Calculus of One Real Variable

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Lecture - 22
Integration Theory of Riemann – II

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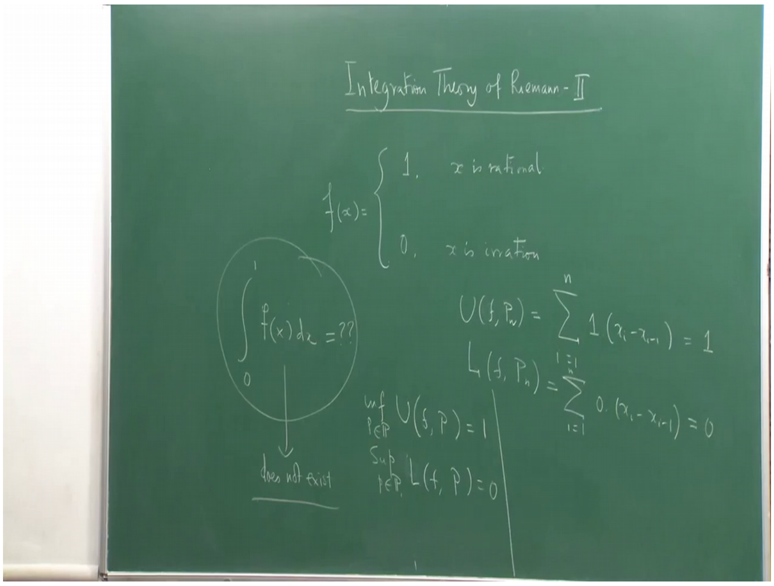

Now it is again the integration theory of Riemann 2. So, and as I told you the story earlier that when Riemann showed this to his supervisor, PhD supervisor, Gustavo dirichlet is very famous in many our parts mathematic including partial differential equations and all those things. So, dirichlet given this function which is nowadays is famous as a dirichlets function. If x is equal to 1, if x is rational is equal to 0, if x is irrational. So, let us look at the function, we let us look at this function and let us assume that we want to integrate this function from one to 0 to 1 fx to dx.

So, it could be any a to b does not matter I am just putting one 0 to 1. So, if I want to find this integral, what does it mean. So, again I want to tell you in the Riemann integration theory if fx is greater than equal to 0. Even if it is a bounded function need not be continuous, but greater than equal to 0. Then the Riemann integral is due for a nonnegative function the Riemann integral also the another name which is called the area under the curve.

So, what is this. So, simply take a partition and construct the upper Riemann sum, you see whatever we have bound whatever we have partition does not matter in every partition there are of course, countable rational numbers are uncountable infinite or irrational numbers.

So, one is always the upper bound 0 is always the lower bound, right. So, basically. So, you will have 1, 1, 1, 1, 1. So, basically if you finally, it will be summation 1 into x i minus x i minus 1 I is equal to 1 to n. So, this is if I have n points maybe if we you would like to write it as u f P n, but then this if you add them up this is nothing, but the interval 0 to 1.

So, the length is 1, but if you write L f Pn, it does not matter whatever be your n is always0. So, if you take the limits of these 2 as I have just showed. So, or if you take. So, it does not matter what is your. So, whatever we have partition this is what is the answer. So, U f P is always one L f P is 0. So, if you take this infimum of the supremum of this and infimum of this infimum this is one supremum this is 0. So, I take the inf over all possible partitions and I take Supover all possible partitions 1 and 0. Now for U f P is not known inf of this is not same as Sup of L f P.

So, if they are not same the function is not Riemann interval in according to darboux rule this is not Riemann integrable. So, this function this function is. So, this does not exist. So, how are we going to show that Riemann integral exists.

