Calculus of One Real Variable
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Lecture – 28
Improper Integrals – II

So, we will continue a bit into a discussion of the improper integral where we have left. We will tell you some about some more concepts. For example, just now we are going to discuss what is called the Cauchy principle value; this due to the mathematician Cauchy; a French mathematician.

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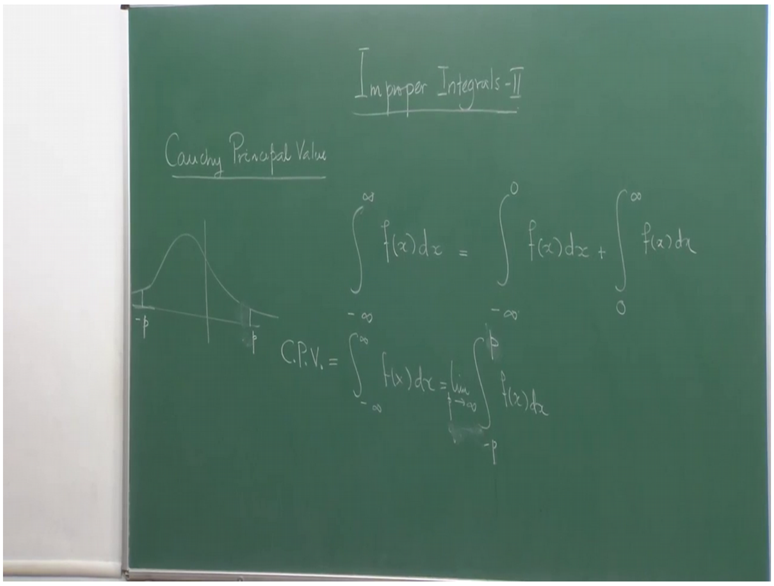

So, the modern version of calculus with that we see is largely due to Augustin Louie Cauchy. And here a very typical way of looking at an integral which is of the form like this.

So, what do you mean by the convergence of this integral, how do you define? That is again a question right. So, somebody would say I know about how to handle 0 to infinity or a to infinity and so minus infinity to a 0 a is 0 standardized for example. So, maybe I would say let me think that I can write the integral if it is at all convergent into 2 parts.

So, each of them are improper integrals and if I can sips, sum them up and if I can show that each of them has a finite value then the integral must have a finite value. That would be the basic intuition. So, what would happen, but Cauchy said that why cannot we look at it in a different way one. So, why do not I now replace minus infinity and plus infinity and integrate by the same integrate this function. That is, you say there is a function defined like this and goes to minus infinity and when up to plus infinity and minus infinity.

So, here you take some minus p and here you take some q. So, basically you have to then write infinity to minus infinity fx dx. So, is this a nice way to do it that is; what was cautious question, minus q to p where limit p goes to infinity and q also goes to infinity?

So, here you are taking limit of 2 the 2 ends right. Sorry every minus p to q. So, when as p to q; so, here we are going to having 2 limits. So, basically when you write down the integral both sides you are taking to infinity. Whether this makes sense or does this make sense that is the question. Because what Cauchy said taking 2 limits can be very pathetic. So, we can more or less have a symmetry and say we will write it like this instead of q will take p here. And we will put here as p and just take p going to infinity.

Now, if this value exists, then we say that the integral exists in the Cauchy principle value sense or cpv. A function in existing in the Cauchy principle value sense does not may not exist in this sense. So, Cauchy principle value sense is a very different way of looking at the integral. So, basically when you have minus into plus infinity then all integrals of this form must exist. It does not matter whether you have put it 0 here or some number a it does not matter.

So, ultimately these integrals should just exist. For what can happen that this integral exists minus p 2 plus p, but when you separate them up and come from one into 0 and 0 to other end it does not simply exist. An example which is given in Goldberg, Goldberg’s book is very it is one of the most elegant books on a mathematical analysis.

