Calculus of One Real Variable
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Lecture – 13
Maxima and Minima

So, maybe I will now start talking about maxima and minima. And many people might be wondering that I am wearing the same dress. Actually these lectures are usually recorded in continue for some 2-3 lectures then which; obviously, each are separate lectures, but they are done in the same setting and in some sense.

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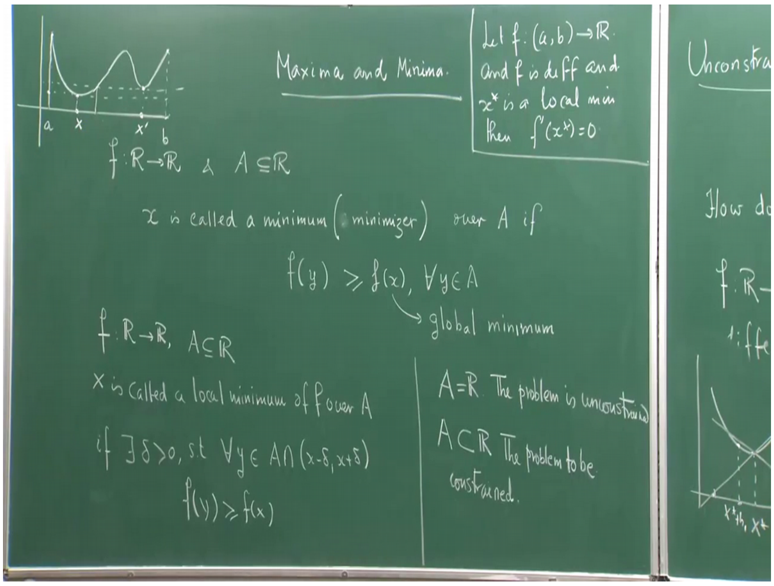

So, teaching is somewhat some in some cases is been in the show business. And I am trying to convince you about things. So, everybody knows about maximum and minimum these terms are very, very common. A common not only in a as a mathematical term, but common as a term that we use in a daily language. In fact, I know a family in Delhi with just 2 doctors; one is named maximum and another is named minimum. It is very interesting for me because I this is a part of mathematics which I have dedicated my life to, and so it is good to speak about this. So, what I mean I have already told you what is the maximum value of a function and minimum value of a function, say I can repeat the same thing once again just to recall you.

So, suppose f I am first looking at this situation f is a function from R to R, and A is a subset of R. So, here I have given this equal to sign at the bottom to show that a could also be R. A is a subset of R and x is called a minimum or sometimes people call also call it a minimizer. So, it there various ways spewak they call it minimum because if I if I go to the real domain of this maxima and minimum called optimization in a joint form if I go to the domain of optimization theory. I would really start calling it as a minimizer.

So, the several ways mathematicians like to call this point. Let us call it a minimum because in calculus they will call it a minima. X is called minimum over A if f of y is greater than equal to f of x for all y in A. So, A could be a look a could be open interval or closed interval a union of open intervals, could be disconnected open intervals, disconnected close intervals anything. When it just it does not mean that a has to be some very specific set or very specific nature.

This is the general definition of a minimum. And I would really ask you to find the maximum, define the maximum. There is another definition of say if I take from R to R and A is the subset of R. So, this minimum that I have defined is often called the global minimum or global minimizer. And the functional value at f is called the global minimum value, for a usual optimization theories they would call x to be the global minimizer and f value to be the global minimum.

So, x in this case is called the global minimum. So now, there is something called a local minimum. So, let us look at the picture of what I am trying to say. So, suppose I have a function defines f from this points a and n is in b. Now if you look at this point at this point you can understand that this point is the lowest has a lowest function value from all values. This point has a lowest functional value, but this point which I am another point which I am marked at this point the function that this. So, at this point this is x and say this is x dash. At x dash the functional value is bigger than the functional value at x, but for some region of the set a, b the closed interval a, b the functional value at x dash is less than all other values in that very specific zone, say in this zone.

So, it is a minimum, but not in the over the whole interval a b, but over a very specific region of a, b subset of a, b. So, such things are called local minimum and we will make it more formal. X is called local minimum of f over A, if there exit delta greater than 0 such that for all y which is element of a intersection x minus delta x plus delta. So, this is intersection f of y is bigger than f of x. Now let me mention 2 important facts about optimization problem. So, if a is equal to (Refer Time: 00:00), then we say that the problem is the problem unconstrained there is no restrictions on the problem. If A is proper subset of R then we you call the problem to be constrained.