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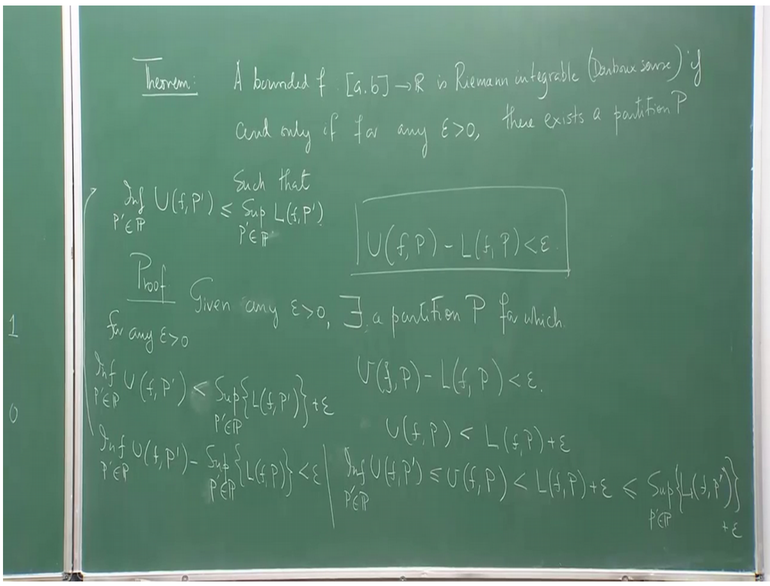

So, if we are now going to reconstruct the definition slightly, and write it in a more workable form into a necessary and sufficient condition make this slightly important enough that we can call it a theorem. It says that a bounded function f from a b to r is Riemann integrable.

Obviously in our darboux sense, but darboux sense and Riemann sense are same we have not mentioned the theorem that I will mention at the end of this class. So, f is a bounded function and it is Riemann integral in the darboux sense then, they are existing a partition p sorry then for any given epsilon or we can write a more complex statement. If and only if we are just trying to separate them if and only if for any given epsilon greater than 0 no matter how small, there exists a partition p such that U f P minus L f P again is strictly e is L f P sum. Now let us try to give a proof of this. You will see it is a restatement is or if you think of it is a restatement of the definition. First I will see assume that given an epsilon greater than 0, I will assume that there is this is a partition p for which this is holding this fact holds.

So, within the proof I will start in this following way. Given epsilon greater than 0 given any epsilon greater than given any eps given any epsilon greater than 0 there exists this is a symbol of, there exists with mathematicians use for short, I am doing it there exist a partition p for which U f P minus L f P is less than epsilon is given to you. So, what I can write that U f P is strictly less than L f P plus epsilon. Now infimum of U f P over all possible is partitions in the partition set is now less than equal to U f P right. Or maybe either I will write as p dash p dash.

So, the infimum is could be less than or equal to this right. And this is strictly less than L f P plus epsilon. But what is L f P it is less than or equal to supremum of l f p double dash or L f P dash p dash in p plus epsilon. So, if I look at this expression, what I get. So, for any epsilon greater than 0 I have just proved that infimum p dash element of p U f P dash is less than or equal to sorry is strictly less than supremum of L f P dash plus epsilon and p dash is; obviously, varying over p. So, you have inf of p dash element of p U f P dash minus Supof p dash p dash element of the partitions at p L f P strictly less than epsilon this is what you get.

Now, this is true for any epsilon greater than 0. So, I can make the epsilon smaller and smaller and smaller and smaller. So, in the limit I can take the lay this limit to be 0. So, so. So, there from here I will simply get the result that inf of U f P dash p dash element of p is less than equal to Sup L f P dash p p dash element of p that is exactly what I will get. Because if I there is the straight write the limit. Now this has to be less than this when this will finally, become 0 I can make this to a 0 the limit. So, when you what are the limit this breaks the strict inequality breaks right this whole thing has to be less than or equal to 0 for this has to be to maintain for every epsilon. Technically if epsilon is strictly greater than 0 then this quantity actually.

Because this quantity is always bigger than this quantity this quantity should be 0; obviously, know you know that already we have studied that this is bigger than this, inf of this is always bigger than this say essentially what we get from here is this and once we get you know that this common value is nothing, but the integral. So, what the integral exists. And now we will make the reverse proof of that the integral exists what happens if there is a integral then whether this there is a partition p dash for with this is true.