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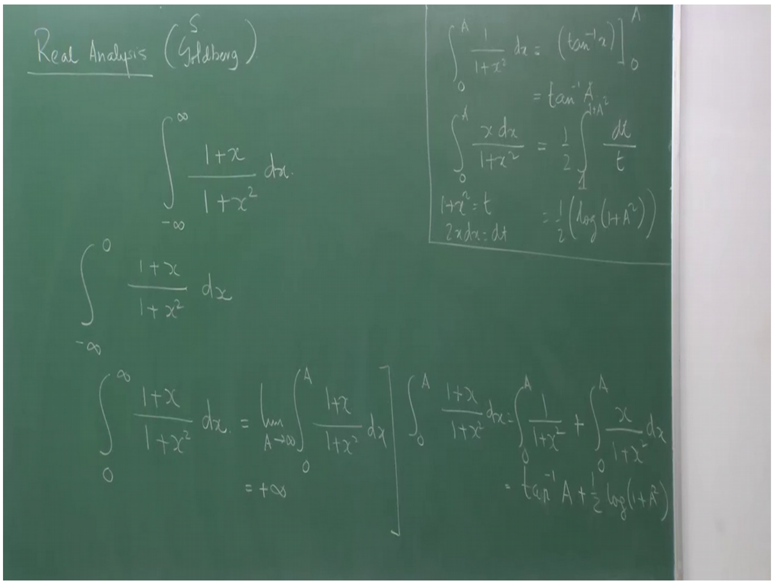

It is called real analysis by Goldberg, but you may not bother to read this at this moment.

So, what I am doing is as we Samuel Goldberg as Goldberg. So, as we move gradually up the ladder, we are going from calculus into more in deathly analyzing the issues. So, we are getting gradually into the domain of real analysis. It does not mean that we are not going to do computations etcetera. We will do it for example, if you look into this integral now you separate them. So, what they look at these 2 parts individually. Does it exist finitely and so if you walk them out in the standard way that taking here taking limit a tends to infinity 0 to a 1 plus x by 1 plus x ax squared dx, then you will quickly realize that this will really not work because I can actually for example, if you write this as limit of a tends to infinity 0 to a and separately work this out.

So, what would this integral give me? So, what would this integral give me? The first one is 0 to a 1 by 1 plus x square dx this element everything to know at this is nothing, but tan inverse x and that we are evaluating from 0 to a. So, it is tan inverse a.

Next one; so, you substitute 1 plus x square as t right. So, when you substitute 1 plus x square as t you will have 2 x dx is equal to dt. So, this would be equal to half of x sorry dot by t and the limit would be from when x is 0 the limit is 1. And x is a the limit is 1 plus a square so this will be nothing, but half log of 1 plus a square of course, is defined log 1 is 0.

So, now this simply means that I have here tan inverse of a plus half log of 1 plus a square so as a goes to infinity if I take the limit of this as a goes to infinity 1 plus a square is also going larger and larger on the log value is going towards the infinity. Tan inverse a as you if you draw the graph of the tangent and if you look at tan inverse a opposite. If tan inverse a goes to infinity a has y goes to infinity basically (Refer Time: 09:56) a goes to infinity tan inverse a also goes to infinity. So, these 2 are only be becoming infinite.

So, this limit is plus infinity. So, if one of them does not exist the other obvious here if you do not even calculate does not have any problem. So, you just calculate one of them. Now interestingly enough let me try to calculate this one. So, now, what I will do, I calculate the Cauchy principle value.

