Calculus of One Real Variable
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Lecture - 23
Integration Rules

So, we have already defined Riemann's integral its basic properties basic associated ideas with it. Now it is time that we put it through some litmus test that is it really the integral. The integral as we know it from Newtonian lines, Newtonian needs that it has some properties like the integral of the addition of 2 functions is same as the addition of the integral of the 2 component function to individual functions.

So, that is a litmus test that if all these rules that we have studied for integration in the sense of Newton can be applied now, can be also showed; for that of Riemann, then the Riemann integral passes the litmus test and then it becomes some sort of acceptable integral.

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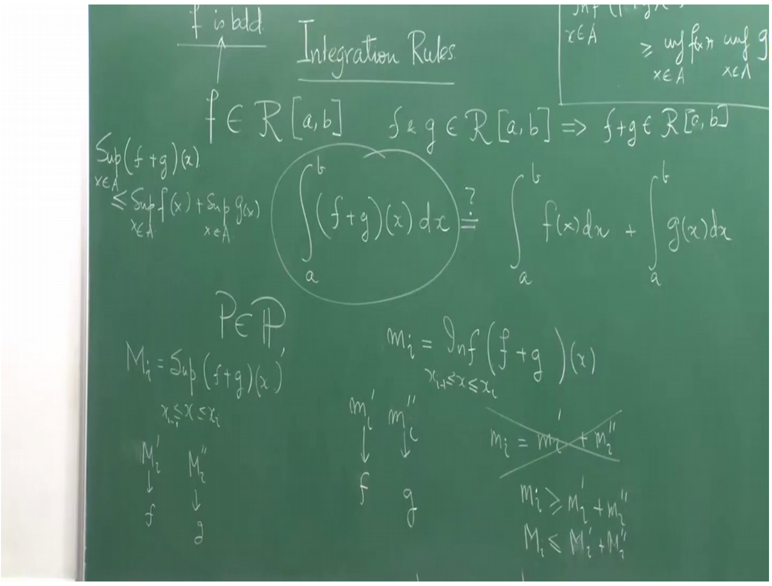

So, we will now write if a function is Riemann integral; we will write that f is in R a, b. So, you can have your own sign invented for this set just to mean. So, whenever I am talking about a Riemann integral function; I am assuming a bounded function; I am not telling that oh; it is something I do not know unbounded hook just it then I am try to prove that if he is bound it obvious you can do that, but that is not what we really doing.

So, this simply means f must be bounded bdd for short, if he is bounded and Riemann integral. So, this is the collection of bounded functions which are Riemann integral.

Now, if f and g; f and g both are in R a b. So, what is the relation between these 2 numerical quantities; now here integration is in the Riemann sense, it does this relationship hold true, you are mine were more mine money tempted to think that oh it definitely does actually does, but we you need to provide a solid backup or rather a proof to the thing that this thing really works.

So, how would you go about doing? So, you take a partition P. So, P is the element of P where P is the set of all partitions as we had discussed in noted in the last class.

Now, let M i be the inf of f plus g over when x varies over the interval or x i and x i minus 1 and let M i dash and M i double dash this is the inf of f over the same interval and M i double dash is inf over g for the same interval.

Now, there must be at it must be very is it tempting to think that m I is equal to M i dash plus m i double dash or this is crossly incorrect, I would rather tell you to go back home and try out this very simple exercise that if you have the int you are taking the infimum of a function f x or f plus g of x over x belonging to some set a does not matter what you get set a. Then this is bigger than the inf of f plus inf of g x element of a you can write f x g x is does not matter.

