Calculus of One Real Variable
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Lecture - 25
The Kurzweil-Henstock Integral(K-H Integral)

Mathematics is a story and like any good story it evolves. So, please take this lecture, as an entertainment lecture, I request the tears no questions should be asked from this part or the discussion because when you take a long journey sometimes you have to have some entertainment. So, these are entertainment you can forget your books etcetera everything.

After the Riemann integral became famous it was later gradually realized there were several functions which are not Riemann integral. In fact, Riemann's PhD supervisor give an example which is now known as the deletion a function which we have written f x is equal to 1 when x is rational is equal to 0 and x is a ration irrational. So, you are asking do you have a Riemann integral is Riemann integral between 0 to 1 answer is no of course, we have done that.

So, but that is that a deterrent in the sense that can all these functions which are now rejected by this Riemann integral theory as integrable are they integrable in some sense because we are no longer looking like Newton or Leibniz as the integral as an area we are thinking of integral as a number calculated in such a way as a limit of sums limit of some telescoping sums.

So, then in 1902; there was this famous thesis of Henri Levesque in from Paris.

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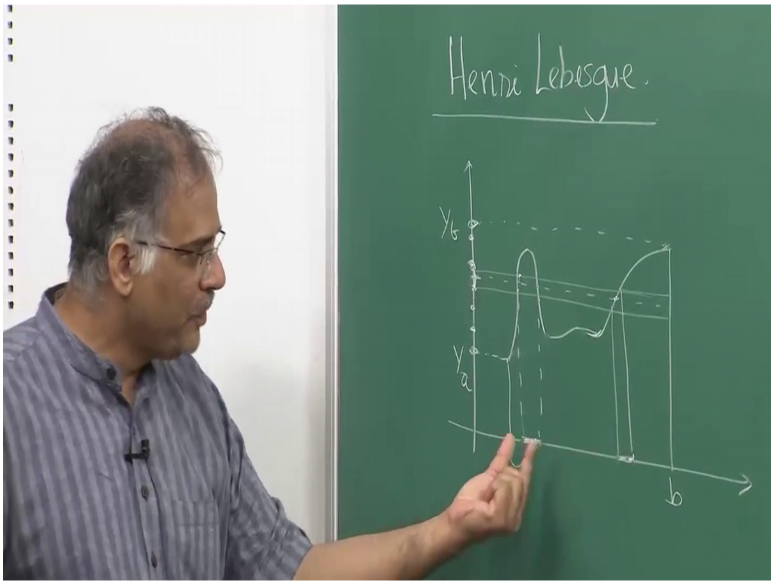

And which he did a very different thing he which he said what would happen if you have a function does not matter maybe bounded function whatever nice function from a to b; I that is instead of partitioning the is a bounded function. So, there is a range there is instead of partitioning the x axis, I will partition the range then instead of partitioning the domain partition the range.

So, let us see if I partition the range what would happen say its y 0, y a, y b, f a, b basically. So, maybe I will put some points here, but again if I have to calculate the area I have to construct the rectangles here. So, idea is this that let me see; if I take this partition or y what are the corresponding xs that are coming in here.

Suppose a function is like this then in this range a values in y this part of xs are also coming. So, we want to construct the rectangles now on the xs. So, corresponding to this you have to now you will now corresponding to this y parts, you will be taking their corresponding x parts which corresponds to these y partitions and then take the functions of functional value in this and then take the; now, here. So, you want you. So, what you will do? You will choose a functional value from here understand it; these are very interesting thing. So, you will these are the y values. So, you will choose a functional values from here you will choose a functional value from here. So, these values look at this point. So, look at this one these value corresponds to this is the point. So, it corresponds to this x this x and it corresponds to this x here.

So, the, but the partition corresponds to this part and this part. So, if you want to make the sum. Now you have to multiply the value of this function functional value this; take this particular value and then you have to multiply by this the some sort of a length of these 2 sets I will just rework it out.

