Calculus of One Real Variable

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Lecture - 30
Application of Definite Integral – II

So, we are coming to the second part of the application on definite integral, we have found areas you will not find volumes. We are not going to tell you the theoretical things how you should go about in drawing picture, we will start with an example from the example it will be clear to you how to operate. So, we will simply learn how to do the stuff just by visualizing those things. See we the here we will study largely from thomass s calculus.

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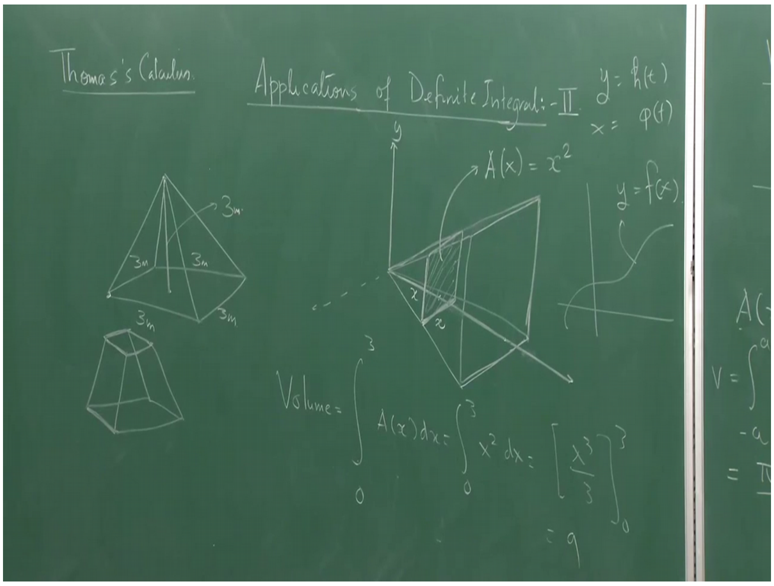

We will take some example from thomass calculus which are walked out by the way and try to explain them. Because here they do not give too much of explanation. So, here the first problem is that you if you have a pyramid. You know what pyramids are you have seen in photos about Egypt and somebody of amongst, you have gone to Egypt.

You are luckier than me it is like a thing like this. So, it base is actually a square. I am telling you to find a volume of the pyramid somewhere I put stuff inside this. Volume the pyramid whose base is 3 meters. 3 meters, how do you do it? Now in calculus everything is looked at the point of view of there of the infinitesimal, that is you take a look at a very thin part of it or small part of it and then try to sum over the small parts. That is the key idea. So, here if you look at it here is the x axis the y axis and as the z axis whatever or whichever axis you want to call and we would like to call that the y axis. So, this is a 3 dimensional figure. So, what is there is that I have drawn the pyramid like this. Just try to make the drawing look more authentic or the put it on this space.

I will try to draw the square draw the square base of the pyramid. So, you draw the square base of the pyramid and add this part. So, 4 parts and if this is square base would look like this and add this part. So, this is what looks the pyramid this. So, here at the end is your 3 meter. So, my drawing might be very bad, but. So, 3 meter 3 into 3 meter. So, what do you do you think of you take the pyramid, it would have been good if you I felt model stress paining, it basically chopped the pyramid up as if you have a slice of cheese and you are chopping it up very finely. So, chopping it up. So, one of these chopped up zones you called ax. Chopped up zones at a point x. And you want to find the area of this right. So, what is the volume? See this all the sectional area if I chop up if I suppose id have the pyramid have chopped the head out, now if the in the truncated pyramid that I will see that is something like this.

Whatever this is actually a square. So, here also what do you see the square. So, square these are the point x say it is a square of length x. And the pyramids are done in such a way there things are done in such a way that if you chop out some x among this will be of length x. So, these are all sides x there is the very structure the symmetry of the pyramid that if you have come x from the origin then these that the base of the pyramid is x. So, this is actually 3 right if the base of the pyramid has x, x sides are x the square then the height of the pyramid is x basically.

If this is 3 meter then this height is also 3 meter this is how this is how pyramid is made. So, what you do you find the area of that cross section. And basically what happens if you take all these small cross sections and pull them up and crop them that gives us the volume of the solid, that gives us a solid and the areas actually contribute to the volume. So, basically what you have to do this is my a of x area evaluated at the point x which is x square. So, it is no. So, the volume is nothing, but. So, I am summing up all the areas, which will actually contribute to the volume. So, here we do it in a continuous fashion. So, integrate from 0 to 3 ax dx, which is integrating 0 to 3 x square dx and you can write the answer obviously.