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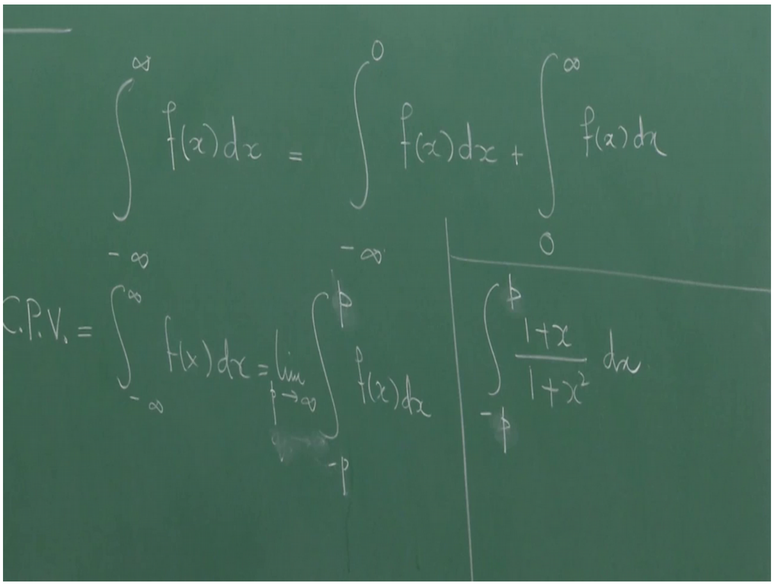

So, now I will have say minus a to a or minus p to p, if you want same formation minus p to p 1 plus x divided by 1 plus x square. How do I calculate this basically?

Now, what I am supposed to do here and this is Cauchy is idea that, suppose if I you can calculate this. So, one might. So, here you might just go into a problem because tan inverse pp you take to infinity and here you will have the same thing p square and then there we are going something log of, yeah you will have tan inverse for one party left tan inverse p minus tan inverse minus we saw that thing might just get complicated.

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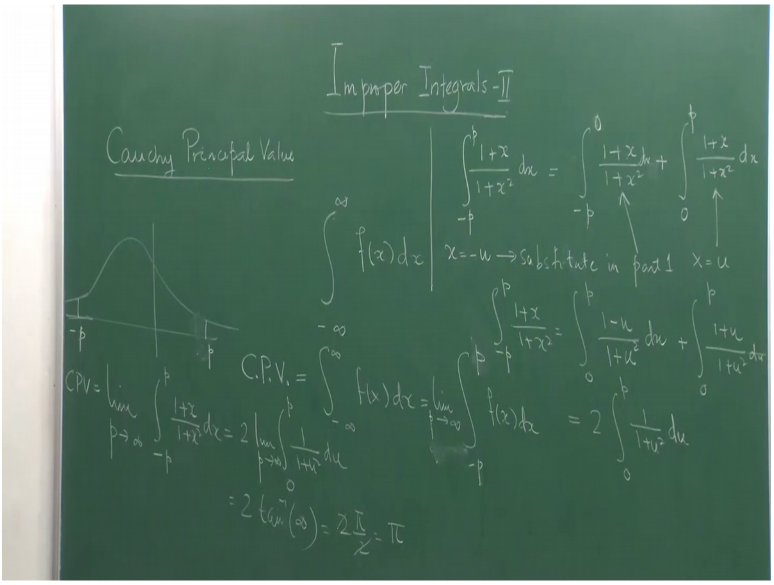

So, let us look at look at this situation in a slightly different way and let us see what we can do with this. What we do is take this function, this one this. So, instead of minus p 2 plus p. So, minus p 2 plus p we will make it slight slightly simpler in the calculation. So, I will write minus p to some s. So, a minus p to 0.

So, what I am doing I am instead of breaking up the interval as minus infinity to 0 and plus infinity to 0 to plus infinity and in trying to evaluate the limit separately I am going to break up this integral into 2 parts. Because that is the easier to do this interval even if I have the system because even if it is a finite integral I can do it separately right minus p 2 0, 0 to minus p. So, 1 plus x 1 plus x square plus 0 to p 1 plus x 1 plus x square dx.

So, what I am trying to do here is that now I will compute this and take p as p going to infinity. I will compute this integral each of the separate integral and see that p is going to infinity. Now I can simplify this of bit. So, if I put x equal to minus you substitute in the first part, one means this parts and x equal to u u substitute in this part.