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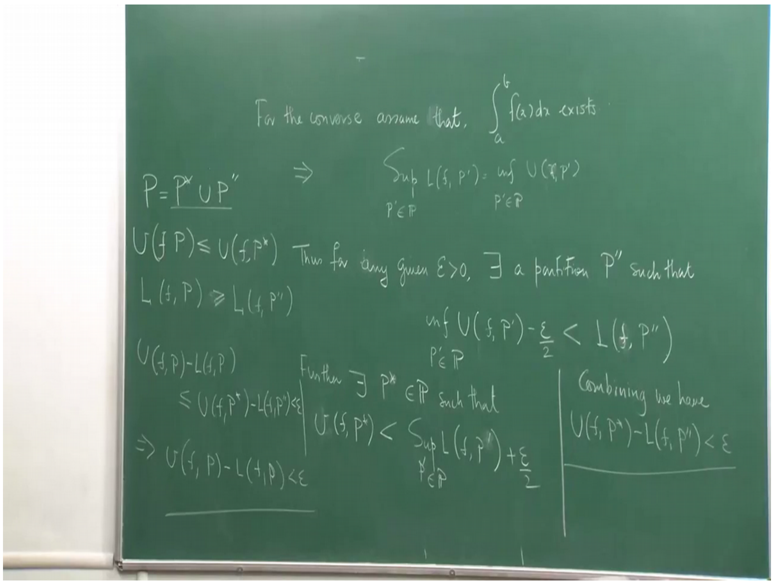

So, now we assume the converse for the converse assume that for the converse assume that.

Integral a to b fx dx exists means we have a finite value, which implies that supremum of p dash over p L f P dash is equal to infimum p dash over p U f P dash. So, that is that is what is known to me right. So, I will now apply the definition of infimum or supremum and we will get some conclusions. So, we will just apply the definition of an infimum supremum here. So, we have come out to here there is a this is a integrable this is what we know from definition. Now inf of this is actually the supremum of this. So, this is a supremum value of this quantity. So, which means that given any epsilon greater than 0 there exists some partition p dash such that if you subtract from the supremum of L f P sum the epsilon by 2 quantity.

There must be some partition for which the L f P that value of the lower sum must be strictly greater that is the meaning of the supremum. That this is the supremum nothing else can be supremum. That is there cannot be anything which is an upper bound which is when there cannot be any quantity which is an upper bound as well as less than the supremum strictly less than the supremum. So, similarly the Sup of this is actually the inf of this. So, there must exist the p star. So, that if you increase the value of the lower bound that is the greatest lower bound. So, it cannot there it cannot be a lower bound. So, there must be some p dash for which u p f dash would be this because this is at the inf of up dash. So, basically that start writing here super lf p dash then which is actually equal to inf of this can replace. Now if you combine these 2 or you will get a get this relationship.

Now, the last part of the result is very simple. You consider a p partition p which is p star union p dash a p double dash. And in this you already know the result you already we have done it that if you take a union the upper sum reduces right. The upper sum will reduce U f P is less than equal to u f p star. And L f P would be sorry bigger than lower sum increases L f P double dash. So, we if you combine them then you will have U f P minus lf p is less than or equal to u f p star minus L f P double dash which is strictly less than epsilon.

So, finally, we conclude and I have got a partition p which is a union if this 2 partitions you have constant we have shown the existence of the partition such that U f P minus L f P is strictly less than epsilon. So, this shows the final conclusion that conclusion. So, this is an if and only if condition for integrablity of f function in the sense of Riemann walked out in the darboux darboux approach. Now let us see how useful this result is. So, to show this we will actually apply it to a problem which I take from spivak, but it is a standard problem it can be found in many texts.

But it is important in to mention the reference that you are working with. So, students would actually know the text which the instructor is looking at. So, I will take a very simple. Very simple problem it would look obvious you will immediately tell me the answer.