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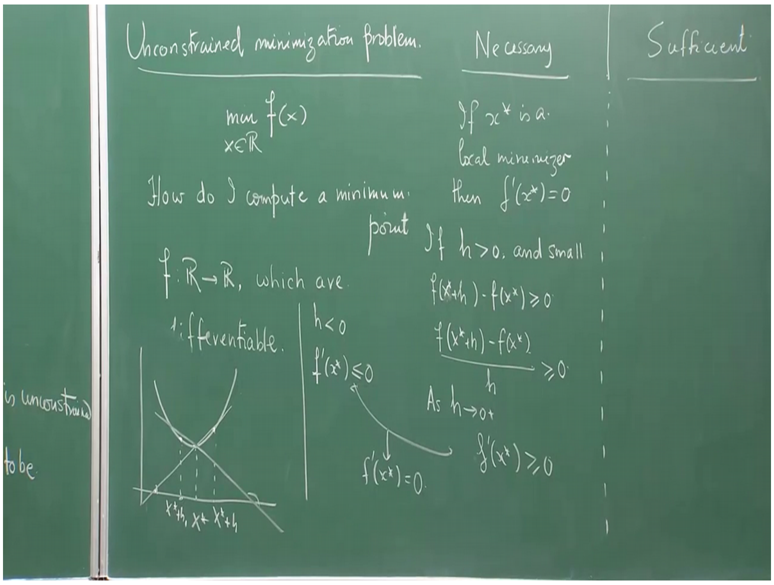

So now; so, our first study would be that of the unconstraint optimization on of minimization problem. Maximization and minimization problems are absolutely ubiquitous to mathematics, and to many many sciences into physics to chemistry to biology to many everywhere optimization to engineering to business optimization place a very, very central role.

Now, here my problem is simply stated as minimize the function f, or if you want to you more precise with the function values where x over R real line. So, what can say about the nature of a minimizer? How do I compute a minimizer? So, my first case is first step is how do I compute a minimizer or compute a minimum point. Of course, there is no guarantee that a minimum minimum value would exist on R because R is not a compaction if any of the function f is continuous.

So, let us be more naive and think that x star be a local minimizer of f, right. Then over the whole R, then how do I actual start computing such a point. And to do So, you need at your basic levels some more additional information. And that information is that of differentiability. So now, we will consider only functions from R to R, which are differentiable. We will not discuss any function which is which has points where the function is not differentiable.

So, if x start local minimizer, what can we say about x star? Does it satisfy any condition? So, if x star is the local minimizer is there a condition which it must necessary satisfies. If it satisfies a particular condition always whenever it is a local minimizer, then we can use that condition to compute an x which we can then test whether it is a minimizer or not. So, there are 2 aspects of this computation of a minimum one is the necessary part and one is the sufficient part. So, we will put this bordering wall between the 2. So, if x star is a local minimizer, then this condition always hold that the derivative at x star would be 0. Because we are only considering differential functions.

So, suppose this is a local minimizer. So, x star is your point, then take any other point nearby say x star plus h, where h is positive basically moving on the right direction. So, you look at the slope of this second, this angle is acute. So, the tangent is tangent value is greater than equal to 0. But if you to make h is a negative and if you come to this point say x star plus h 1, where h 1 is negative then you look at the slope of this second here the angle is obtuse. So, the slope is negative.

So, if you. So, the key fact about this point a local minimizer is that. So, within some domain with in some delta neighborhood, if you keep on moving x to the from moving if you keep on moving towards the right from x star, the slope of the seconds will remain positive. While if you move to the left the slope of the seconds will continue to remain negative and that is the key negative or 0, whatever that is that is the key idea behind the fact. So, so basically the derivative if I think of it is a continuous function which is continuously varying the slope is continuously varying. So, it moves from positive to negative. So, there must be a point where it must be 0 and that x star is the point where it is 0 that is the key idea.

This is the geometrical idea, but let us just do it. So, what happens that if h is positive and small when that is within a given neighborhood. So, that is the meaning. So, f of a plus h, sorry; x star plus h f of x star plus h minus f of x star is greater than equal to 0. Now if I divide by h because h is positive. Now if you take the limit as h tends to 0, h tends to 0 plus. Because the derivative exist it does not matter which limit you take it will always you will get the derivative. See you will get f dash x star is greater than equal to 0, but now if h is negative; now h if h is negative. So, here this would remain same, but when you divide by h this inequality will change. So, you will take the limit as h tends to 0 to prove that and when you combine this 2, you will get that f dash x equal to 0. So, if you combine this 2 to get f dash x star is equal to 0.