So, what would happen then it will have minus p 2 p 1 plus x 1 plus x square would become 0 2 p, 1 plus u by u 1 plus u square d u right.

Student: Minus u.

1 x is equal to sorry, 1 minus sorry this will become 1 minus u by d square and this will remain putting x equal to u it will become just the same thing.

So, here if you put x equal to minus u dx it will be minus d u. So, minus would come here. So, 1 minus u 1 plus u squared d u. So, when x is equal to minus p x is equal to minus p u becomes p and when is 0 it remains 0. Since it is the minus 1 a sign the integral positions of the integrating points that is the all end points the change the interval sweeps. So, it becomes 0 to p.

So, now if you add them up; so, what will you get? You will you will get 1 plus 1 2 u, u will cancel. So, you will get 2 0 to p. So, how; so, this integral has now come down to this nice little form. Now if I take limit p tends to infinity minus p to p 1 plus x 1 plus x square dx. So, this is actually the Cauchy principle value cpv.

So, then what I am trying to now write this is same as 2 times limit 0 to p p tends to infinity 1 by 1 plus u square d u. So, basically it is 2 into tan inverse infinity actually. I shouldnt write that and inverse infinity it is essentially it means 2 into pi by 2. So, cancels an answer comes to on 2 pi.

So, you see the limit exists finitely. And hence the Cauchy in the Cauchy principle value since this integral has a value, though it does not have a value in the standard sense. So, so this is a very interesting approach, but it is right a I think most people on given bother modern texts from the event talk about what that is also in one way. So, in those days convergence was a very important issue in convergence of integrals and people wanted to make even not. So, good looking nice integrals they wanted to design ways by which they should have value just like we were wanted to increase the number of functions which could be integrated.

Now, we are going to talk about how to test for convergence and divergence. So, our example will start investigating the convergence of an important integral right and important. So, will tell now will discuss from standard text which I have already mentioned in the beginning Thomas Finney or Thomas as calculus, which is a standard. So, we are going to integrate it us reduce you know that, if I take the indefinite integral of this integral just 0 to it integral of e to the minus x squared dx.

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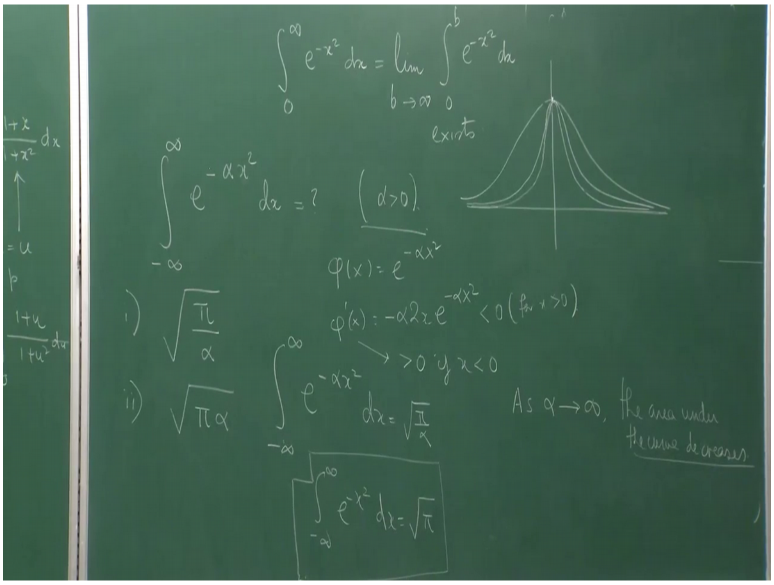

It does not make sense because there is no function whose derivative is this, but can I calculate this does this integral have a value is it finite that of course, is a question.