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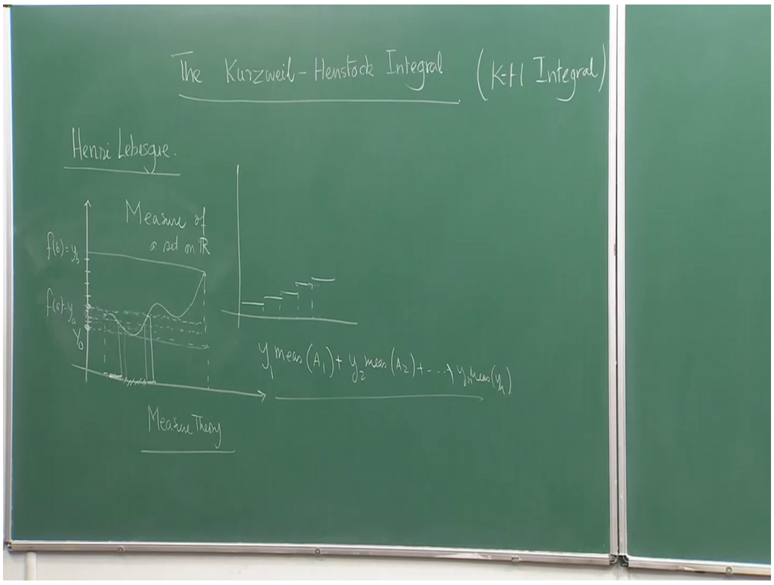

So, let us just see what Henri Levesque has done and how this a; nice idea came about of measure it could be for very bad looking functions also.

So, maybe I will just do it for nice looking function. So, that you do not have much. So, or maybe I will just do it. So, here is my range y a y b. So, this is f b equal to y b and f a equal to y a, but my range is not really y a range actually goes from a minimum point. So, this is a minimum value so, but this is my y a, right. So, these f a is y a.

So, this is my starting point of the range which is a my y 0 and stopping at y b and y a is in between. So, what happens? So, I now partition this to small parts now if I want to see where the excess of this partition lie this particular first partition. So, you see here this is one point this is one point all the corresponding to this x; the partition value is lying this point.

So, this is the corresponding excess on the x axis on which it is lying similarly if I take the next partition next partition point let me tuck tagged it let me just push it through. So, what happens here, where are; what are the values which are lying here the values which are lying this partition is this part and then they up to the end this range of x.

So, what this part of x function, values are lower lying within this second partition, but. So, I am partitioning the range I am not partitioning. So, this is my second one of range partition am partitioning the y axis. So, what the idea is; so, this is the partition of the y axis. So, now, I am seeing this is the x part for the first partition for the first partition, this is my x parts for the second partition this is my x part. So, they are not just one interval they are basically union or 2 intervals.

Now, I would; what I will do? How I will form the sum or the Riemann sum; if you want to call it. So, I will take some point here and I will take some point here; so, which are the tags on the partition. So, I will just see where this point lies. So, this point lies here. This is the x, it also lies here does not matter. So, then for this also for the first one I find I choose some y value which is a functional value and for the second also I choose y value which is a functional value and with those values; what I have to do? I have to multiply the length of the interval, but for this second case where is my interval I have to union of 2 intervals.

So, you can; so, why do not you write the f dash x i into the union of 2 intervals, but that is not my wish I my wish is to have a function value and into one interval so, but then what is that interval what is the difference between these 2 point x i minus x i minus one that is the length of interval. So, how do I call this union of these 2 things as interval it is essentially a set.

And now I do not know; what, how will I define length of this set will; I call this as the union of the 2 intervals is the length; shall I tell this length plus this length into the function value which would might make sense. So, essentially here because the function is nice and continuous you are getting such union of things, but if you have functions like this; now very bad staircase functions then things might look very bad when you partition the y axis.

So, I have to now force. So, I can get very strange kind of sets. So, for a given partition of y; so, if I take an interval of partition on the range the corresponding xs might form a set which is very; so, strange. So, there would not be any particular way to define its. So, then Riemann sorry, then Levesque thought the length is what; it is a measure some kind of a measure of the space occupied by the line.

So, can the next natural question. So, what I get is given a partition y is in the partition of the range. So, corresponding to y there are certain xs for which f x would be in that partition. So, I basically what are these I am finding the x s for which f x would be in this partition [FL]. So, what is happening is that those sets the set a in the interval a, b the subset of interval a, b for which f x is some in some partition.

So, I want to multiply a some the some value of take some f x value from here and multiply with the length of a say, but if the set is. So, strange set could be strange enough that you will not be able to find its length; it will not there will be no meaningful way to talk about its length. So, how will you talk about the length of such a set?

