Calculus of One Real Variable

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Lecture - 27
Improper Integrals – I

So, welcome back to our lecture of the sixth week, I think this second lecture, but we are going to speak about what is called improper integrals. Improper integrals are a very important class of integrals, because here we will speak about integrating over unbounded intervals. Whenever we have spoken about integral of from a to b f x dx a and b is a closed and bounded interval, which is not true.

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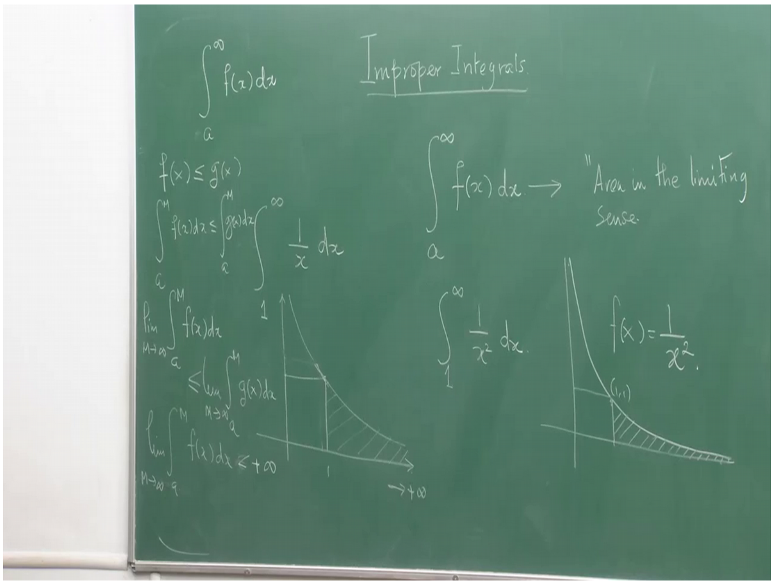

And we ask ourselves that what is meaning of this question. Now, for example, I can ask you, is there any meaning to this question of area under a curve from the area under the curve point of view, it might look very strange because for example, I am talking about an integral of this form. So, here you look at this parabola. So, here is one, this is one, so yeah this all right. So, this is 1, 1 this point. Now, we are trying to understand whether this area makes sense.

So, you might be saying oh come on this area is going continuously towards the infinite how can you have a finite area. So, does this has any meaning or not? So, here is one and it goes towards the infinity, but I can say ok let me think about little bit more slightly different problem. Of course, I am not putting x is equal to 0, then this function has no meaning. So, this function would look slightly different then starting from one, this is 1, 1, this point 1, 1.

So, does this area has a meaning? Here it does not look that you really have a finite area here it might be that you will have a finite area, but you might say no, no; however, hard you try this one by x square will continue to come nearer and nearer to x-axis, but it will never go and touch the x-axis. So, how can you say it will area would be finite.

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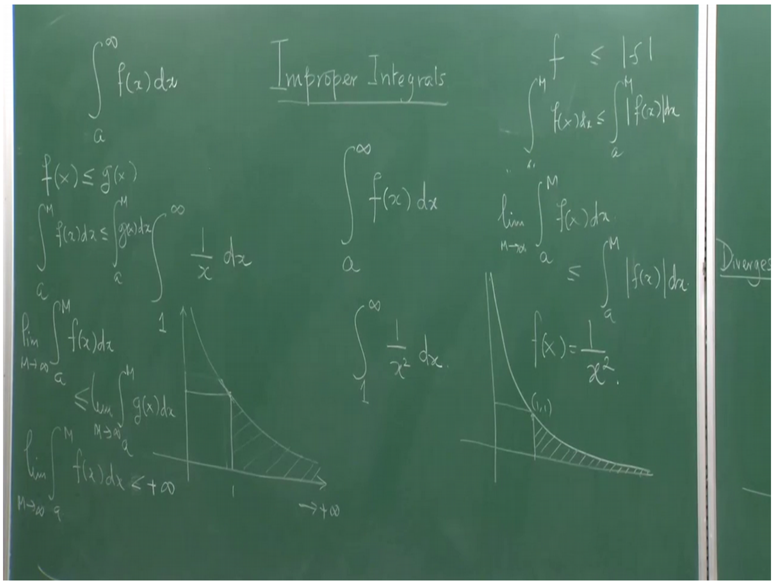