So, this is actually the formula. So, this is something you should prove. So, what I actually have here is that m of i is bigger than m of i dash plus m of i double dash. So, let us look at the supremum. So, I can write M i capital M i as a supremum of f plus g maybe I should write f plus g x; if you want to be more you want to look more precise and x is lying between in the interval x I; x i minus 1, then I can; I will say m i dash and m i dash are the supremum over the same interval this is for f and this is for g do not say that the sum of these 2 is equal to this; this is crossly incorrect again,

So, again go back home and look into just try to prove this simple formula way is a subset in R. So, the super f plus g is less than super f plus super g. So, what I have here is another thing. So, what I know I can write from here that m of I is less than equal to capital M of i dash plus capital M of i double dash.

So, what does this give me if you are very clever you if you just clever on to see through it, you will immediately realize that these things appear this appears with the lower sum of g this with the lower sum of f and this with the lower sum of f plus g.

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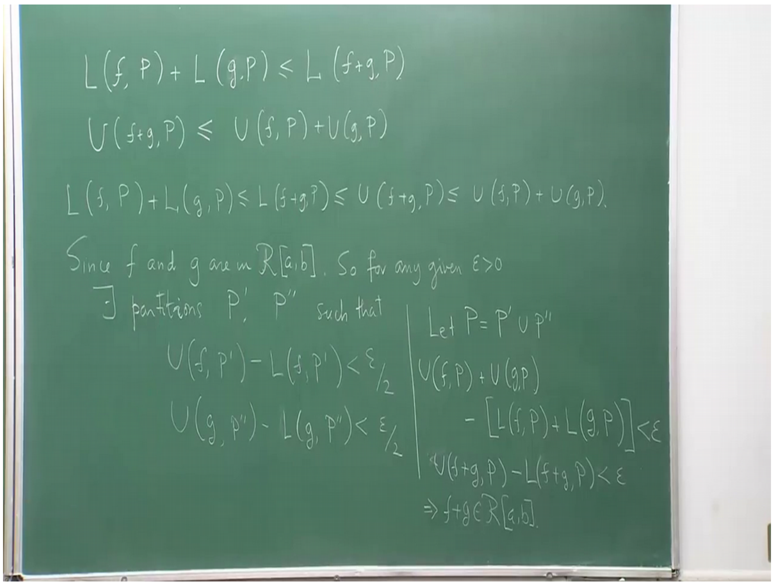

So, what I have in effect proved is lf P plus L g, P is less than equal to L of f plus g, P while for the upper case, it is the reverse upper case you will prove that U of f plus g, P for any given partition P; it is less than U of f P plus U of g, P this is a key idea because without this you cannot do much, right.

So, what you have actually proved you have proved that L f, P plus L g, P is less than equal to L f plus g, P this inequality becomes handy later on f plus g, P which is less than equal to U f, P plus U g, p. So, just combining these 2; so, this is what you have at this moment.

Now, f and g are assumed to be integrable; right that is what we have assumed f and g are the Riemann integral. So, since f and g are in R or in R a b; what would it imply; it would imply see; now you are using the darboux format to do everything because we have show is told you to told at the end yesterday that darboux approach and Riemann think are essentially other the same thing.

So, they give the same value for every the class of functions which the integrate are the same. So, now, darboux approach is easier to handle. So, we are using the darboux approach. So, since f and g are in R a, b. So, what I have. So, there exists partitions. So, so for any given epsilon said greater than 0 there exists partitions P dash and P double dash such that U f, P dash minus L f, P dash is strictly less than epsilon by 2 and U g, P double dash minus L g, P double dash strictly less than epsilon by 2 that is what is the idea is.

Now, consider a P; let P be the union of P dash union P double dash. Now I leave U 2 to the this formidable exercise that then what would happen then U f, P minus lf P is less than this and U g, P by P U g, P minus L g, P is less than this and hence less than this. So, what you can prove is U f, P plus U g, P this minus lf P basically now adding up plus L g, P this thing is strictly less than epsilon this is what you can finally, prove and consequently what do you prove.