So, he started thinking about the general notion of measuring a set on a real line and that guild gave rise to the notion of measure to the notion of measure of a set on R and which finally, blossomed into a very important theory called measure theory. So, he found that once he did this, he could actually integrate functions which are not integral in the Riemann sense, but this idea of measure of sets became very important because there were in lot of cases we really have to measure.

When we come to 2 dimension; of course, measure of a set is the measure its area; if the set is nice looking, if it is not how do you think about it. So, then he said that based on the idea of the length of a line, he said that the measure must satisfy certain properties and he listed those properties which are the axioms of measure and any function which satisfies if when I when acted on a set satisfies those all those properties can be termed as a measure of that set.

And now what you are basically doing; now you what you do now; how the sum will be done function value into measure of the set for which for which for those x s for which f x would be in the same partition as this function value this y value. So, you will have some y dash from some y 1 from the first partition y 2 from the second partition multiplied with the measure of the set. So, that all the xs in that set has the function values in the partition in which y one is contained, right. So, that is the idea.

So, the sum in this case would be for y into measure of a one plus y 2 into measure of k 2 and so and so forth; up to Y n say. So, Y n is one element from this set, but measure a one is a set such that for all x; f x here; f x lies in the partition in which y one lies. So, so this whole game has now changed. So, he had to do the same sort of summing same sort of idea of rectangle building.

But coming from a very different perspective allow you to build new mathematics and that that gave rise to better theory and integral which was effective enough for many many things. So, it became the official integral. So, for Riemann got left out 31 still is still 31 is still troubling many mathematician lies with his Riemann hypothesis, but the question then came measure is a concept not. So, easily understood is a concept not. So, easily handle able is a concept which is very not so easy to digest.

Because we have some very fair notions of geometrical idea of measure of sets or objects which are already ingrained in our mind to think of measure as an abstract concept as a function which satisfies certain axioms or properties is something not so easy. So, as one might think many mathematicians who use better theory might say; oh, what is the problem; that is just like that.

You are again getting used to things in mathematics when you are used to things in mathematics; these things wont look problematic, but if you are not used to those things, but you want to look at them and try to see whether its easier than what you have read. For example, in the Riemann case major theory would be slightly hard not to crack compared to the way you have we you have been fed with Riemann stuff.

Coming from Newton do Riemann was not a very difficult thing, it was the same game done in slightly different way for any enlarge; the class of functions from continuous to bounded, but here he does something much more different at I enlarge the class of functions which is Riemann could not even handle, but to do that his tools very strongly deviated from which Riemann had taken it is not the standard path.

This question was addressed by these 2 mathematician one a polish mathematician and one a English mathematician Carswell I think the name will immediately tell you is polish and hence talking English mathematicians who in the 1960s spoke about ask this question what Riemann sis was an extension of Newton's idea which is a very clear idea what is the meaning of integral first as an area and then trying to extend it.

See we are if you have seen we have actually we are making a very nice historical development on the integration theory now the; so, they asks the questioned cant Riemann's approach be modified without getting into this measure theoretic approach see measure theoretic is the huge deviation from what you have known.

So, cant the Riemann's idea we modified see when you came from Newton to Riemann it did not bother you same partitioning the interval constructing sort of rectangles in their mind and instead of constructing a rectangles he is doing the same thing, but he is not telling it if the function value is non-negative is the same thing as what Newton was doing.

Newton did not understand much about bounded function, they did not bother they were only looking at continuous curves fine. So, so it is not a very difficult thing to accept fine Riemann was the genius I who we were are not he caught it we did not; that is all, but the case is very different when it comes to Levesque to think like this is not so easy because integration theory is linked with the notion of area that this whole notion of area and building rectangles is so ingrained into our mind; nobody will ever think that you can you can also partition the y axis the range.

And that is why to make this break from the traditional thinking Henry Levesque would always be a giant of 20th century mathematics; mathematics in general, but these 2 mathematician much less well known asked a very pertinent question you can say; these guys are just asking questions from a educational point of view not really because that is the pertinent questions that can my basic geometric intuition remain when I want to broaden the idea of the integral broaden the class of integration can my basic geometrical idea which Newton had propounded can that remain because that is the idea with which Riemann started.

So, the question is can Riemann's approach be modified and that is what these 2 people have done there are several books written and several mathematicians are trying to push this integral maybe one day it will also be one of the accepted official integrals in mathematical community along with Levesque integral. So, we are not going to discuss Levesque's integral because it needs tools which are not for this course.

