**Calculus of One Real Variable**
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**Lecture – 14**
**Lagrange Mean Value Theorem**

Today, we are going to talk about a very important idea call the mean value theorem or Lagrange mean value theorem named after it is originated the famous mathematician Joseph Louis Lagrange.

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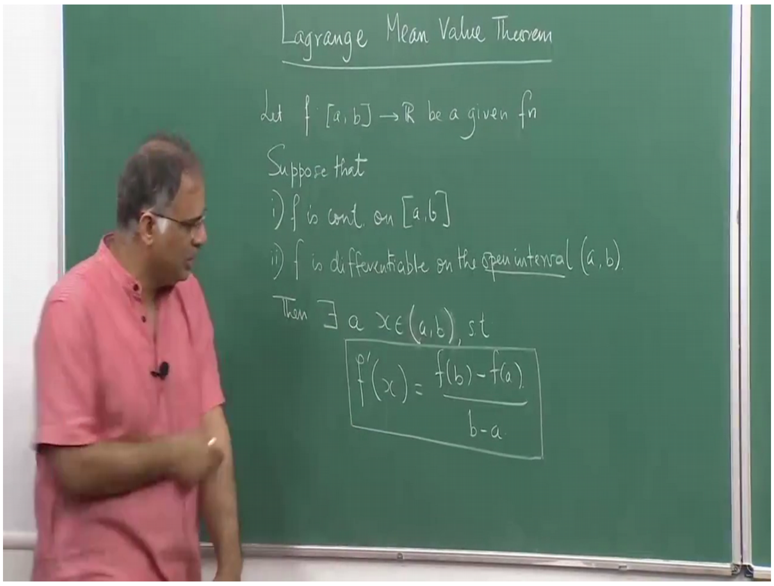


Mean value actually means average value; I understand, most of us are know what is the meaning of average its simply says that average, averaging actually is a proper policy, it works, now what does it say? What does a Lagrange mean value theorem says? It says let f from a, b to R be a function we are given function.

Suppose that, number 1, f is continuous on a, b cont means continuous this as a short forms I am using tends to improve the economy of writing and number 2, f is differentiable on the open interval a, b. So, I am suggesting it has to be on the open interval a, b right; the differentiability if these 2 conditions satisfied then there exists. So, this is the symbol of there exist, x element of a, b actually in the open interval a, b of course, I will just x such that f dash of x is equal to f b minus f a by b minus a. Now this is the statement of the Lagrange mean value theorem. So, I will first show how to interpret this idea what it means it has one mathematical interpretation and it also has a physical interpretation. So, let us first look at the mathematical interpretation of what, what does it mean?

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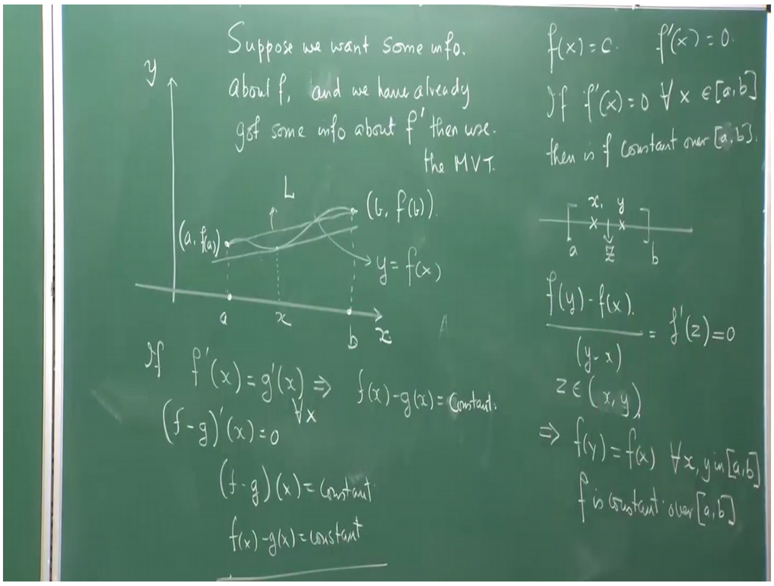


So, if I have a function; I am trying to draw the graph of that function, say a to b may be this is a continuous and nice functions here.