I ask you to prove another result at home, let f is now defined. So, here I am optimizing over a subset. So, it is a constant optimization problem, but the set A is an open interval and f is differentiable and x star is a local minima. So, I am using this word h this is small means it is the local minima should stay with in some given neighborhood delta neighborhood. So, because if A is equal to (Refer Time: 00:00), then y should just belong to x minus delta to x plus delta because the intersection would be just this and x star is the local mean, then x star x star is equal to 0.

If I make this to be a closed interval this result would not continue to remain true. So, whatever I have on R, I will have the same story on the open interval how as far as the necessary condition goes, but it would not be the same story over a may it would not be a same story if I have closed interval. So, I will just show you this fact. So, there is a huge paradigm shift if you move from one open interval to a closed interval.

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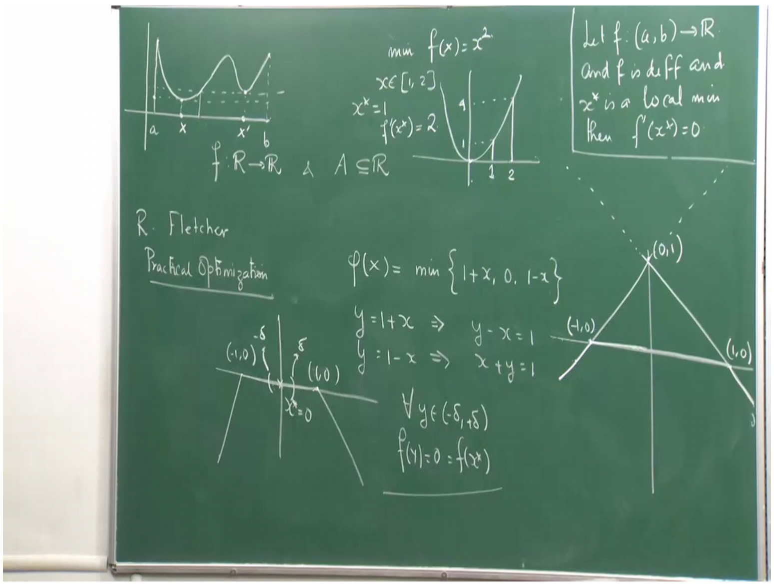

Suppose if you take the minimum want to find the minimum value of f x equal to x square y equal to x square, over x belonging to the interval 1 2. You very well know that we can draw the graph of this, I equal to x square. So, from one/ and this is 2. So, at one

(Refer Time: 17:45). So, this is 1 and this is 2, one/ the value is one/ and 2 the value is 4. So, you know over 1 2 the minimum value is obtained at one x equal to 1 is the minimum value. So, in this case x star it is a global minimize one, but what is f dash x star, a dash x star is 2 because it is 2 x and you put 2 x star and if you put x star equal to 1. So, when x star is one and f of x star is 2 and it is not equal to 0.

So, whatever you know for this and the whole real line and open interval does not hold true when you are in the setting of a closed interval. And so, in that case you have to think about something else. We will come in to that later we because we have a separate section on optimization where we will think about this. But now what about the sufficient condition we will come to this later, but here I want to tell you about some important fact, which is usually never told in a calculus course, but this is something you should realize at the very beginning and then your understanding of as I told you of calculus would be much better.

Now, here we are talking about local minimum local maximum and all those things, but here, I will give you an example where the global maximum of a problem can look like a local minimum. So, this is the very standard result and this proof I this function I have taken from the book of fletcher, it is one of the most well known it is called the wizard of optimization. As in the name of the book is practical optimization published by Wiley. So, here it gives a very interesting example just consider the example phi x which is minimum of 1 plus x 0 1 minus x. So, what does it mean? See you plug in an x and compute all this 3 different functions. And whichever a whatever is the minimum value among them that you return as the value of the function phi x. Now what does y equal to 1 plus x represent? It represents the line x minus y equal to 1, and what does y equal to 1 minus x represent it represents the line x plus y equal to 1, and y equal to 0 represents the x axis. Sorry, it is for y minus x is equal to 1.

So, if you draw the graph let us first draw the graph. So, you know this y minus x and x this these 2 lines would meet at 0 1, they will both pass through 0 1. Because when x is 0 in both cases y is one/ and when x is one for example, when right. So, for example, when y is 0. If then this case x is one/ and when y is 0 here x is minus 1. So, this represent the line y minus x equal to 1. So, minus 1 0 and then these goes through 1 0, and this is my x equal to 0. So, take any point here. So, that is the minimum value? The minimum value between these 3 minimum. So, if I evaluate this functional value. So, this line is going like this, this line is going like this.