What we are going to do? I am just going to show you just I have to in my own collection; I have 2 books I do not even ask you to go and read this books, but just for the namesake which books I have I had followed one is modern theory of integration by Robert Bartle; he is very famous, he has a very famous teacher and has a very good book on analysis mathematical analysis.

And this is the way it is a lecture notes called by Australian mathematical society lecture says number fourteen the integral and easy approach after Kurzweil and Henstock. So, we will just follow what is shown here, this would is slightly more technical for you, but we will try to follow what is in here and we will try to define the Henstock integral and then we will try to show that one integral which is not one function which is not Riemann integral is integral in this sense.

So, what he does? So, let us go back to the Riemann integral and rethink not in terms of upper sum and lower sum it is actually this darboux approach that we take that confuses us Newton did not take that darboux approach; he take that sum and then take the limit you Riemann did the same thing the darboux actually better little spoilsport he made this easy L P, f; U P, f and everything got jumbled. So, so they want to go back to that same limit taking business make the sum take the limit, right.

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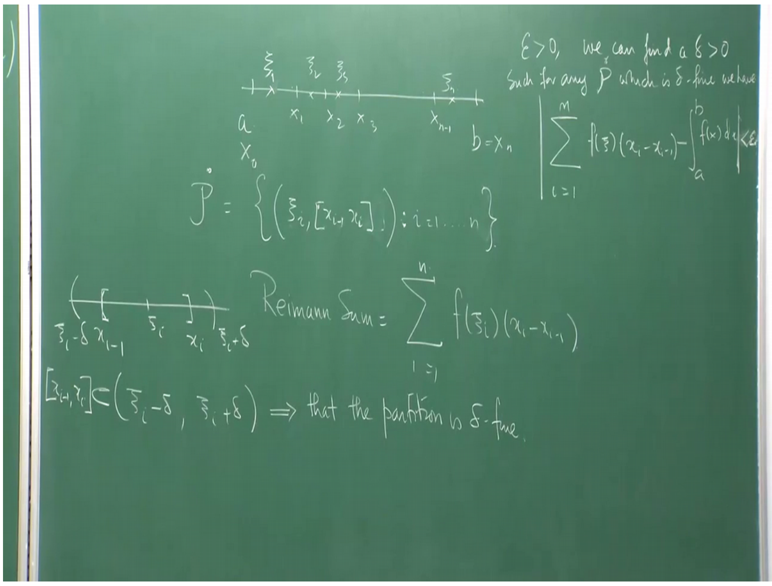

So, what you do in a. So, this is the partition P. So, if I write this as x one and this is: x 1, x 2, x 3, x n minus 1. So, I can start with the indexing right, I can start with x i 1 as one element I choose from here x i 2, I choose from here x i 3 I choose from here x i n, I choose from here and s on.

Once I do tag some points there tagging is something you know you know it very well in the age of Facebook, I do not have to tell you what its tagging your tagging photographs day in and day out. So, so when you tag a point here. So, this is called a tagged partition. So, tagged partition is usually given like this the tag partition always expressed in this form these are tag partitions; so, the partition element and the tag.

Now, what did Riemann do? Sorry sir, I made a mistake you are pointing x is x i minus 1 x I, right. So, this is a partition, this is a tagged partition. So, if P dot is the sign which many people use and he is also using which is standard; so, what does; what did Riemann actually do? What do you; what is the Riemann sum? The Riemann sums; now what you want to do you want to show that it is less than. So, there is a number which you can call an integral between a to b; if the difference between this and that number can be made as small as possible.

But how do you get to that number you got by definition or Riemann you get to that number by increasing the number of ns. So, once you increasing the number of ns that partition points the part length of the intervals becomes smaller and smaller. So, given a delta you can always have a partition. So, that each of the intervals have a length less than delta 2 delta, right. So, given an interval given in tagged partition, right like this was x i I, x I; x i minus 1.

So, I can have a delta such that you have here x i i plus delta and here you have x i i minus delta and x i i minus delta to x i i plus delta this actually contains this interval x i minus 1 x I; you can always do that once the points increase given a delta; you can increase your points in such a way that you can always get this if you can do that then such a partition is called a delta fine partition.

So, if you can always do that this would this would imply you that the partition is delta fine what is the meaning of partition with delta fine the largest length of the largest interval is less than 2 delta that is simply; this is a big name given to very simple thing the mathematicians are very capable of doing such things.