So, you I have already proved that this is bigger than this and minus of this is bigger than this. So, what you have finally, proved you have proved that U of because here this part is bigger than this part. So, this minus this is less than epsilon. So, you take this to this thing to this part and add it with epsilon and you can apply that this plus this is less than this. So, ultimately you will get U of f plus g, P please check the calculation minus L f plus g, P is strictly less than epsilon.

So, what you have demonstrated that if f and g are is an R a, b; it implies that f plus g is in R a, b. So, now, this integral now becomes meaningful. So, that is I put a question mark here now this integral becomes meaningful now the question is whether this is equal to this; this whole thing that is first step. So, this is the shows one this actually is showing that f plus g is in R a, b. So, it is Riemann integrable function R a, b.

So, that part been over we will now transfer ourselves. So, that to this part so, that you can see the working that side. So, now, let us look at this now we will use this thing that we have proved now what we have proved is U of f, P plus U of g, P; right.

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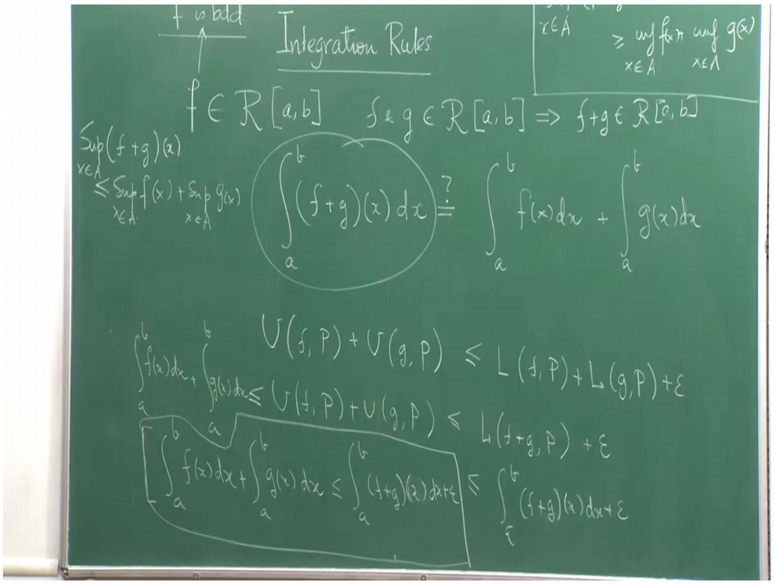

Is less than this is less than L of f, P plus L of g, P plus an epsilon that is what we have proved because once you have wanted to prove that thing? So, this information is there.

But this again as you have proved here. So, what do you have from here what do you have you have the following you have U of f P plus U of g, P. Now I am applying this fact on this. So, it will give me L of f plus g, P.

So, the integral is the supremum of the lower bound right plus epsilon. So, what is this; this is nothing, but integral of this is less than integral of a to b f plus g x d x plus epsilon and what do I have here each one of these are bigger than their respective integral. So, this part is it is bigger than integral of this because this is the infimum of this over all the partitions plus this part is integral a to b you agree to this. So, this is what you have proved, right.

So, what you have proved integral a, b f x d x plus integral a b g x d x is less than equal to integral a to b f plus g x if f plus g, x; sorry, if f plus g x d x plus epsilon that is what you have proved.

Now, now this part been done. So, you have this equation now. So, this equation is there. So, epsilon is there you have proved this equation now prove this inequality. Now I will prove something else which will assume. So, I am just removing this part and let me just see what we can do with the remaining part. So, here is the same sort of explanation.