9 cube by 3 which is 9. Now let us look at a more interesting example take the graph this root over x.

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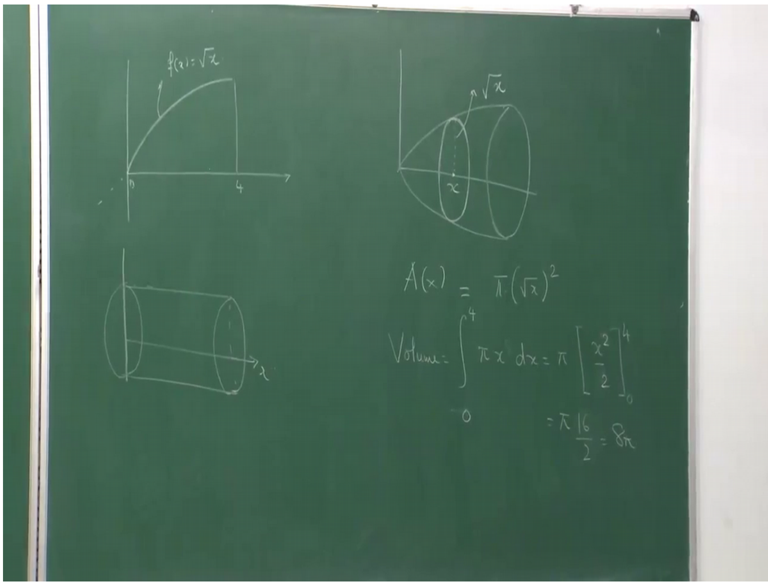

Take this function root over x from 0 to 4 that is what we had studied. Here fx is equal to root over x. Now suppose I rotate this suppose here is my hand and I am rotating it like this. So, in that space it is like if you take this book do not think of the fact, but just think of this. So, I am rotating this book. So, it here it is a rectangle. So, I when I am rotating this, just in your minds eye just see what happens what does it generate it generates a solid of revolution it generates a solid. So, if I have a if I have a rectangular region and if I generate it if I move it in the space if I rotate it around the x axis. So, I am rotating around the x axis then for our for our rectangular zone, I generate a cylinder.

So, these are called solids of revolution. So, here also if you rotate this one you will possibly generate an object like this a part of egg sort of thing on end of a think. So, how do you find it is volume. So, once you have rotate around the x axis basically your whole thing goes in a circular motion. So, if you now cut the slices if you have a slice here. At x this slice how do you find the area. Because now it has become a circular slice it is no longer a square slice, that you see here it has become a circular slice. So, once it has become a circular slice what are you going to do what would happen. So, once you have a circular slice. So, now, this is your x this is your root x. So, now, you have a circular thing of radius root x. So, basically you will have pi root x square right. So, area a of x. So, now, if you against club up this area together.

You put those areas one after another they will contribute to the volume and form the volume. So, this is what you have to see in minds eye on that is what I have been telling you from the beginning that is what I believe. And mathematics has got lot about visualization without geometry mathematics does not just progress. Now if you look at this? What is a of x pi of root x whole square pi whole square. So, my volume of this particular solid is integral sorry 0 to 4 pi of x square dx. It is pi of x cube by 3 0 to 4. It is pi of 4 cube means 64 by 3.

Oh sorry square root oh sorry x pi x, I make a mistake right x square sorry x square pi 2 thank you then. So, it will become square root of x is x. So, it is. So, so it is pi of. So, it will become what it will become 4 square 16 by 2. So, 8. So, it will become 8 of pi. So, you see the area becomes. So, simple 8 of pi. So, now, if I have a circle and then I rotate it along the x axis what will be formed.