So, essentially you are controlling this. So, what is what is the Riemann integral Riemann integral simply means that given any epsilon greater than 0 given any integral; we can find a delta greater than 0 such that for any P dot on a tag partition which is delta;

fine, we have these are definition of the limit actually that limiting definition is retained in terms of this delta fine partition this is less than epsilon.

Now, this delta is now a fixed positive quantity what if I could control this delta what if I could change this delta for each every epsilon then I could take out some superfluous point which are not many of them, but at which the function values are peaking and allowing not allowing us to control the distance such is a case where the notion of Gaugeous very introduced by Henstock and Kurzweil very separately.

So, I will do it; I do not have time to actually give you an example of why you need to talk about Gaugeous.

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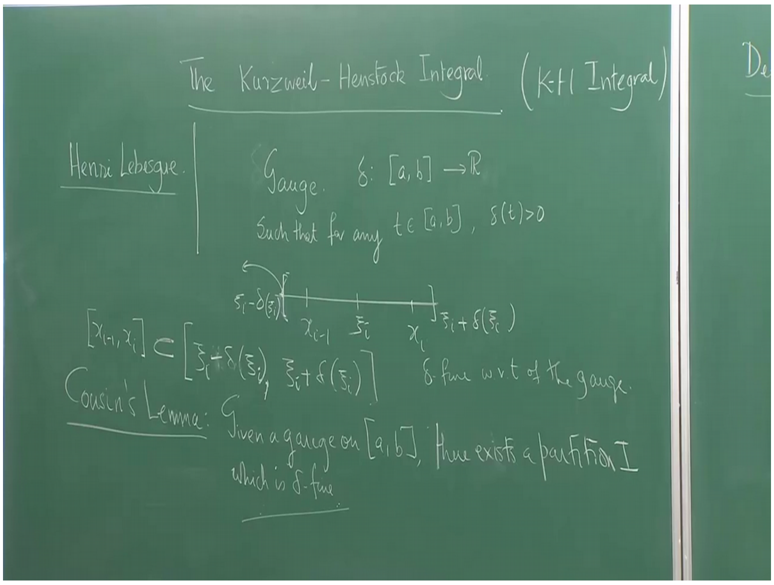

But I will show you when we do that do that particular example. So, they define what is called a gauge what is a gauge; gauge is nothing about a function from a, b to R plus such that for any.

This x i is always need not be exactly in the middle x i could be a x i could be this; x 1 could be also x i which could be the tag for both the partition both this one and this one it could be is x i 1 and x i 2 both that can also happen a 0 could be the tag also or a could be the tag. So, that for any t element of a, b actually I should write it directly R plus plus R or just write R into R such that for any element of a, b delta t is strictly bigger than 0.

So, for each tag point now I will have a separate delta here what is the meaning of delta fine a single delta should work for all the here; I am; I do not need a single delta to work for everything, I will need just a gauge, I will use for everything it will change; so, just a little bit of thought process. So, now, we will define in the same way. So, if you have an interval say x i minus 1 to x i and if you have a have a x i.

So, what is that interval which is controlled by the gauge? So, to gauge is that is that is controlling that interval is you can give it as open interval you can keep it as a closed interval does not matter it just its up to you these guys have taken it as open closed interval I sometimes prefer to put it as an open interval it does not matter. So, this is your control controlling interval right and it should be.

So, this intervals x i minus 1 x i should be inside it actually what it is telling that every interval would be controlled differently here every interval is the controlled in the same way there every interval would be controlled in a different way. So, this is a very key feature of the gauge which is not there with other things right so, but how do you know that if I give you a closed interval a, b right which is non degenerate a is not equal to b.

Whether such a gauge exists; whether you will be able to for every tag construct such an interval; so, for every tag that you have says x i i here. So, is x i i. So, every interval that you have which is tagged for every tag now the whole interval is tagged; we are only talking about tag partitions. So, given an interval a, b; how do you know that there will be a function such that for whatever be your tagged partition you can always use that that x i value and control the and so, that this interval will actually contain this interval basically this open interval is essentially opened this interval does not matter open or close is not a big issue.

So, this thing is that is there a gauge for which I can do it for any x i. So, we just now what is here delta is not just a fixed number the number can change, but the functional form is same. So, there is a function that functional form is same for every tag that you use, but only for every tag the x i the del x i value changes.