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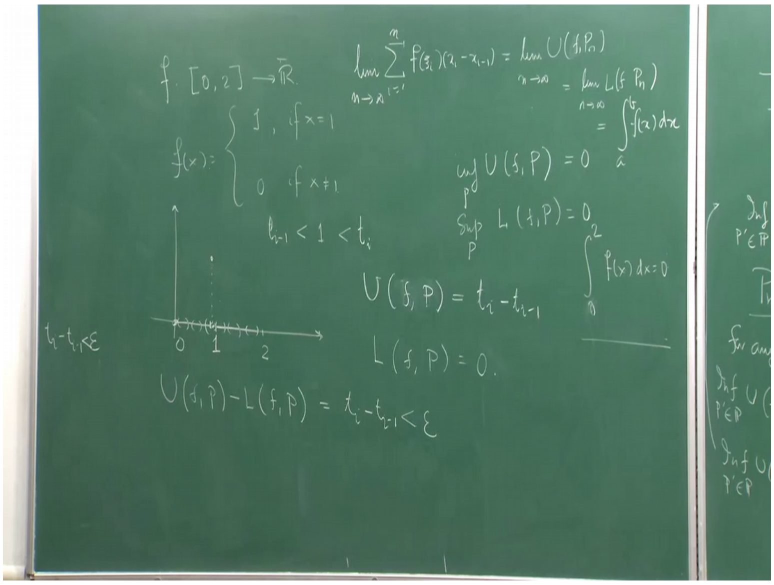

So, the function is like this. And fx is defined as follows is equal to 1 is equal to 0 now. So, what is if I draw the graph it is just like this. So, at the point one it is one. So, of before that it is 0. There is a discontinuity. Now what will you do you have to understand this one will either be a partition point or it will be within some partition point. It does not matter. So, there will be suppose I take a partition point say ti and one is here. So, it is upper sum whatever be the partition right, it the upper sum is u p f sorry U f P is equal to t i minus t i minus 1.

Assuming that I do the partitions at the i it is in the ith interval. It could be in some other interval does not matter. Similarly, the lower sum it does not matter whatever is a partition is 0 the lower sum is 0. Now what is the difference between U f P and L f P, but I can make, I can make the partitions in such a way that I can make this ti minus ti minus 1 to is as small as I like. So, given suppose I choose an epsilon and I can always choose 2 points t such that around the point one such that ti minus ti minus 1 is strictly less than epsilon. So, it does not matter whatever epsilon I have; I can always construct the partition p in which the ith interval ti minus ti minus 1 would be strictly less than epsilon and that would contain the number 1. And so, I will immediately get this and from that theorem I will immediately conclude that f is integrable.

Now, what is the value of this? You can immediately say it will 0 come on what is the value see how you do it here. See what is happening here is that if you take the inf of U f P, the info U f P is 0. Because I can make this as small as I like bring I and I minus 1 closure and closure 2 1 from both the sides. So, you inf of U f P is actually 0 and the Sup of L f P is; obviously, does not Sup of L f P is actually equal to 0. So, hence these 2 are same and. So, integral 0 to 2 fx dx is equal to0. So, you see it is a very simple way to calculate of course, you can say oh I can look at it and say it as our integral area is 0.

But here it is a very rigorous approach that if there is a there is a discontinuity at one point it does not matter I can still compute the Riemann integral. So, if a function is bounded and, but still has discontinuities at some finite number of points, and it still remaining integrable. So, that so discontinuity at finite number of points can be forgotten. So, for a Riemann integral function discontinuities at finite number of points is not the issue right. So, here is a is very this is a very important lesson to be learnt from this very simple example. Now I will state this result to the darboux result which one of the best ways it has been stated is in a book called principles of real analysis by allie brandison book in show. I will just show you in the camera.

So, that you can because nowadays people take immediately you will just take a short of the book rather than writing it. People do not waste time nowadays this is evolution actually. So, what they say what they prove is the following they prove that that integral in the darboux sense and integral in the Riemann sense are same. So, let me just tell you maybe I should going to do waste time, I can just write it down. So, integral is the darboux sense and integral in the Riemann sense are same that he says that, limit of maybe I should just simply write this any time to infinity, f i x i minus x i minus 1 i is equal to 1 to n is same as limit of u, U f P n, n tends to infinity is equal to limit of n tends to infinity of course, the proof is involved as on book proof is not done.