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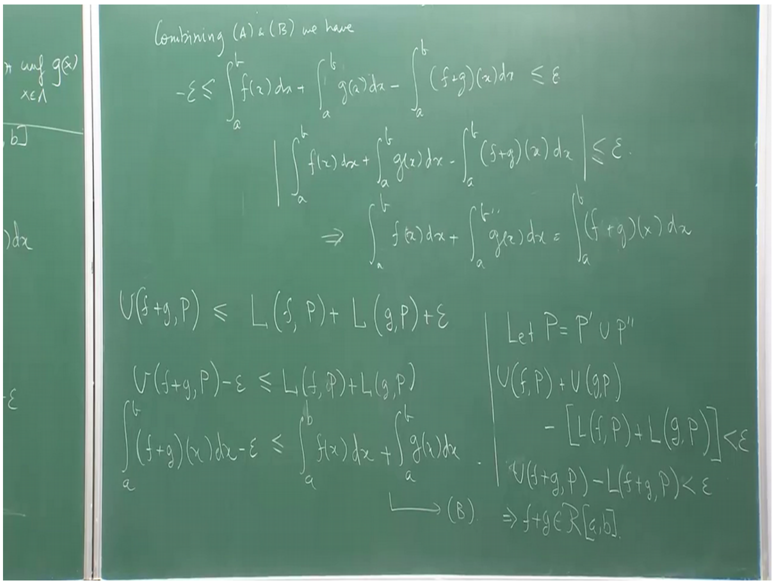

So, from here I can write using this fact from this equation please note U of f plus g, P which is less than equal to this and hence it is less than equal to L of f, P plus L of g, P plus epsilon. So, U of f plus g, P minus epsilon is less than equal to L of f, P plus L of g, P and you know that you g, P is bigger than the integral f plus g because f plus is already proved to be Riemann integral. So, a to b f plus g x d x minus epsilon is less than equal to each of these are less than the in each are individual integral this is less than the integral a to b because the integral is the inf of this.

So, the integral is the soup of this. So, the integral is essentially. So, the upper bound. So, similarly for this one by the very definition of the integral you have this. So, what are you getting? So, this plus this is this plus epsilon and this minus epsilon is also less than this. So, this plus this is less than f plus g plus epsilon and f plus g minus epsilon is less than this. So, combining these 2 equations; maybe I can drop this part now. So, I will put some numbers here. So, maybe a and b. So, I will put; sorry; this equation as a inequality and this inequality as b. So, combining a and b; a and b we have integral a to b f x d x plus integral a to b; g x d x minus integral a to b f plus g x d x is lying between epsilon and minus epsilon.

So, if I take epsilon to be 0; if I make this over any epsilon I can make epsilon 0 from both sides. So, by the sandwich theorem which you know about limits if the limits are 0 from both sides, then the limit of the quantity inside must be 0, but this quantity is free of epsilon. So, this; the limit is the same as this quantity. So, or if you want to be more precise this simply means integral a to b; f x d x plus integral a to b g x d x minus integral a to b f plus g x d x; this is less than epsilon.

So, this; since this is less than any epsilon greater than 0; it does not matter whatever be the epsilon the only non negative number which can be made as small as I like is 0. So, means this thing this absolute value of this thing is 0. So, the any quantity use absolute value is 0 is 0. So, it showing that integral a to b f x; f x d x plus integral a to b; g x d x is equal to integral a to b f x; sorry, f plus g x d x.

So, you see it needs a lot of work to really go ahead and prove please calculus this important rule for integrals which you which we knew earlier, but now we have done it for Riemann and you see that it actually works. So, Riemann has Riemann's integral passes the first test.

Now, there is something else which we will not; we want to the proof there is no much time that to spend on proofs, but this result that I want to now write down my. So, let me take a point c which is lying in a, b say maybe a midpoint or something like that.