Now, I will give you a slightly puzzle a game to play a little bit. So, I conjecture 2 things. I say alpha is positive and I say that this is something I would like you to I give you a quiz question. That is, I say that it is I say maybe many of you who are preparing for jee or some other examinations whatever you want worried you might tell me might like this sort of questions mcq, for give you to set of questions and you really have to answer to me without doing integration, what is the correct one. How would I really think about it. Suppose I am lucky enough and I can draw graphs of functions. See always understand e to the power minus alpha x squares graph is not the graph of e to the power x. One has to understand this very fundamental fact.

Obviously if I put x equal to 0 it will become e to the power minus alpha and (Refer Time: 20:23). So, the question would be how do you guess what would be the right answer. So, to guess what is the right answer let me try out the following. Let me tell you if you draw the graph you take any software like Mathematica or maple or if you have an access to of course, I am sure many people watching this program does not have an access to. Just try your calculator and try nowadays they have the exponential value in your this math calculators and so you can just try out and draw some few points and try to extrapolate actually what should be the form.

So, usually it will be a form like this. Set x equal to 0 this is alpha. So, alphas value is this. Sorry x equal to 0, it will I make a mistake in x equal to 0 it will be 1. Now what would be the this would be the usually. So, it will become set x equal to one it is e to the power minus alpha. And this functional value and x equal to minus 1 alpha it is e to the power minus alpha. So, once you know this that plus and minus 1 it has a value it has a symmetric value. And then you try to know the nature if you take this function phi x and say take x positive. And let us try to take a take the derivative of this.

So, let us try to take the derivative of this; what would happen. So, phi dash x is equal to minus alpha 2 x e to the power minus alpha x square so this is always a positive quantity, but this is not for x positive this is strictly less than 0. So, it is a decreasing function. Similarly, when x is negative it is. So, phi dash x is positive if x is strictly less than 0.

So, it is natural that for starting from to point 0 the graph would fall down. So, what would happen, when if I ease now increase the value of alpha. I always have to keep at x equal to 0 it would be one. So, this length cannot be changed. So, what I am supposed to do now, if I change the value of alpha increase it the graph would become sharper like this. The more increase I to of alpha no sorry my drawing might be very bad more sharper the graph becomes. And the area under the curve decreases. So, as alpha tends to infinity the area under the curve is decreasing. So, area under the curve is no decreasing, but what is this product phi alpha.

So, area under the curve finally, goes towards 0 as alpha would increase now phi alpha means as alpha increasing this is becoming bigger and bigger. So, this can never represent the area. So, 0 minus infinity to plus infinity e to the power minus alpha x square dx is root over pi by alpha. So, minus infinity to plus infinity e to the power minus x squared dx is equal to root phi. I will tell you what tomorrow a bit more about the computation of integrals and why, how you do it?

So, this is one way that you actually do it, whenever they are not one way this is where you can argue things out I will show you ways by which you can actually compute these integrals without actually computing them. See tomorrow well spend some time talking about some interesting integrals because it is not always about applications. Mathematics is also sometimes about mathematics and what I want to tell you tomorrow would actually help you in doing good computations; you will try to compute integrals without actually computing them. We will not compute them, but we will argue by looking at the structure of the function and call the etcetera and will actually get to their current value. So, that will be a part of the training also itself.

So, now will go back to the more mathematical question of how do I what is the how whether that this is there, which means I am asking whether this is finite which means I am asking whether limit. Now it might be slightly confusing look slightly difficult can not do much about it, but let me tell you can look at the whole thing in a very simple way; now that is why knowing graphs are so important.