So, let us try to define this integral within the limits of what we know. So, if we want to do that we only know about finite stuff, so maybe it is not a bad idea. You can try it out that you take the following definition. So, what do you do and this is what approximation is all about some sort of an when you think of that immediately you say I know about the bounded stuff. So, I will say let me try a to m, f x dx where m is some finite number. And then I keep on increasing the value of m, the more I increase the value of m, so if I take m, this integral is not equal to this integral, because here I have a finite area, area might or might not have finite area. So, as I keep on making the m bigger, I come more near and near to the value of the integral if there is some value.

So, we define this integral to be in this following as limit m tends to infinity integral a to m fx dx. So, you calculate for every m and then find the limit. If this limit exists then that particular value would be called the value of the improper integral. So, let us have this two tests one for one to infinity 1 by x dx; and another is one to infinity 1 by x square dx. So, let us first for this case compute. Let us look into an integral of this form 1 to m, 1 by x dx. And then so this integral would be nothing but logarithm of M or ln M the natural logarithm log to base c actually minus log e of 1 and log e of 1 is 0.

So, basically this if I at all have to define then this should mean this, and you can say oh come on guy you just draw the graph of log, graph of the log function is like this at 1 it is 0; and then it keeps on growing up and up and up and up. So, what is in there. So, what have you done there. So, you are showing at this is actually plus infinity, the limit finitely does not exist. So, this integral does not give us an area, the under this function when on the limits are unbounded, one of the limits is unbounded one side does not give me a finite area which looks quite natural by the diagram.

So, do we say that this thing diverges, this integral diverges, this is a technical term. You may not be so worried about this technical term that this is diverging, and this is doing that. So, let us try out this integral. This might not be quite knew to anybody who has been just out of high school because in high school you do not learn all these things. So, here we are obviously doing things very differently. So, you need to look into that aspect.

Now, what is the issue at hand is the following. So, let me do the same calculation here. So, I take an integral 1 to M 1 by x square dx, which is integral 1 to M x to the power minus 2 dx which going by our standard formulas x minus 2 plus 1 minus 2 plus 1 1 to M. So, this is nothing but what, this is nothing but, so first you put, so this become x 2 this becomes minus 1 this becomes minus 1. So, basically it becomes minus 1 by x 1 to M. So, essentially this is now becoming 1 minus 1 by M.

So, what is the limit? Now, if I want to define let me limit of 1 minus 1 by M, this limit as M tends to infinity what is that, so that limit is one because M because bigger and bigger, this goes to 0; and because m is going to infinity from the from the right side. So, this goes to 0, while we have one here which is we have M, so that is limit. So, it is amazing.

So, we essentially telling you and in some limiting sense in quotes in some limiting sense the area under the curve 1 by x square area under the curve f x equal to 1 by x square actually right area under the curve f x equal to x is 1 by x square actually gives us an area. From 1 to infinity you want to calculate the area under the curve it has an area and which is 1, you cannot see it has an area in the sense of the area that we know, but an area in the limiting sense. So, you have to understand here this simply means area in the limiting sense. You must understand that you might not always have this a to infinity kind of thing right. So, you need not always have this kind of thing.

So, how do you show that an integral is convergent means finite. So, if you have say one integral a to infinity f x dx, how are you going to show that it is finite. One of the good ideas is the following, suppose f is such that over that range if x is less than equal to g x then you know that integral a to M for any M f x dx, it less than equal to integral a to M g x dx. So, once that is done, what is the aim, the aim is the following

Now, you take the limit on both sides provided the limit exists suppose the limit exists here this is finite that this in this convergent. So, this integral is actually called a convergent integral and a to M f x dx is less than equal to limit M tends to infinity a to M g x dx, then what is happened, what happens here. If this is finite say this is finite, so this is strictly less than infinity. So, then limit of integral a to M f x dx is less as m tends to infinity is also strictly less than infinity So, if this is a finite quantity, and hence it is also convergent.

