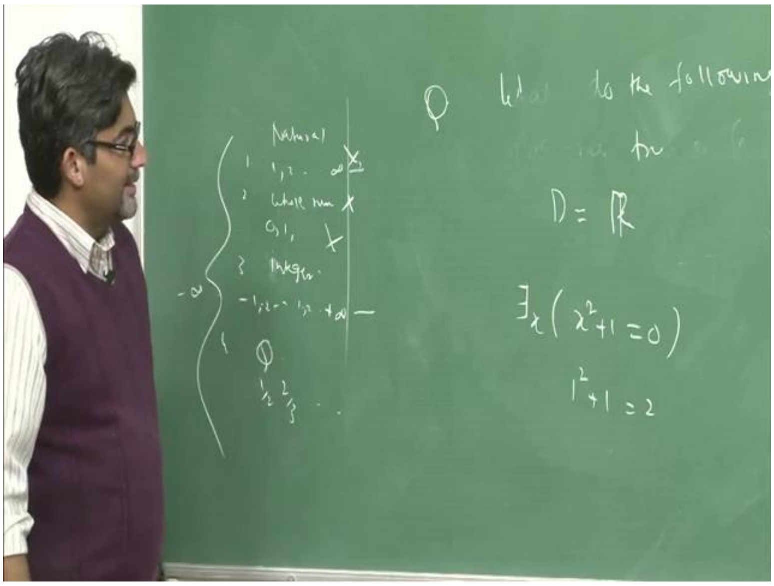

This is the domain real numbers, which is written in this sense. So, now, if we take for example, such as there exists some x x square plus 1 is equivalent to 0. So, now, in this case, is there any real number which satisfies this particular kind of property. For example if we take 1 2 3 etcetera and all, natural numbers, then suppose if you take 1 square plus 1 is equal to 2 is not equivalent to 0, it does not satisfy this particular kind of thing. Or you take 2 or anything into consideration, any natural numbers you are going to take into consideration is always it is not equivalent to 0.

So, now, coming back to the whole numbers, if we take 0 into consideration 0 square plus which is; obviously, equal to 1. So, there also it is not going to satisfy the whole numbers also. I mean it is not true in any domain. So, now, let us consider the integers, it consists of even negative numbers also. Suppose if we take minus 1 and minus 2 whole square for example, let us consider it to be 4 4 plus 1 5 which is not equivalent to 0, even that also it will not hold. And then this is the integer sand, even if you take into consideration rational numbers and this is not going to be equivalent to 0. And; that means, that this formula x square plus 1 is equal to 0 it does not hold in any structure and all. So, the formula which does not hold in any structure is considered to be contradiction.

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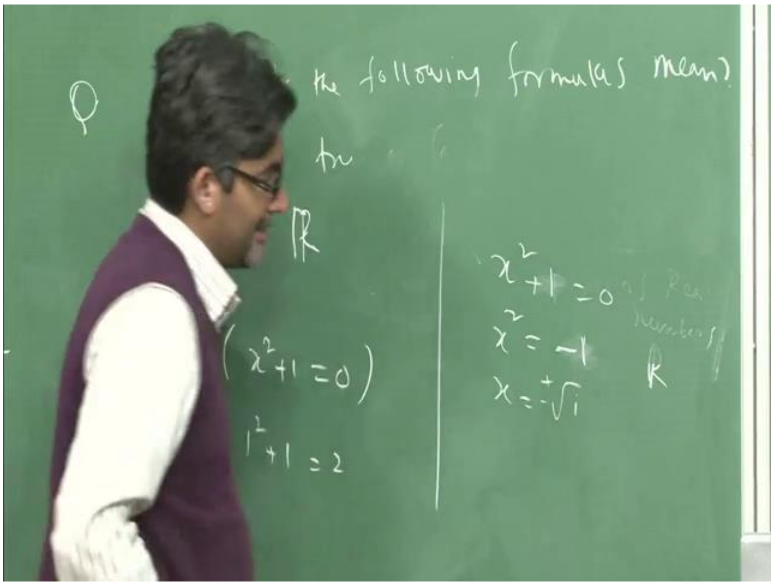

So, x x square plus 1 is equal to 0. For example, if we just talk about only x square plus 1 is equal to x square plus 1 is equal to 0, usually we write it as x square is equal to minus 1 and x is equal to something like plus or minus i x it is a complex number and all, which it is different from the real numbers. So, there is no model or no structure, which satisfies this particular kind of formula. That means, this formula has to be contradiction.

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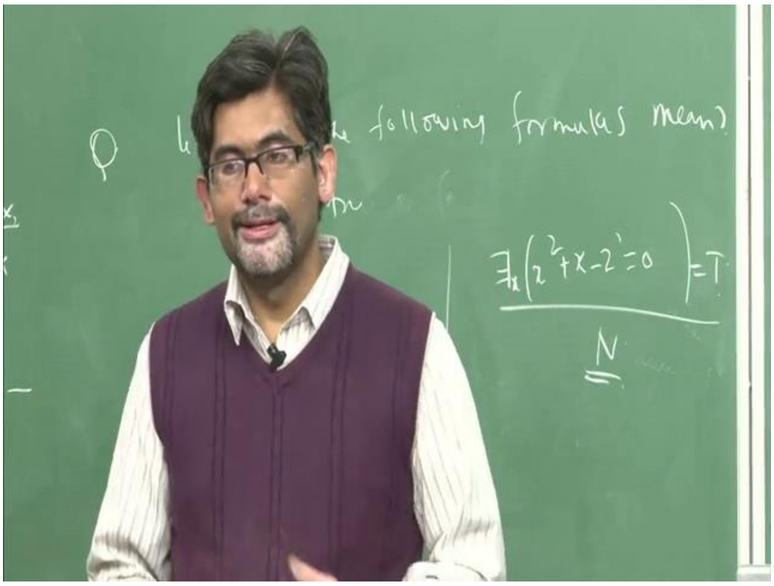

And now if you take into consideration some other examples such as, there exists some x x square plus x minus 2 is equal to 0, whereas, this is going to hold in some cases or not this is what which we are trying to see. So, if we take natural numbers into consideration, if we substitute 1 for it, what will happen? 1 plus 1 2 and 2 minus 2 is equal to 0; that means, it holds in at least such a in the case of natural numbers. At least 1 instance this formula is going to be true, then this is going to be the whole formula is going to be t here. So; that means, this formula is going to be true, at least its true it holds for at least natural numbers, then that particular kind of formula is; obviously, true .

So, in this lecture, what we have seen is this that, we started with the semantics of predicate logic the definitions. And then we have seen with some examples, when a given formula is true and when a given formula is considered to be false. The same formula is considered to be true of some kind of domain, which is considered to be false and some other kind of domains.

So, in the next lecture, what we will be talking about is; some important decision procedure method, which is called as which we have when using it in the case of in the context of prepositional logic, that is, the semantic tabloids method. And using semantic tabloids method, we will be dealing with some of the important logical properties such as, when group of statements are satisfiable with respect to the predicate logic, when a given formula is considered to be a tautology, when a given formula is contradiction etcetera. All this in future we will be talking about in the next class.