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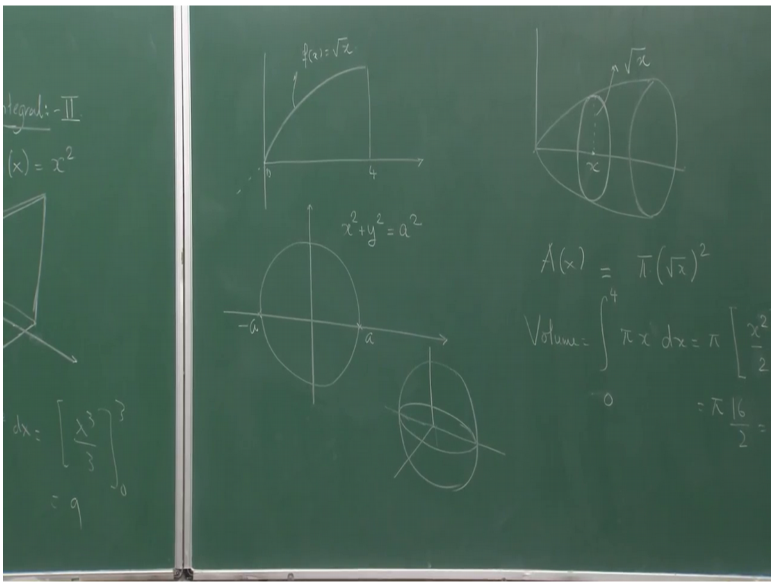

So, I have a circle now I have a circle x axis y axis. So, here is my circle. So, x square plus y square is equal to a square this is a and x axis there is minus a. So, if I rotate this in the air, you can think take a circle object and try to rotate it do it in your home.

So, cut out the circle from a paper and just try to rotate it and see what it generates it will actually generate a sphere. So, this is actually generate a sphere of radius a, whose every cross section is a circle we can have a cross sections of big circles or radius a right. So, how do you know can I use the same method to find the volume of a sphere. You see the same type of ideas we are using to find the volumes is the same type of ideas there is no. So, let us try it out with the circle with this sphere. So, now, we are going to find the volume of a sphere.

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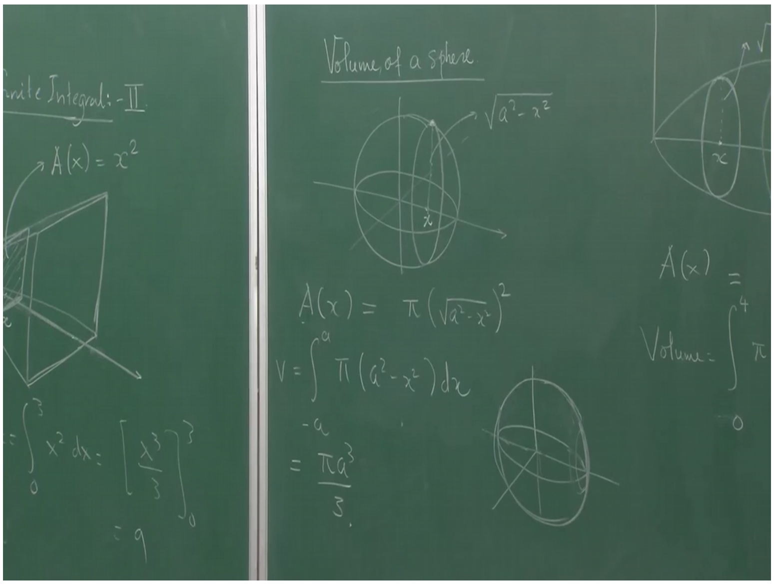

So, what do I do with, how do I look at the volume. So, I take any point x. So, x is my sphere here my x axis. So, here is my sphere my drawing is not exact. Please forgive me for that.

So, here is I have cut out a section at a given point x, but you see any section that I cut out corresponding to a x if I hit. So, what I cut out I cut out a circular section. So, this is my x what is then what is my y. So, if I cut out any section you just take up just think of that your you have your handout orange and we just cut it off. So, you cut out a section and just cut one part of just with the knife or even apple you just cut one part off or a stick is orange that is better cut one part off. What you see you see the pill and you can see the oranges inside. So, if you take any point x the middle which is the centre of the circular disk. And you just hit the top what do you hit you hit another circular disk which is our radius a. So, this point this height is nothing, but root over a square minus x square.

Because any point on this great circle is of the radius a is of the question excess a is of the form x root over a square minus x square. Because any great circle has the equation x square plus y square is equal to a square because what is this you are what is the sphere it is a rotation of those great circles you are rotating you are rotating that circle in the air. So, at every passing moment you are you are basically at every time instant you are passing the your circle is there in space. So, now what is my a x.

So, great circle you know, if you take a sphere take a globe, if you have a globe in your home, look at the equator is a great circle. Just take a path perpendicular to the equator through it is some of it is central points is a great circle. So, take the take the equator take the radius of their. So, that is the radius of the earth right assuming that the earth is a sphere. So, that the radius of the earth. Then any circle that lies on the surface of the sphere with the same radius is called a great circle. So, I am sure you must many of you have heard the terminology or many of you haven't heard the terminology. So, but that is it when you can go ahead look at the globe at your home and try this thing out.