Now, what it says. So, if you look at this point of the graph then this is nothing, but a f a. Now if you look at this point on the graph of the function. So, this is your x and this is your y so this function is nothing, but y equal to f x. So, this function is actually y the graph of the function y equal to f x. So, at this point it is b f b so what is important is the following. Now, what does this thing means now if you join this f a; f a and b; f b by a straight line. So, call this straight line say L call the straight line L then slope of L is nothing, but f b minus f a by b minus a. Derivative has a very interesting explanation in the sense that which I think we have not told you in your class in your lecture on derivative, but derivative has a very interesting interpretation. So, the derivative is nothing, but the slope of the tangent line at a point x f x. So, here if you are having a function is differential say and this is the point x f x and this is the tangent.

So, we have also mentioned it, but this slope, if this is theta then tan of theta is a derivative at x f x. So, that is I, I would ask you to go back and think why a derivative at a point x f x would actually be so because what does the definition of the derivative says.

It says that you if you take some point here which is say this is x then this is x plus h then you will have some point whose coordinates are this coordinates would be x plus h and f of x plus h. So, basically you take 2 points and you basically look at the second joining it the chord. So, you do it as you make x smaller the chord finally, comes and aligns with the tangent and hence in the limiting case the tangent slope of the tangent at the point x f x becomes a derivative.

So, here what it says that slope of the line L is such. So, now, I can always find the m x in the interval a, b such that the tangent drawn at the point x f x to the graph of f x is exactly parallel to line L, because what it says that there exists an x such that this is equal to f x f dash x so x is some element in a b. So, which means that the slope of the tangent to the graph at the point x must be same as l. So, these are 2 different lines, but they have the same slope in with hence it must be parallel. So, basically there must be a tangent to the graph of the function f which is parallel to line L that is the meaning of the statement. Now let us give a physical explanation of what we have just written down as the Lagrange mean value theorem the physical explanations are following.

So, suppose a chord starts from time x equal to 0 and n z time x is equal to t in that the chord has moved a certain distance. So, in this axis is the time axis and this is the distance axis, there is a time axis and then there is a distance axis. Now, so at the point x equal to 0 there is distance travel is 0, 0, 0 at a point x equal to t the distance travelled is say so the distance is a function of time let us call it f of x the distance in f of t. So, this is nothing, but t f of t. So, now, how do I calculate the average speed in over this distance? So, the average speed is nothing, but say if I want to calculate the average speed. So, the average speed is nothing, but a distance travelled in this time interval divided by a time taken and now assuming that the path which the path of the curve which has been described is a nice function with differentiability and continuity property just as it.

So, what does it say that there would be a point so here is a line joining. So, what did says that there exists a point by the mean value theorem such that the tangent to that to that point on the graph, say this is some x. So, the tangent at that point of the graph is parallel to this line joining 0, 0 and t f t which means that during the travel there must be a time point when the curve travels at the average speed there must be a instant where the velocity exactly equals average speed of the car this is quite and natural thing to happen.

So, that is exactly the physical meaning is a very kinematical problem is a physical meaning of the derivative right so we have some ideas. Now, what does this mean value theorem does we are yet to prove the mean value theorem and we will do the proof at the end, but let us first see what sort of health the mean value theorem can do to us in our studies. So, when do you apply the mean value theorem what is the situation when you apply the mean value theorem?

Suppose you want some information about f some information or info in your language info about f, but and we have already got some info about f dash then use the Lagrange MVT or just then use the mean value theorem. I will give you an immediate example which you will be quite surprise to see many of you might not thought about it if the function f be such that f of x is equal to constant c then it is very simple for into prove without much difficulty that f dash of x at any point x is just 0, it is just 0.