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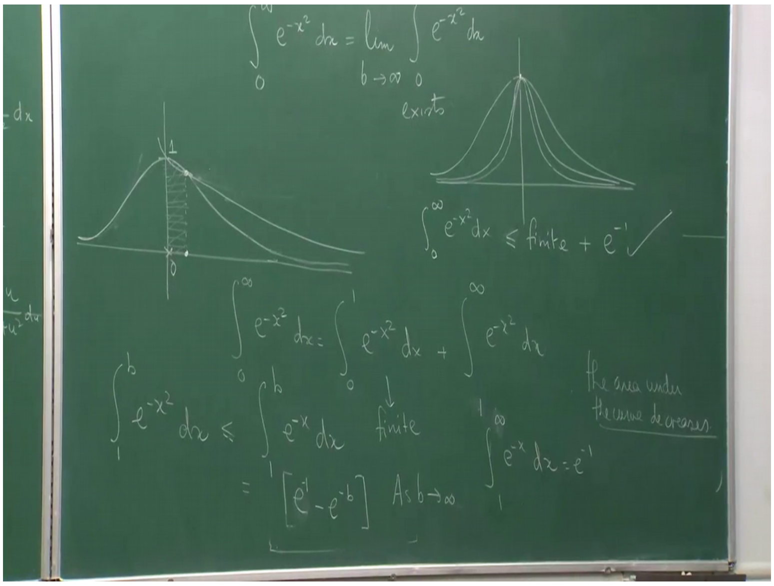

So, this is your one for I am just drawing the crowd this is your graph. So, this is 0 and this is one.

Now, what happens is the following that you, now you observe you may take up to 1. I will tell you why I am doing up to 1. The reason is follows at x equal to 0 e to the power minus x is also 1. And the e to the power minus x graph comes in like this. So, e to the power minus x after 1 actually goes above this graph. So, then it makes a much more easy for me to evaluate look at the nature of this integral by the comparison test that we had learnt in the last class that you can actually now compare them. Plus, e to the power minus x is our thing where we have an elementary integral right. So, you can actually have a derivative you have a antiderivative.

Now, the this part which is 0 to 1 you see to the power minus x square is a finite continuous function, and maybe you are not able to calculate the value just like that we can; obviously, calculate it numerically which we will which will be one of the topics of tomorrow numerical calculations. So, you see here this area is finite. So, basically if I want to look at this I will look at it in 2 in this way. So, this I already know as finite plus I will look into this; sorry.

So, now let me look into this integration. So, you again know from the graph that this is nothing, but, what is this e, e to the power what is this integral. So, it we. So, it c it will be e to the power 1 minus e to the power minus b that would be the answer now as b tends to infinity this, sorry e to the power ha sorry e to the minus 1. Thank you. So, it will be actually e to the power minus e to the power minus b minus e to the power minus 1. So, that will be minus and minus is plus it is the minus 1. So, as b tends to infinity e to the power minus 1 is not affected because it does, it does not depend on b here e to the power b tends to infinity e to the power it will minus x goes towards 0. There is a curved graph. Actually and this is I think the graph is what. So, it should be like this. Actually it should come like this this were. So, so here oh this is also going down, but it is going much toward 0 much faster to the (Refer Time: 31:03).

So, what would happen then integral 1 to be e to the power minus x 1 to infinity e to the power minus x dx is equal to e to the power minus 1 that is answer. So, which means that 0 to infinity e to the power minus x squared dx is some finite quantity were actually less than some finite quantity plus e to the power minus 1. Because e to the minus x square 1 to b dx is less than equal to this. So, when you write the limit they will remain to be less than equal to, hence this integral is actually convergence. So, it was meaningful for us to discuss all this stuff. So, I will show you how just without doing actual calculation you will actually solve we will get the answer root pi by alpha will do it tomorrow.

Now, there is another test apart from this comparison test of showing that fx is less than equal to gx and if you integral 0 to infinity gx is finite, then fx is also finite something called the limit comparison test.