So, the equator is a great circle. So, equator is this one like the earth. So, any circle for example, this circle you would draw a circle here or take some circle here. So; obviously, you cannot draw a circle here because it will become it will have the curvature, but if you draw a circle, if you cut off like this if you have a circle like this then this this is not a great circle, but a circle like this one which has the same radius is a great circle, see what we are trying to say that even if you chop off some part the y point from that x is always lying on some great circle. So, because ultimately the spheres the spherical volume is a union of great circle. So, you have one circle which is your great circle and then you are basically the equator then you are moving it like this.

They equal the equatorial plane of the earth this one and then you are moving it like this we just rotating it around the x axis right. So, that is what is happening. So, ax is nothing, but pi into root over a square minus x square whole square. So, now, here volume is integral from minus a to a pi of a square x square minus a to a pi of a square minus x square. I would like you to calculate this to give the answer pi a cube by 3 you see. So, things just by some geometric imagination simple geometric imagination, you could achieve. So, much So, much things using the calculus. So, that is that is what. So, gives me a gives you much more joy to think about this subject that, many things like washer method is that.

So, many things, but everything cannot be really figured out in all these classes. Everything need not be done at because you still can do something. So, we will now try to look at lengths of parametric curves right. That is what happens sometimes given a curve which have learned at high school I can bring. So, what is the curve means for me some curve line in the xy plane and usually I would like this to be defined as y equal to f of x so, but then and many times it is useful for example, to introduce a third variable t. And express y as sum ht and x as sum something phi t. So, then we call this a parametric representation of a curve. So, with respect to t. So, let me now look into find the lengths of curves which have parametric representation.

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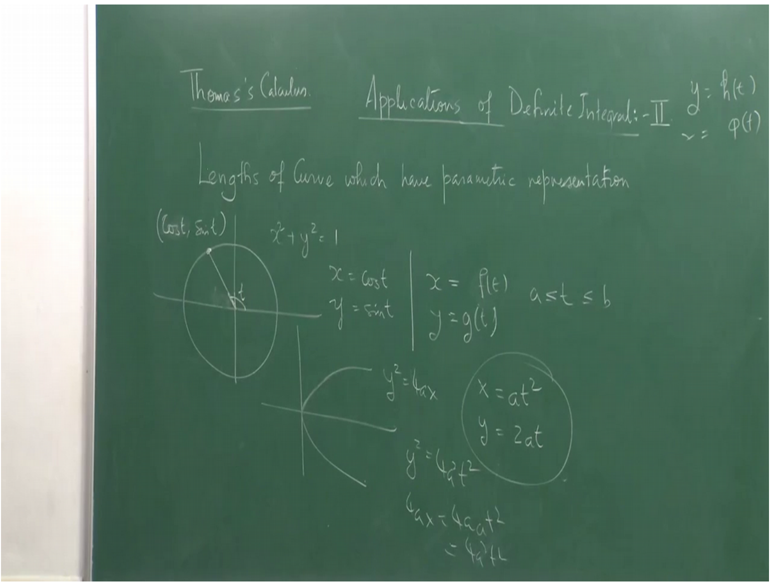

So, examples of curve which your parametric representation. This is very useful it is parametric representation is not only useful for this calculus courses, but is also useful in very advanced mathematics like algebraic and geometry. So, assume all if you have a circle. So, you know suppose this x square plus y square equal to 1. I can represent this curve as x equal to cos t plus y equal to sin t. So, if bring the variable t. So, so. So, I basically. So, I have x is written as sum f of t and y is written as some g of t. I will just go by the notation of this books of that it is if you somebody uses it you can; obviously, understand this. So, x is ft y is equal to gt.

So, this is the parametric representation of a curve, t can be in many cases restricted between b and a width, but not necessarily in this case t is free any t that you choose. Of course, you can say I can represent t can be represented from minus pi to plus pi in this case also will be the same. So, these are this is a called a parametric representation suppose you people have also studied a parabola, y square equal to 4 ax. So, x equal to at square and y is equal to 280, because you see now if I put my square y square is equal to 4 a square t square.

And if I put 4 a x for a x is equal to eighty square is equal to 4 a square t square. So, this is a parametric representation of a parabola. So, this is your ft and the user gt. So, curves can be represented by parameters. Now if you can represent a curve by parameter then it becomes very easy to actually compute the length of this curves. Like every any point on this curve say here can be represented not by just x y can be represented as cost t sin t right, t is actually the basically angle. Now if you have a curve defined by a parameter like this, this way parametric representation.