That that should be really done in a much advanced class for graduate on a advance undergraduate class for even under the advanced ago some graduate class in mathematics. And this is same as the Riemann integral of darboux. So, remanns approach whatever cause of functions is possible you can actually integrate by darboux approach you can integrate by Riemann approach. So, reimanns approach and darboux approach to the Riemann integral are they are the same thing. So, so darboux approach make sense because this is much more easier to handle. And let me also tell you that a continuous function, if you take a continuous function it is the bounded function and any continuous function is Riemann integral, but we are not doing the proof.

Because a proof needs some notion called uniform continuity. So, these little things would actually be provided in the notes. Now what is uniform continuity or what are the main results how it is applicable to the Riemann integrability. So, the whole idea is that we have not used this idea of uniform continuity because that is the only place where you only use it. So, we have not just done it. In the next class will show that using that darboux approach is very simple to prove some rules of the calculus and we will show that if we have something like this.

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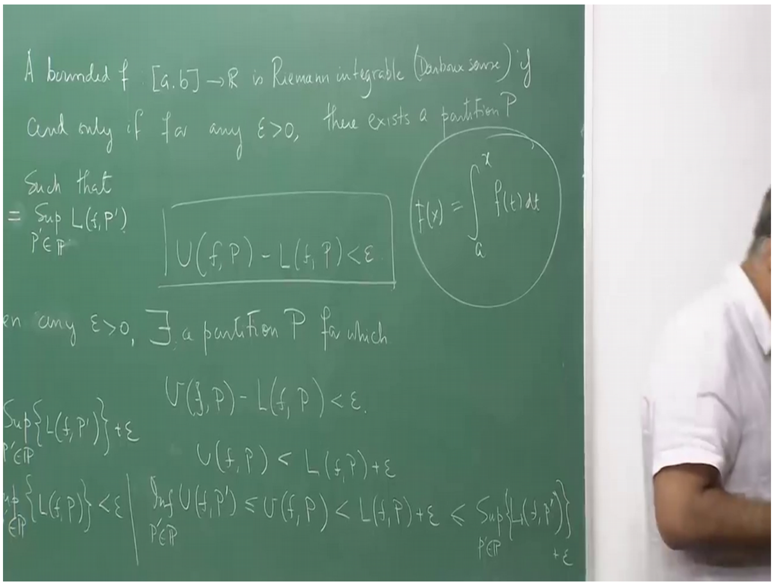

So, you have a bounded function and you have so this is a Riemann integral function and if I can write it like this you can prove that capital f is calculus hey sorry capital f is a continuous.

And once you prove that that would really lead you to the fundamental theorem of the calculus which you have already learned for the Newtonian situation the newton's approach, but we will learn it for the mode general approach we that is for the mode general approach the fundamental of theorem calculus actually holds. So, this is important and once we finish that those are the 3 accuses which. So, that tomorrow we finish that part. So, maybe first part would deal with that may be the we will deal the whole thing in one lecture. And if time permits tomorrow depends on what is the scenario then tomorrow, I would tend to give you a more of a entertainment lecture you can say every course should have into entertainment lecture we believe.

it is on how this idea of Riemann's this idea of tagged partition has been actually generalized making the partitioning more flexible to act to integrate functions which are not integrable in the Riemann's sense. And actually it is better than the official integral of mathematics that is Lebasque integral, but Lebasque integral is. So, much as I told you. So, much in better in our psyche as mathematicians we tend to forget that something else could be better.

So, that is the integral video to polish mathematician called (Refer Time: 28:35) and (Refer Time: 28:37) Rolf (Refer Time: 28:39) is a British mathematician. So, they invented this simultaneously, independently that they could actually will not prove things there much we will try to give a definition shows some examples are something which does not is not Riemann integral what is interval in that sense. In fact, some integrals which of which to are not integral able, but in the sense of Lebasque which is the Lebasque integral, which of course, you do not need to know about it is also integral in there that sense and this is something which is there is only one book written about that class of integrals is due to wattle Robert wattle.

So, who was the famous book nowadays wattle shewhart what. So, Robert watleys book I think Robert wattle is in the live anymore. Robert watleys book he says that this is the true modern theory of integration and it should be more famous it should be more known maybe it is not known today maybe after 200 years that would be taught in every class. So, with this I would end my talk and I hope that you have enjoyed it.