Let me see if there is any way to show that any such function where we can do such a thing. So, let us now show you an example of how we can actually use what we have just discussed. For example, if you take the integrals sin x by x square, and you want to integrate it from 1 to infinity, see this is not the easy integral, I do not think this is an elementary integral, in the sense that I do not think there is a function whose derivative would be this thing is not so easy.

So, when you have here we had functions which has an indefinite integral this function, but here I do not think you have an indefinite integral, it looks quite complicated to me. So, what you do maybe there is it may be a I can just figure it out immediately what looks the look has makes you feel it does not. So, what happens is the following that here you now look into the fact that sin of x is always less than equal to 1, it does not matter it is always less than equal to 1; it is less than equal to 1 by x square. So, 1 to infinity sin x by x square dx is less than equal to 1 to infinity 1 by x square dx. Of course, I am just writing I am not I just skipped the limit part there separate because I know that this limit exists this is one. So, what we can say that this integral, this integral is less than equal to 1, and hence it is a finite valued stuff.

So, now, please understand that no idea cannot be you cannot have an area integral if you take area under integral it cannot be minus infinity. So, such things do not happen. So, please be careful about that. You might ask the question it might be point minus infinity still it is less than one, but no there is nothing called minus infinity because you are computing the area you can compute a small area from 1 to M also. So, basically you cannot have you draw the graph of sin is where x square, you will immediately realize that you cannot even if you take a little chunk you cannot have any you will have an area under the curve, but you will not have a area which is minus infinity they are going to be negative areas. So, areas are any always greater than equal to 0.

So, not of greater than equal to 0, in the sense that if I take the modulus of sin x for example, if I take the modulus of sin x then we are talking about non negative function; so, we are talking about areas. So, this is always greater than equal to 0, this mod of sin x is always less than 1, the same store you go through. So, this is one way of doing of course, the thing would be minus infinity to a and you can write down the same stuff and you do not really have to bother. I would like to say that there is a very important concept here it is called absolute convergence.

What is the meaning of this absolute convergence, you know this f is less than equal to mod of f in f of x for any x. So, if we integrate out say from a to M f x dx, so for any proper integral it will be assumed that is integrable. So, see all these functions f that you assume the Riemann integrable in over an interval, now here it would be yeah. So, now if this integral is finite I mean this once I take the limit if this integral becomes finite then we say that the original problem is also convergent and original problem is also finite. So, if you take the limit of a to M, M tends to infinity, so there are functions which are convergent and but not absolutely convergent. So, if this is finite, so this is strictly less than plus infinity this part, then f would also be finite.

So, when this happens. So, whenever I can integrate I can get the improper integral of mod of f x, then I will immediately show that integral of then integral of models will also exist. And hence it will be what is that it is a the then we call that the integral when f the integral f is actually absolutely integral a to b, a to infinity f x dx is an absolutely convergent integral.

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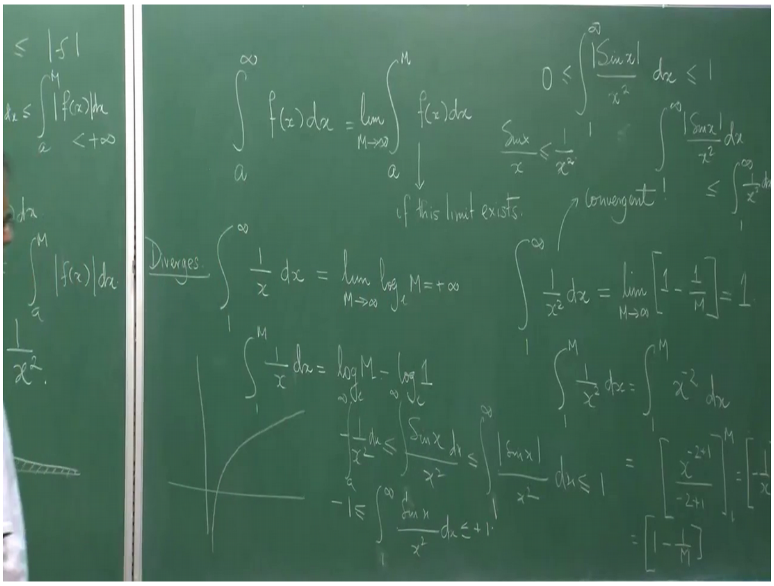