The can there be a tag given a little close non compact interval this closed and bounded interval a closed interval is there a tag when; sorry, is there a gauge which through which I can do this thing this controlling of the intervals the answer is yes and that is done given by Belgian mathematician called cousin they would call cuisine, but we Indians know this spelling and we will call it cousin it is called Cousin's lemma.

Actually using Cousin's lemma you can prove other things you can prove even in many-many other issues like extreme or many you can prove that a function bounded on interval as lower bound upper bound all those things can be prove using this. So, I here if I have an interval I and have a gauge for which for every inter; every sub interval of the partition of the tag partition; I have this feature then I call that sort of partition is that there this is delta fine with respect to the gauge.

So, now Cousin's lemma said that if I define a function delta on if I define a function delta on the interval a, b; the delta is the function from this which satisfies this, there is delta is a function from a, b to R plus plus actually, then there is a partition I of a, b which is delta fine they will always exist a partition I which is delta fine. So, actually my earlier statement should be I made a mistake; should be written when a given interval a, b or delta fine gauge exists.

When a gauge exists which is delta fine a gauge exists which makes an inter which makes a partition delta fine. So, given a gauge on a, b there exists a partition I; partition I which is delta, fine. So, here delta is no longer a number it is a function delta, fine with respect to the gauge now how are you want to define. So, basically you have to show the existence of a partition which is delta, fine.

So, given a gauge; there is always a delta fine partition; obviously, the next question would be which would be natural that of course, you can ask that given any partition I the question is there a gauge with respect to which it is delta fine that is a question that is if I have a partition or tag partition is there a gauge which I can define a gauge with respect to which is a which is delta, fine, it is obvious because you can always define a constant gauge.

But given a gauge which is not constant the more relevant question which Cousin's lemma answer is that can you have a partition which is delta fine with respect to this gauge not a number, right. So, now, we are going to define the K-H integral for those who want I can just show you the pictures of those 2 mathematicians it is from the book maybe you would understand something maybe I can just put forward I do not know whether how much visible it is to you, but maybe you can they can zoom in the camera and they can show you something.

So, here are the 2 mathematician here is the English one Henstock you can understand and here is the polish one Kurzweil; so Henstock and Kurzweil. So, here we define the integration and then we give an example and finish our discussion. So, our major discussion of integration is over next are just applications.

We show some techniques of how to integrate some ideas of improper integral some other applications calculating volumes areas a surface area and all those things and then into other issues of analysis.

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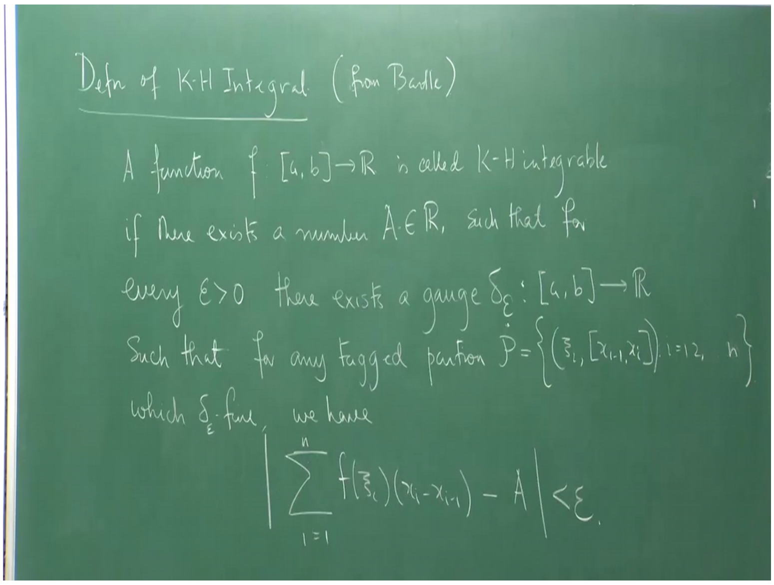

So, here I would just write down the definition from Bartle. So, you see; now I have extended the idea of the Riemann integral just by taking his approach slightly modifying it, but it is not very hard to understand.

So, the definition is from Bartle's modern theory of integration a function f say from a, b to R is called K-H integrable or generalized Riemann integrable is called K-H integrable; if there exists a number a element of R such that for every epsilon greater than 0, there exists a gauge delta e.