So, the minimum value between these 3 this y x is z x equal to 0 this and this is the number here. So, up to minus 1 0 this is the minimum value, and after minus after that between minus 1 to plus 1, 0 is the minimum value. Because both are above 0. Again after 1 0 this is the minimum value. See if you know look at the function carefully the function is this minus 1, 0, 0, 1. This is 0 1 something like that sorry. So, this is 1 0. So, that it is look like an inverted tub inverted tub. So, look at this point x equal to 0 n. So, x star equal to 0. So, I will take a very small delta neighborhood around it. So, that that delta neighborhood completely remains with in this line minus 1 line joining minus 1 plus 1. So, this is my delta and this is my minus delta [FL], that is what it is; that it what it does.

Now, I want to tell you something very interesting here. Now for whatever why I take in this interval for all y element of minus delta to plus delta, what do I have? I have f of y is equal to 0, which is equal to f; f of x star which is f 0 where x star is 0. So, hence it satisfies the definition of my local minimum, over the of the function over x equal to r, but is the point x star equal to 0 really a local minimum. What is the what is the real property of this point it is the global maximum of the function. Because of the function value all the functional values are either negative or 0.

So, here. So, just these definition is enable to really distinguish the situation where where a global maximum can be misinterpreted as a local minimizer because at this point also. F dash x star equal to 0 right. So, in that case one defines what is called a strict global minimizer or strict. So, there is a concept of strictness coming in. So, how do you define a strict global minimizer. I will just write that definition and will then. So, I am writing for minimum because I am more comfortable with it everybody does what he or she is comfortable with it. But I would like you to also try out the things for the maximum those who are students from economics would like to try out put the maximum yourselves because that is what you would require.

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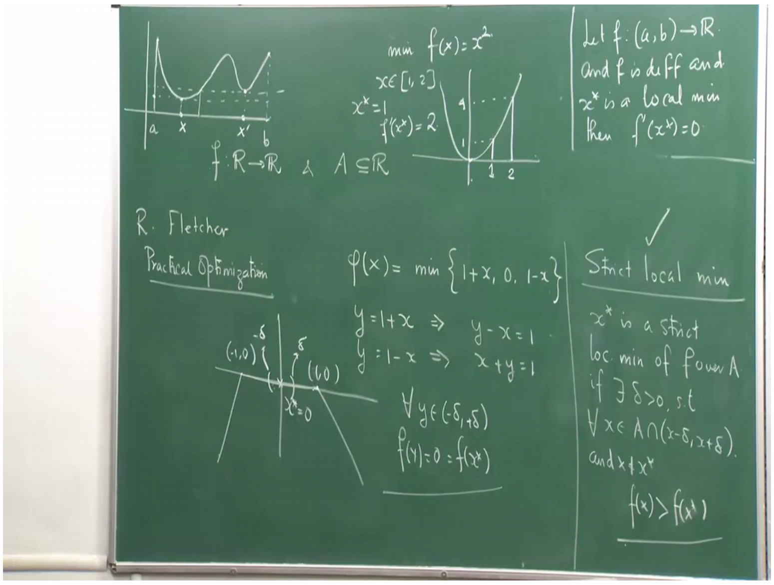

Now So, strict local minimum. So, what does it mean? X star is a strict local minimum of f over A if there exist delta greater than 0 such that, for all x element of a intersection x minus delta x plus delta and x not equal to x star, we must have f of x strictly bigger than f of x star this is the key idea.

For example if you take x equal to 0, for a y equal to x square, this is a strict global minima. In fact, because if you take any go to any point in on the right side or left the function value is strictly 0, there is no other point on or for which x square is 0 other than 0. So, that is So, 0 is the strict local minima. So, this strict local minima is something important that takes away this sort of very strange looking points on a out of you discussion.

So now once I have got f dash x equal to 0 that does not immediately tell me whether that is a maximum or a minimum.