So, here is an example you can simply take this example for example, your sin x by x square is less than equal to mod sin x by x square. So, in fact, if you look at it the sin x is also bounded by minus 1 by x square. So, if you integrate it, a to infinity, if we integrate it a to infinity dx. So, this is anyway you have just proved to be less than 1. And here I can again integrate this side also, a to infinity dx, here we will immediately prove that will be minus 1. So, the integral here is lying between sorry 1 to infinity. So, 1 to infinity sin x by x square dx is lying between plus 1 and minus 1. So, this integral is called absolutely convergent. So, this whole integral is actually called not a to infinity suppose if this exists.

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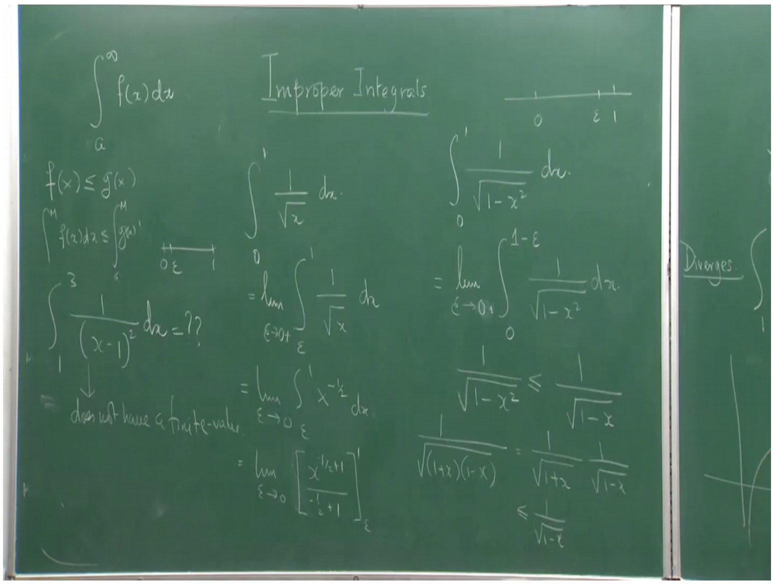

Integral a to infinity mod f x dx exists then integral a to infinity f x dx is called absolutely convergent. So, this is one aspect of it. Now, you know you can have things in slightly different way, your integral could be very strange integral could be within two within a closed and bounded interval. But at one of the points or some point in between in the integration range the function blows up or is not properly it is not defined in that case what would be how would you define integrate, this improper integral

So, for example, let me take this example from the book by Samuel Gold Work on Real analysis, I do not think that you really have to bother much about the book because this anyway a very common example of such situations. That if you have a situation like 0 to 1 1 by root x dx. Of course root x is differentiable and defined at 0, but 1 by root x is not. So, it will become meaningless thing if 1 by 0, if you are looking at the left boundary lower boundary, but in between 1 by root x is absolutely properly defined. So, how do you handle this situation?

So, again we have to bring in that simple idea that let me try one thing I increase the value of epsilon because I know that within 0 to 1, except zero nothing for nothing of for all other point of function is well defined. So, if I take some epsilon positive and then I try to evaluate this and naturally I would expect that if this limit as epsilon goes to 0 from the positive side 0 plus this let me do exist then I would be able to call that limit value as a value of this integral. So, here everything is done in the sense of a limit you see. So, the whole of calculus is essentially based on the concept of limits, so that is something you have to get into your mind very, very clearly, all these things are derivative integrals are all names given to special kind of limits.

So, limit epsilon tends to 0, epsilon to 1 x to the power minus half dx, I am actually doing the computation. So, limit epsilon tends to 0. So, it is epsilon to no. So, it is x to the power minus half plus 1 by minus half plus 1, so its limit epsilon tends to 0 from epsilon to 1. So, you will have here x to the power of half root x of 2 root x. So, 2 root x 2 root x epsilon to 1. So, limit epsilon tends to 0, 2 minus 2 into root epsilon. So, if sum tends to 0, the answer is 2. So, you see you can give a finite number to it, and we call two as the value of this integral.