But if I ask you the reverse question that if f dash of x is equal to 0, if f dash of x is equal to 0 is a constant. So, the I am asking of the reverse question if f dash of x is equal to 0 for all x say in a open interval say all x element of m v or whatever or what the whole r does not matter then is f constant over a, b. So, that that is the sort of questions we are going to ask now here as I told you the [FL] is this that if you want some info about f and you already has some info about f dash then use the MVT then use a mean value theorem that is the key idea of when you want to use a mean value theorem. So, how do you do it? So, take the interval a, b on which the functional the derivative is 0 and take any points say x and y in a, b any 2 points any arbitrary one does not matter which point then by the mean value theorem f y minus f x divided by y minus x is equal to f dash z where z is an element in the open interval; sorry; x y.

So, now, but in this whole intervals of z is somewhere here, now in the whole interval a to b for functional values of f dash is 0. So, f dash z is also 0. So, this would imply at f y is equal to f x for all x y in a, b because a; this is this has been chosen arbitrarily so it does not matter. So it is, so, whatever point, do whatever ant 2 points you take they would have the same value. So, if I if you take any other point z it will have the same value as f x and same value has f y which means f is constant. So, this is the first these are application of the mean value theorem, right, because that this is the way very important and interesting application for example, if you have another simple application of what you have just. So, once we know this I will show you another simple application of this. So, simple application is the following.

So, if f dash x is equal to g dash x, then it implies that f x minus g x must be equal to 0, equal to constant. This is an obvious application of this idea because that is if f and g have the derivative same that the can only differ by a constant the algebraic non constant parts must be equal. So, this, what does it mean f dash x is equal to g dash x this implies. So, that f minus g dash of x is 0 and from the above result this is so these 2 for all x of course. So, from here there the result above that we have shown that if f dash is equal to 0 for all x and f f must be constant. So, f minus g of x is constant and by definition this is nothing, but f x minus g x and that is constant. Calculus is about change and derivative is the most important tool which encompasses that idea that we are talking about things which are changing.

Now, we will do the proof, but to do the proof we will come in a stepwise fashion. So, I will keep the theorem up here and rub and start working on this spot. So, we are going to do the proof in 2 steps first we will discuss result by a little long French mathematician called Rolle whose claim to fame since to with this theorem called Rolle’s theorem which gives us an easy way of proving the Lagrange mean value theorem, but though if you know Lagrange mean value theorem the Rolle theorem can also be proved its essentially some sport of a special case of that.

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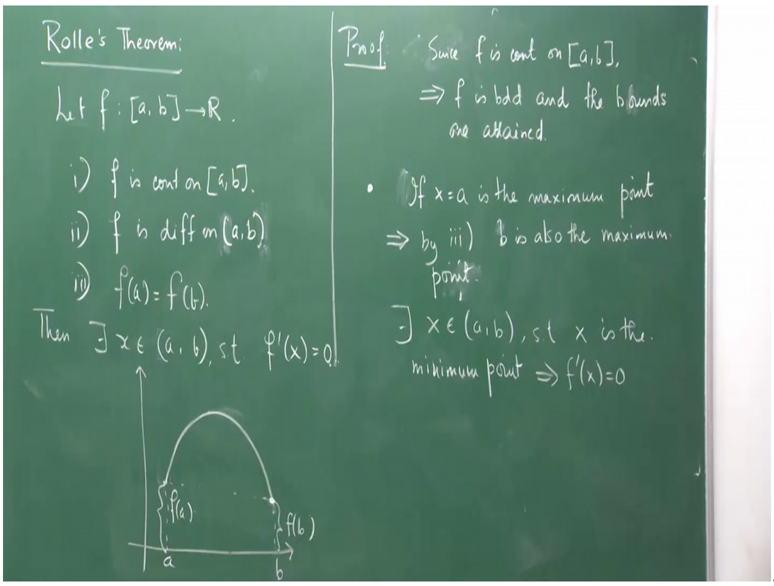

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So, let us first prove the Rolle's Theorem and then apply the roles theorem to prove the Lagrange mean value theorem. So, what this, what are the ingredients of a Rolle's Theorem? What does he say? What are the things that are there and what does it say. So, as before let us have a function f from a, b to R, here you will observe that in all the cases we are assuming the continuity at over the whole interval a, b closed interval. So, at the left and right end points I am essentially talking about left continuity and right continuity and left continuity.