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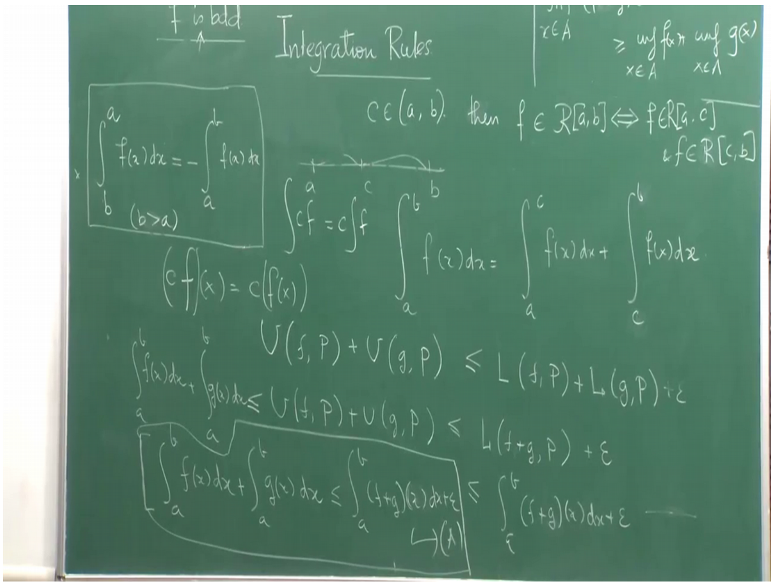

Of course, I can take c is equal to a and b what; there is they will become degenerate points and they will unnecessary complexities she do not require c is in a b and then f element of R a, b; f is Riemann integral if and only if f is element of a, c and f is element of sorry R a, c and R c, b means f is Riemann integral over a, b if and only if I take any point between a and b f must be Riemann integral over each of the separate component separate part. So, if I have a, c, b f is Riemann integral over the whole if f is 1 integral over this part and each of this individual part in if f is Riemann integral over each of this individual part then f must be Riemann integral over the whole.

So, it needs certain time to prove this it is; I would leave it to you to really work this out and in that case; you can actually show that integral from a to b if x d x is equal to integral from a to c f x d x plus integral from b to c f x; sorry, c to b f x d x.

Now in our analysis we have always assumed that the integral when you are evaluating the integral a to b this integral from a to b is valid when a is bigger than b that has been the approach, but again just like the way we did for the Newtonian case we can make a definition the definition is the following the definition is that if a is not bigger than b that is b is not bigger than a a is bigger than b. So, b to a; what does it mean how do I define it this is defined as minus a to b f x d x. So, only take this are definition this is true when a is when a strictly; sorry, when b is strictly bigger than a then if I write the integral from a bigger value to a smaller value this is same this negative of the integral from a to b; it is same as the area explanation that I had given you the same idea, but now just we have picked that definition up and put it in here.

So, once you do that you once you have that definition in then this c been a requirement to be inside a and b is no longer valid you can just put c anywhere and get the result and of course, you can show for Riemann integration theory that if I multiply c with a function any number you construct a function c f which is defined as. So, c of f x is c f of x is nothing, but c times f of x.

So, in this case; so, the integral of c f is c times integral of f; I am just writing very short and things, but these are very common things that you can learn up yourself from any books or even the net important result that is there which for which we do not need any extra styling or extra information, but this is an important result which we would not prove because the proof would taking a lot of time, but we are going to state this and because this idea would be important when we prove where we start to have a look at the mean value theorem or Riemann integrals and their applications.

So, you can look into the proof these are the things; let me tell you, I am going to give you notes, I will start giving it out maybe from tomorrow, I will give my some TA notes of the first chapters, but if I start giving; I am trying to build up the notes, but the course duration. So, short that such a huge thing cannot be covered I mean I would not have time to write down the whole notes in every detail possible way. So, wherever some proofs are not been given in the class the proofs would be given for example, these proofs of these ideas are from Spivak, but here proof styles, but here I have done more detailing work which is not given in the book.

So, for the proof, we are just done. So, what I am trying to say is that I will give my TS whenever whichever proofs are not given whichever ideas are not introduced in the class in detail those things will come in the notes. So, they would not be coming. So, rapidly, but they would be in the notes they would be there for a long time and I am sure you can learn things even after the examination it is not that examines the end of each and everything.