Now this is always not the case that for such situations where you do not have this infinity business as a bounded interval, you will immediately come and hit into nice things. For example, if you take you could let me just check the example from Goldwork; it is for example, this integral x minus 1 square. So, if you look at this integral 1 to 3 1 by x minus 1 whole square what about this there x equal to 1 the thing does not work. So, whether this is a convergent integral or divergent integral means whether this exists or it does not exist. So, whether these are the finite value or these do not have a finite value, so that is something we have to look into.

So, this I can leave as a homework and tell you this one does not have a finite value. So, there are certain issues. For example, so there are certain integrals, which are convergent, but not absolutely convergent that is their modulus is not convergent, but the integrals themselves are actually finite such intervals are called conditionally convergent, but we will not discuss about them at all here right, this is not the situation where we need to discuss them.

So, sometimes the limit can be on the opposite side for example, where you have on the upper part there is a problem, on the upper end not on the lower end. Now, what about this integral? So, you understand at x equal to 1, it blows up. So, the idea possibly would be to first calculate out 0 to 1 minus epsilon and then compute the limit of this. So, if this limit is finite then of course now you here epsilon comes from 0 minus because here you have 0,1; and you have dropped by an amount epsilon because you know between 0 and epsilon the function is well defined.

So, 1 minus epsilon has to go to 0, 1 minus epsilon has to go to 1, so epsilon approaches one from the right side actually, so epsilon approaches 0. So, you are taking a positive quantity off right one epsilon sorry epsilon goes to zero plus sorry, so 1 minus epsilon. So, 1 minus when that is epsilon goes to 1 from the negative side, so you can also write epsilon going to 1 minus right. So, basically we have to compute this integral, but it is not so easy to compute this integral that I can assure you. Why, because you need to know that always it is not possible this is the kind of integral which comes up often in physics in the study of vibrations for example, pendulum motion it is called these are all elliptic integrals and it is not so easy to compute them.

So, what we can do is to first show that it is absolutely convergent. Basically, what we do is that we look into this. First I would ask you to check whether this is correct is this correct, is this correct, not correct? When x is lying between 0, 1, 1, this is not correct. Why do you think so, because I can write this thing as 1 by root over 1 plus x into 1 minus x. So, it is equal to 1 by root over 1 plus x into root over 1 minus x, but these are quantity bigger than 1. So, it is root is also bigger than 1. So, 1 by that is a fraction. So, it is fraction of this quantity and hence this is less than 1 by 1 minus x.

Now, if I can now evaluate that this integral from 0 to 1 to infinity and then do something then you get the result. So, I just ask you to show that from zero to one or zero to one minus epsilon 1 by root 1 minus x dx is actually 2 minus 2 root epsilon. Let us check this calculation out and then if you take epsilon limit where it goes to zero limit as epsilon goes to 0, what are you going to get this will become 2. So, essentially you know knew that 0 to 1, because we can now again integrate both sides and take the limit on both sides 1 by root over 1 minus x square dx is less than equal to half or less than equal to 2. So, it is a finite integral, it does converge. Of course, this is naturally bigger than zero these are non-negative functions, so this bigger than zero, so put the bound, so it is finite.

So, there are some other issues which we will discuss in the next class and give you some interesting examples like gamma function and beta function which are part of this course on integrals because they are very important class of integrals which arise in physics in engineering repeatedly in statistics for example, gamma function plays a major role. So, we would look into some such functions and also we will tell you how to compute some integral using some of you these sort of integrals using heuristic techniques, but for the time being, we are pretty fine with this basic idea. So, we will start with some basic idea tomorrow for the Cauchy principle value in the next class and then we will go to talk over gamma and gamma integrals and beta integrals, some interesting class of integrals and some nice formulas I will write down for you just for you to have fun. But I hope you have an overall understanding of what is what we had tried to explain today.

Thank you very much.