So, but I do not bother about the derivatives at the end points that is, that is I am I am happy without that. So, let us have the following number 1 point f is continuous on a, b to f is differential on a b. So, I am just giving the short and continuous differential differentiable and number 3 is an important assumption it says that f of a must be equal to f of b. So, if this 3 conditions are satisfied then there exist x in a, b such that f dash of x is equal to 0. Now, how do I prove this prove this result now you will see whatever you have learnt in the classes previously will now come to your help. So, this pictorially this result looks like this. So, this is your a and this is your b.

So, f of a; this f of this is the f of a and this length is f of b and f of a is equal to f of b. So, it says like if you start rising from here you would again have to fall back to f b. So, there must be a point where you reach the maximum. So, you climb the hill and get down on the other side or it could be like this. So, you descend to valley and get up on the other side. So, you will have a minimum. So, at those points f dash x would be equal to 0 which you already know now the proof depend on this result which again is a very important one which you already have studied since f is continuous, since f is continuous.

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Sorry, differentiability would be always of the open interval please that is a I would say wrong writing, but I am mentioning it a, over and again that I have also made a special lining which has to be a special mod, that it has to be on the open interval, since f is continuous on a b. So, f is bounded above and below that is what is a result say f is bounded f is bounded. So, it implies that if it is bounded, if is bounded and the bounds are at end. So, there exist an x in a, b for which f x is the maximum or f y and the f, if x is the maximum there is some x dash for which f x dash value is the minimum and bounds are at end and the bounds are at end. Now suppose first case that the maximum is achieved at say the point a.

So, if x equal to a is the maximum point then b must also be the maximum point right, if x equal to a is the maximum point then b must be also the maximum point, because f a is equal to f b right and because a function is continuous there must be a minimum value in b 2 end. So, it will be implies that it implies that by 3 b is also the maximum point, but as you know because of the fact that the bounds are at end there must be a minimum value also the function there must be a x point which is the function attain the minimum value, but then that x cannot be lying either a or b it has to lie at some point in between. So, which implies there exist x element of a, b such that x is a minimum point.

And this implies that f dash x is equal to 0 of course, if there are a both the minimum and maximum lies in the interior in the in open interval a, b then anyway f dash x is equal to 0, the same argument can be given is say x equal to a is the minimum point then f x equal to b is also the minimum point because they are equal. So, that is the prox idea that this equality these are (Refer Time: 25:25) and hence it does not matter right. So, basically we write this result as f dash x into b minus a is equal to f b minus f a, then if I put f f f b is equal to f a then what would happen f dash x into b minus a is equal to 0 and if b is not equal to a then f dash is must be equal to 0.

So, that is exactly the thing so, you can think of it is special cases of that result, but that is used actually to prove this fact. So, you can argue for the other cases also I am not arguing and now I am going to prove to you the actual result right. So, now, we are trying to prove the real Lagrange mean value theorem and with that we will finish our discussion and this proof is from a (Refer Time: 26:11) book which have been using of course, everything is not I am telling from (Refer Time: 26:26) where the things of. So, elementary you can just skip on telling and these are very standard material, but some proves are taken from specific books and which you really mention that and second d now how I prove it, what is happening here this is a.