So, this gauge would depend on the choice of your epsilon. So, that is something very important such that for any tagged partition. So, there will be a; we are say delta fine tagged partition by Cousin's lemma. So, there can be infinite delta fact fine tag partition there will be actually. So, if you one to one you can do the other.

So, for any tagged partition P tilde P dot sorry neither is a. So, that for any tag partition P which is delta, fine. So, delta fine essentially means this for every interval this should happen. So, the length of interval should be less than 2 twice of delta. So, which is delta epsilon fine we have summation I is equal to 1 to n say; so, n here if x i i x i minus x i minus 1 minus a.

So, this is the definition there are some mistake here which is minus. So, basically you have here these 2 endpoints. Thank you. So, this is x i i plus delta epsilon I and this is x i i this one is x i i minus delta epsilon I. So, those who have copied it as plus; obviously, did not make sense because it that this would be of length 0.

So, this is obvious typographical error writing error basically. So, this interval should contain this interval. So, though length of this interval should be less than twice of delta epsilon I; so, that is the meaning of this. So, there is a definition. So, we recast the same definition of Riemann now with this. So, we are there is an upper sum lower sum business I cannot think of any upper sum lower sum description of this thing.

Because that will again take you or take away the delta fineness that control that you want to have that control will go maybe there is you can actually divide some upper sum lower sum theory, but I do not know; I maybe I just want to think about it, but it is not really required. So, now, what I am going to do is to go give you this example and end.

The famous Dinchilet as an example which Riemann could not prove that is Riemann integral it is not. So, sure is an one which it that is integral now in terms of this K-H integral.

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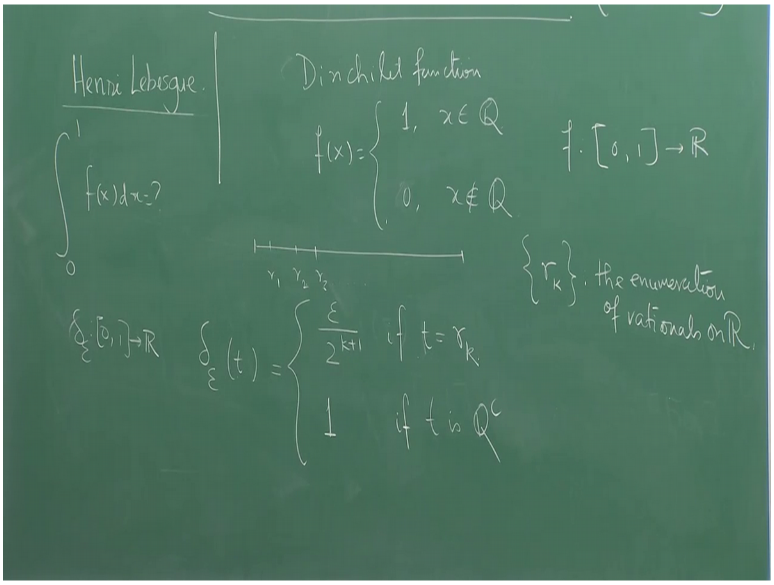

So, it took the polish and English brain to outwit the German at least in this integration business. So, here is this Dinchilet function and x is rational 0 when x is not rational what if x is rational; obviously. So, what do you do you take a partition, right.

Now, in the interval 0 1; so f in this case is defined from 0 1 to R. So, now, on the interval 0 1; so, we are trying to see what is the meaning of f x d x in the Kurzweil sense; now in the interval 0 1; you know that there are rational numbers are countable, you can line them; line the rational numbers up as R 1, R 2, R 3, R 4, R 5; whatever.

So, when you tag. So, in your in your in your taggings. So, some cases are tag would be rational number some case tag would be irrational number. So, upper one can say why you do not I just take a partition in such a way and that I will only tag the rational numbers every partition will have a rational numbers why only a finite number of.

So, I only tag the rational numbers that is also one way, but a more general way is that you have a partition you just tag arbitrary it could be rational it could be irrational [FL]. So, so you have this the rational numbers are there. So, you they could be enumerated, right. So, R-K is the enumeration of rational numbers this is the enumeration of rational numbers f rationals.