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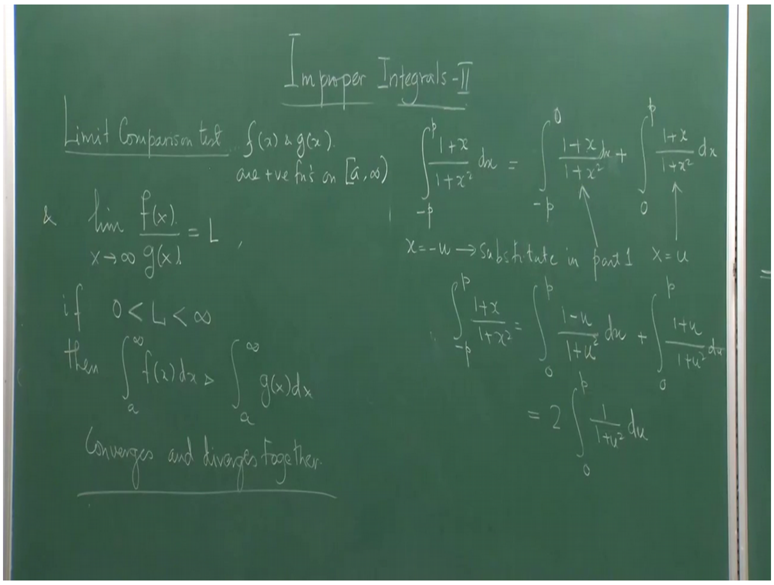

So, this is something also important. So, the limit comparison test says the following. So, I am writing the statement from Thomas infinite Thomas has calculus what is the standard result and finding too many books on analysis. So, it is limit. So, you have 2 functions f and g right. And you do not know whether fx and gx or what is their nature.

So, I will have 2 functions. So, here I will have f x and gx are positive functions, are positive functions on say some a to infinity take. And limit fx by gx as x tends to infinity what is their behavior. So, the limit should be a non 0 limit and a finite limit. So, limit as x tends to finity should be l and if then integral a to infinity fx dx an integral a to infinity gx dx converges and diverges together it converges and diverges just together.

So, I will just give you one example and before I in the course. So, if one of them converges the other converges if wanna put them diverges or the diverges, that is it; that is the meaning. So, let me take this one example and let us see what happens.

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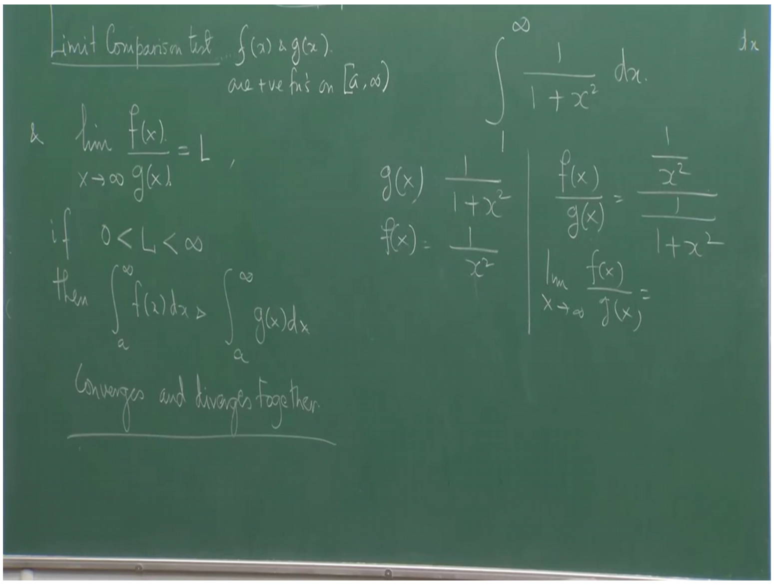

So, this example is from Thomas Finney and in will look into this function this integration. So, your fx is equal to; obviously, this is continuous on 1 to infinity and this after one.

So, this it is a continuous function what we know that somehow that x square is less than equal to 1 plus x square. This is positive. So, one by 1 plus x square is less than equal to one by x square. So by comparison test also you can prove that, but. So, one by x square could be might g x suppose I choose gx equal to one by x square here. Of course, you know the comparison, but there suppose you do not even know then then you will take fx by gx. I do not know which whichever were you want to check then in then it will become maybe you as to book sizes that we should change it took that is we should take this as fx and this as gx. This is these are gx and this as fx does not matter maybe some easier computation.