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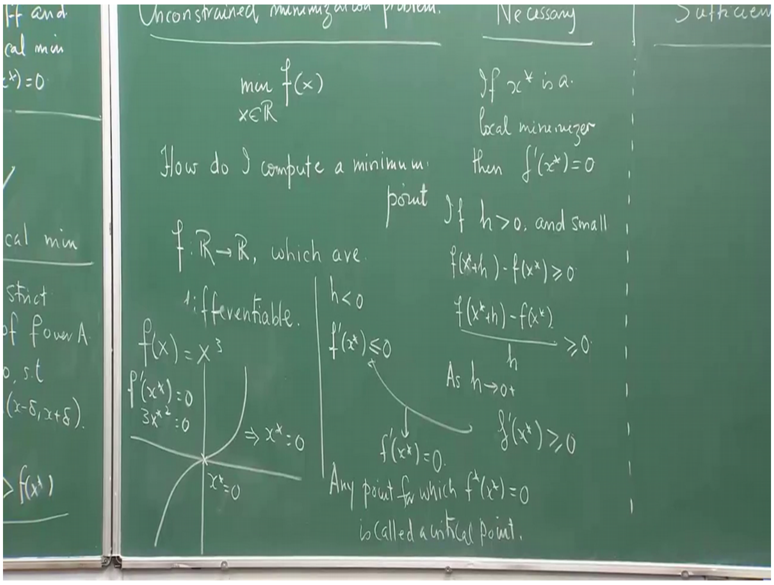

For example if I take the function f x equal to x cube, and I draw the graph of this. The that I had been advising from the very beginning or the advocating that you should try to draw graphs of functions because that makes the nature very clear. Because now you have mathematical packages for and those who have accessed to it and use those mathematical packages now if you compute f dash x star x star equal to 0. So, you will 3 x star square equal to 0, which will imply that the only solution that you have is x star equal to 0, but look at x star equal to 0 the function values on the right side is bigger than it functional values on the left side is smaller than it. So, it is neither maximum or a minimum local global does not matter.

So, f star x star equal to 0 does not say anything about the true nature of the problem. If x star was really a minimizer it would have satisfied this definitively. So, I have computed out of point which is satisfying this. So, usually any point for which f star x star equal to 0 is called a critical point. Now what I want to tell you is that there is a difference between a critical point and minimizer. Here x star equal to 0 is a critical point, but it is not minimizer or maximizer in x case of x cube. So, what additional information that you need that if I find and point x star, such that f star x star equal to 0 the only thing that I can tell about the point that it is critical point of the function or a stationary point according physicist like to call it a stationary point.

If I have that point there is no way I can say that that is a minimizer. So, there must be some additional conditions required. And for that conditions I need f to be twice continuously differentiable. So, let f from R to R be twice continuously differentiable. Let x star be a point let x star be a critical point, and that is an that is f dash x star equal to 0. Then and sorry, and f double dash of x star is strictly greater than 0 then x star is a strict local minimizer. This is a statement which is never given in any calculus classes. This is something important this is the true measure of the local minimizer that you get, if you put this additional condition you get a strict a local minimizer. You just do not get a local minimizer, you have that additional stronger thing coming to you. It is it is a free gift actually.

So, where is the proof that this would happen. So, for the proof we have to wait for one one more class that such things that this would actually give me a strict local minimizer. So, this is the basic idea of unconstrained optimization. So, when we will study a there is section a class on optimization, then we will look into the issue what would happen when you would have constrained situation like the one that we have given how do we select out points there. So, that would be given in a separate lecture where we will because this whole issue of maximum minimize very important and it has huge applications. So, that is something we will really do.

So, with this I would really like to stop my class. I hope you have very basic idea of minima or maxima. Maxima you can figure out your you self how to write the same old story for whatever I have the written the story that I wrote down for minimization, the same story works for maximization. But you have to write write it down in a proper way, in this case you would have strictly less than 0 then will have a strict local maximizer. So, you can define things like that also.

So, with this I end this class and hopping that you are actually progressing well in understanding any problem you can write to the forum and we will answer you back, my tiers would answer you back. And please be please feel free to ask a question, please note that no question is stupid enough that it can not be asked. Of course, do not ask us what is your name sort of thing, ask us some question which is relevant to this courses and lecture.

If you have some criticism about the lecture you want the lecture to have some component, please let us know and also I would like to assert once again that the philosophy of this course is not really to give make you prepare for some examination. Because you will never learn things by preparing for examinations, you only learn the concepts really when you are free in your mind and concentrate on leaning the concept.

So, our whole questions would be actually conceptual, based it is based on the very basic concepts. We will in the tests and examinations for those who would give you test the basics we are not going to ask you very complicated question where you have to do strange maneuvers. By doings mean a very complicated mathematical computation does not tell you that you have very great mathematical mind mathematical minds are once who can who works more on the concepts.

Thank you very much.