So, here I will not give the proof because the proof would be not so easy to comprehend immediately, but I will send; I will get this proof and the proof of the other one done through I will write these proofs up and they will be scanned and put up and then you can really look into it very carefully and try to understand them because if I explain them; I have to explain them fast because of time constraints and then you lose the idea the idea is this here is the important result and that is with this we wrap up today's discussion.

So, if f is element of R a, b [FL] and x is a number between a and b. So, and say x in any number between a, b define f of x as integral a to x f x d x, then f is continuous on a, b of course, using this and this fact you can prove that integral a to a is nothing, but 0. So, f of f of a is 0 here and then it begins.

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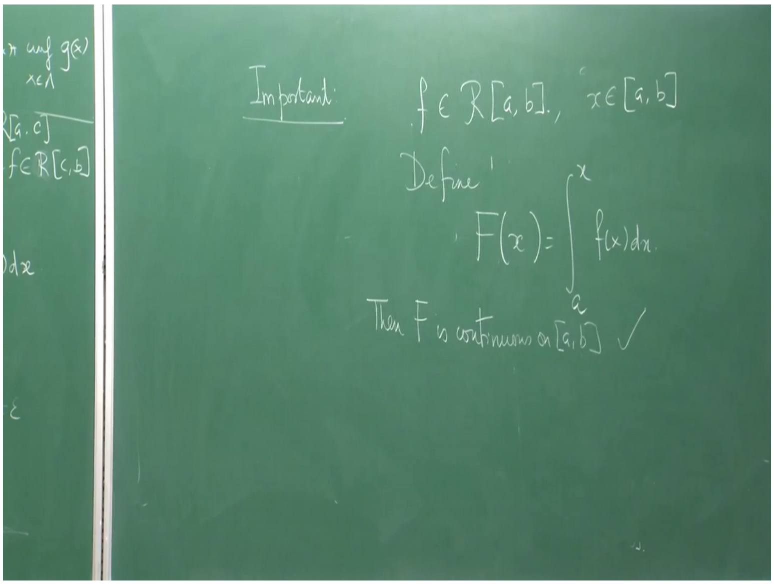

So, this is very important because function f need not be continuous, but the what you can call as the anti-derivative if you want to call it this one is actually a continuous function this is something very important to keep in mind; the proof will be given in actually in the notes.

So, with this we end the class today and tomorrow's class, we are first going to talk about the mean value theorem and its application and then we see that how the integral of Riemann can be that is very recent development in our 60s, I would say 30-40 years not much. So, we are trying to also push in some recent research ideas into the class. So, that you get a feel of how mathematics develops; how mathematics progresses because many of you have the idea that mathematics when it; when I learned it at the first year engineering level or first year level in any profession of course, mathematics ends there goodbye you just solve problems. So, mathematicians are people who are solving problems from calculus books or some books of linear algebra or some complex analysis books.

This is what is what one things about mathematics this is what one feels that mathematics is that is the general idea about mathematics in our community, but I want to bring in this lecture because I am evolving the course rather than following strictly to what I initially thought because I want to evolve it, because I want to show that mathematics is not a dead subject or calculus or integration all these ideas are not just dead which has stopped somewhere people are still evolving them and that is very important that this is evolution.

So, here I am trying to show you how this even Riemann's idea which was some one hundred plus years ago how that was evolved that is very important this evolution of ideas that mathematics is not something which has ended once we have given the first year calculus lectures you can pack up and so, mathematicians are usually useless bunch of people who were just given money to teach calculus that is the general idea and many people have actually told me this.

But that is really not mathematicians are there to dream and bring mathematics to a level which was unseen earlier. So, that is what I really want to show you. So, that is why I am putting in this lecture which is really not a part of a traditional calculus course; if I write a calculus book sometimes in my life; I do not know how I will ever do it, but I would definitely put a chapter on the hence talk integral hence talk hoods will integral which is also called the generalizing Riemann integral; so which would be part of the section on the fifth week of your studies.

Thank you very much.