So, you know that this is not Riemann integrable. So, now, you define a gauge functional on 0, 1, 2 or in this way see you know whenever; there will be when you tag a irrational the functional value would be there 0. So, that interval is useless whether you control it you do not control; it is of no meaning only the rational one has to be controlled.

So, you define this as the following. So, x i by 2 k plus 1 if t is equal to R K. So, t will be some H, nah; t will be some rational number which is R 1, R 2 could be R 1 could be R 3 or could be R 4, right. So, you go on like this. So, you start from 0 t d 0, d 1 and then they go on. So, whenever you get a rational number if it is R 1. So, it is 1 by 2 square basically g P series nothing else you need a one words in somewhere.

So, and for the irrational case; it does not matter what you put these just can put a constant because you do not need to control it; it will be 0 it left no effect on the sum. So, put one if t is irrational t is in the complement of q which is a irrational; I hoping that this definition is already done.

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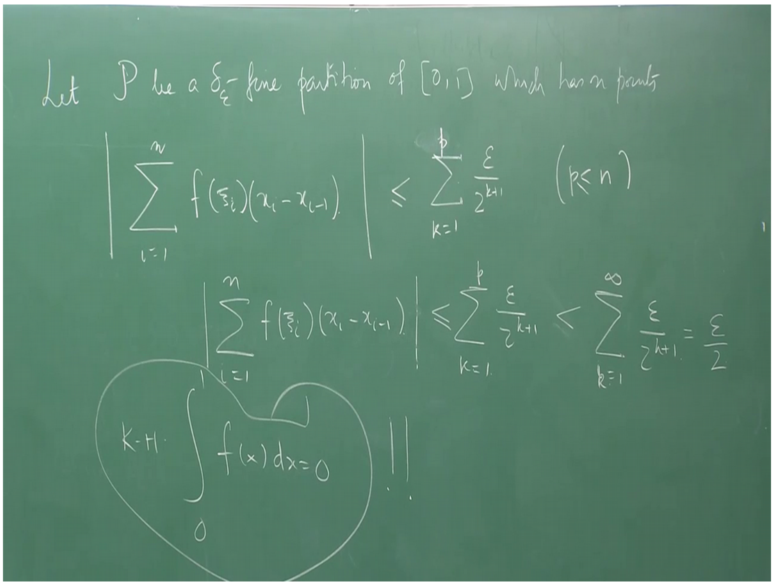

So, we can now actually work this out. So, let us look at the sum there are I equal to 1 to n.

Now, let us consider any. So, there must be a partition which is delta, fine. So, let us take a partition of n points which is delta fine. Now let P be a delta, fine delta epsilon, fine partition of 0 1 of 0 1 which has n points does not matter. So, let us look at this thing, right; let us look at this.

So, only contribution is coming from rationals and that would be less than equal to summation; whatever be the number of rationals I face. So, it will be less equal to K equal to infinity to the cardinality of rationals or I need not write say I have face say I could face say P number of rationals out of n tagging say P; P less than n, right, P strictly less than P number of rationals P could be equal to n could be less than n; this I will definitely have right this sure.

Student: End point.

Endpoint; now first part thing is given x i 1; x i 2 x i n nah first x 0 to x one the tag is x i one. So, last one the tag is x i n. So, no I am telling it could be n P is less than n some that some tag could be irrational. So, irrational things do not matter rational things matter, right. So, so why did not I take rationals at the very beginning I might not have a partition which is tagged all by rationals which is delta, fine.

So, if there is a delta fine partition which has no rationals nah is obvious 0, finished if it is tagged by some rational it is still less than this where P is strictly less than n, but this is again less than. So, what you will have. So, I equal to 1 to n f x i i x i minus x i minus 1 x i by 2 k plus 1; this is strictly less than. So, k equal to one to infinity x i by.

Now, if you go by this thing this will become one in a g, P series; so, k equal to 1 to whatever. So, this will become. So, this will be equal to epsilon. So, half right epsilon by 2 or we starts with one right half plus half into 1 plus that would be 1. So, it is half; so, its epsilon by 2 that is my chosen epsilon.

So, what is my number here it is 0 because this is this minus 0. So, any delta any delta fine partition any partition which is delta fine that is controlled by this delta and you can show this. So, it implies now the in the; this sense and that is it job done.

So, I hope you enjoyed this entertainment lecture where we successfully integrated something which was not Riemann integrable without using the measure theoretic approach, but have using the simple extension of Riemann's idea.

Thank you very much.