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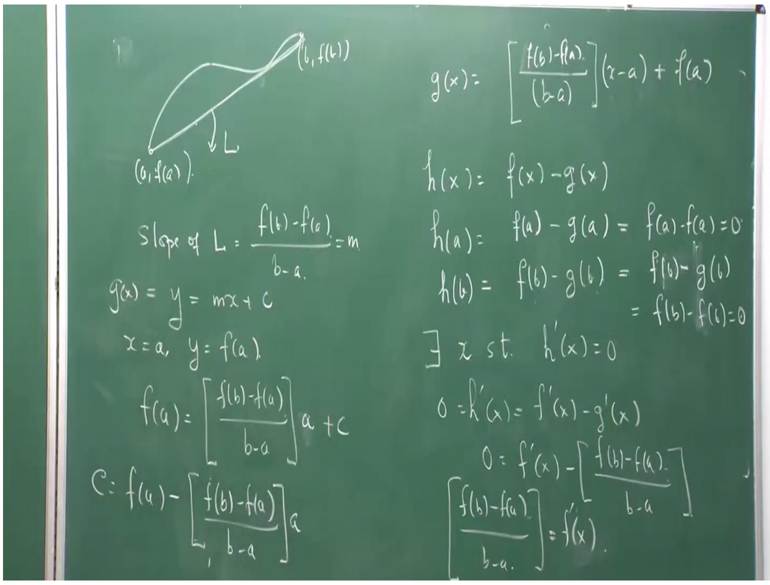


This is my, a f a and this is my f b some sort of a line. So, this is a f a and b f b basically given any function value.

I would create a function where I would like to know how far. So, this is the line I essentially want to know if at any x how far is this function from this line right because even if you think about a travel right the best way to travel from point a to point b is to move at a fixed speed and go, but because of the chord does not go at a fix speed you essentially. Now a curve like that the curve that you saw in the case of the physical explanation; so, here basically we are not trying to take for any x from f x your trying to know the distance of this line. So, what is this line? So, this line has a slope this given line if I call it l. So, slope of L is equal to f b, minus f a by b minus a right this is a slope. So, this is this is your m.

So, what is equation of this line this equation is y equal to mx plus c right this is or I can call this as g of x if you want. So, what happens when x is equal to a right then y is equal to f a. So, f a is equal to you know already the slope which is f b minus f a into b minus a into x plus c. So, what is c c is nothing, but f a minus. So, x is a sorry f b minus f a. So, when x is equal to a y is equal to f a naturally and then you putting in this line because a f a is lying on this line and so this is this is my constant. So, so finally, my function g x would look like f b minus f a by b minus a, into x minus a plus f a. So, basically I would like to know the difference between f x minus g x that is the key idea. So, I construct a function h x. So, now, what is my? So, this function is; obviously, because f is continuous in closed interval a, b and it differentiable in open interval a, b h is also satisfy the same properties now what is my h of a h of a is equal to f of a minus g of a.

So, what is g of a? If I put x is equal to a; this is 0 g of a is equal to f of a. So, an h so, f of a minus f of a is equal to 0 now what is h of b h of b is you have put. So, it is f of b minus g of b. So, this is b if I put x equal to b, b, b gets cancelled b minus a, b minus a gets cancelled f a gets cancelled with f a and this is nothing, but so just a moment it becomes equal to f b. So, g b f b minus g b so, so f b minus in this case if I if you put if you put g b. So, g b, so g b is nothing, but f b because if I put here b b minus a, b minus a cancels f a f a cancels. So, there is g b. So, g b is nothing, but f b. So, you will have f b minus f b which is equal to 0.

So, now, h is a function which is having continuity over the close interval differentiability over the open interval and h of a is equal to h of b equal to 0. So, which means there exist x such that, h dash x is equal to 0 what is h dash x h dash x is equal to f dash x minus g dash x. So, h dash x is equal to 0. So, 0 is equal to f dash x minus what is g dash x if I take g dash x it is nothing, but this is constant. So, it is nothing, but f b minus x minus a derivative is one f b minus f a by b minus a.

So, finally, we will get this result that f b minus f a by b minus a is equal to f dash x and that is the proof of the Lagrange mean value theorem. So, with this proof we end our discussion today and we will continue in the next class where we will talk about the applications of this particular result and you will see a lot of things will come and we will start talk about increasing functions and a huge amount of application because there you see because you will have knowledge about the derivative. You can draw conclusions about the function as called simple applications of the mean value theorem and heap up by this simple result can give us huge amount of information about a function once you know something about its derivative. So, this is something very important and these 2 lectures have to be if you if necessary go back to them and listen to them often on because these are very important chapters very very important lectures and very important results.

Thank you very much.