So, this will become what one by x square so the limit x tends to infinity fx by gx is equal to 1 the answer is 1 here, but I know 1 to infinity one by x square converges. So, 1 to infinity these also converges. So, this actually does not really require the image comparison test, but it has been done, but what requires the limit comparison test here is an example integral 1 to infinity.

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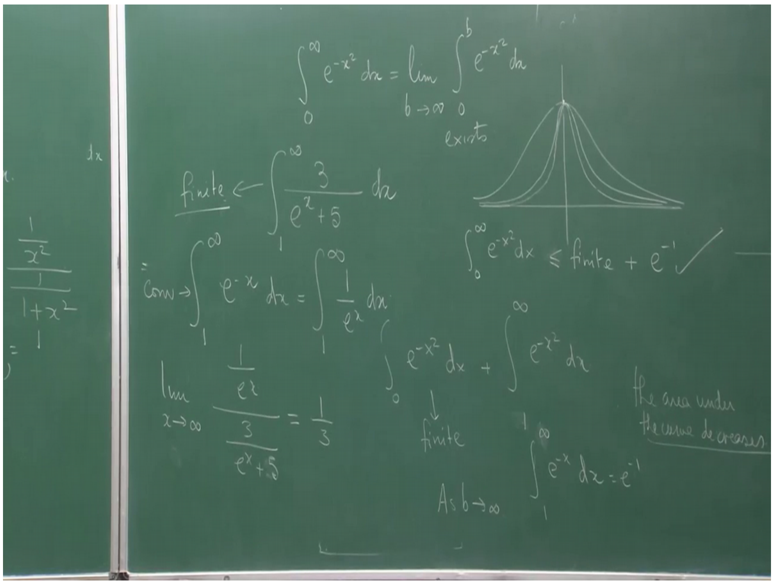

So, in this case use the fact that 1 to infinity which we have just done e to the power minus x dx which is equal to integral 1 to infinity 1 by e to the power x dx this converges this we have already done, this is e to the power minus 1 is answer.

So, take this to be your fx, then 1 by e to the power x by 3 by e to the power x plus 5. So; obviously, you can hear it is not. So, clear because you can say that e to the power x is less than e to the power x plus 5. So, 1 by e to the power x plus 5 is less than 1 by e so, but if I multiply by 3 on this side I can I do not know how to compare now. So, this fx less than equal to gx that approach which position to showed in the last lecture cannot be applied here. So, here we really have to take the limit. So, limit as I will not do the limit operation, I will just write down. So, so that you can figure out the limit yourself, limit as x tends to infinity is equal to so is one third actually. So, then it is in that range. So, because already we know that this integral converges, this integral converges means this is finite and hence what this integral is also finite. So, with this we end todays lecture here. And tomorrow we will start by telling you how to compute which is the last the last 2 lectures for the 6 week is left and the 7 th and eighth week would be some few applications of integrals and then going on more on to sequence series power series and all sorts of things like that. And so of Taylor series and how those things come.

So, what we are going to do we are going to go to go back to differentiation again. So, and we will see the interplay between differentiation and integration. Tomorrow I will show you for example, in applications I will show you for example, how can you use integration whatever you have learnt to show that pi is irrational. So, those sort of things that is this is the proof by your Niven, you want Niven. So, so I will show those kind of things tomorrow.

It is tomorrow will actually try to use integration in many ways trying to compute integrals and doing other things trying to see how numerically integrals are computed so; obviously, we cannot do the numeric write over here, but we will show you the algorithms how there developed and all those things. So, this will be our discussion for tomorrow. And so let me close today's discussion it is already 43 minutes 13 minutes it exceeded actually. So, I hope you had what some idea of doing the thing. So, improper integral discussion ends here.

Thank you.