So, how does one find what is the length of this curve. It is actually pretty simple let us see how do we do we derive the length of this curve. And maybe in the next class we can talk a bit, about it is applications not to this, but to also. So, next classes basically you should be in the 7 th week of your last, but one week of your lectures.

Student: length.

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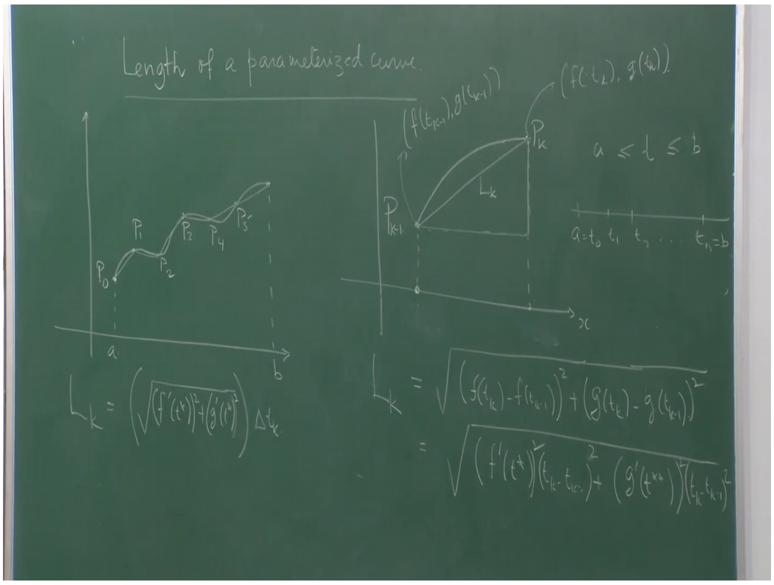

So, let us now find the length of a parameterised curve you will see here the beautiful application of mean value theorem and you see if you take any curve which is lying on the plane. And then you can actually by straight lines piecewise curves. So, that is why if linearity is such a because our minds are very linear you know our minds cannot think in terms of curves or minds always think in terms of straight lines. Suppose is your curve like this. So, what I do I actually make a partition of this say from a to b. Basically I make several partitions. So, this point I called p 0 I will put another point say p one another point say p 2 another point says p 3, p 4, p 5 and so on. So, here I am joined this point by straight lines.

This partitioning points on the curve and then I get an approximation of the length of the curve which is not exactly the of length. So, you can understand if I keep on increasing my partition points, some way the length will come. So, let me look into a very small cross section, where you have the point p k minus 1 and pk. What is this point? What is this point. So, what is happening we are basically now breaking up the interval. So, t is lying in the interval a to b. Not x or y please understand t. So, x and y. So, there is no one we are not showing in relation between x and y directly, we are showing we are telling that x and y is linked, but the indirectly linked they are linked essentially with a third variable t. So, it is a third variable on which we play the game.

So, I break it up into sub a is t 0 and t 1, t 2 whatever; however, you do in Riemann integration tn equal to b. Now this is tk minus 1 is tk minus 2 basically those 2 point right. It is actually y equal to fx, but just take these 2 points. So, these x and y. So, so this point corresponds to the point tk minus 1. So, this is and this point corresponds to the point tk. So, here my coordinates are f of. So, x is x and y. So, f of tk minus 1 g of tk minus 1 and here it f of x of tk and g of tk. So, basically I am co instead of looking at this curve, I am looking at the length of this straight line. Which is nothing, but the change in x by change in y. So, the length of this line as I calls a delta l we want to call this length of this line as delta l or lk if I want to call whatever you want to call does not matter at all.

If l are notations I am not. So, much worried about notations till you know the stuff. So, lk kth piece basically first piece second piece third piece forth piece. So, on for the kh piece. So, length of lk if you look at it is nothing, but the change in x is just pythagoras theorem the root over f of this part f of. So, x coordinate as change in x coordinates. So, these are change in x coordinates. So, f of these x coordinates f of tk minus f of tk minus 1 whole square plus g of tk minus g of tk minus 1 whole square. Now apply the mean value theorem do this. Each of them fk minus tk minus 1 gk minus gtk minus 1 assuming then f and g are nice functions. So, assume that f is a smooth function right and f and g have continuous derivatives in the interior of a b and assume that f and g are nice functions.

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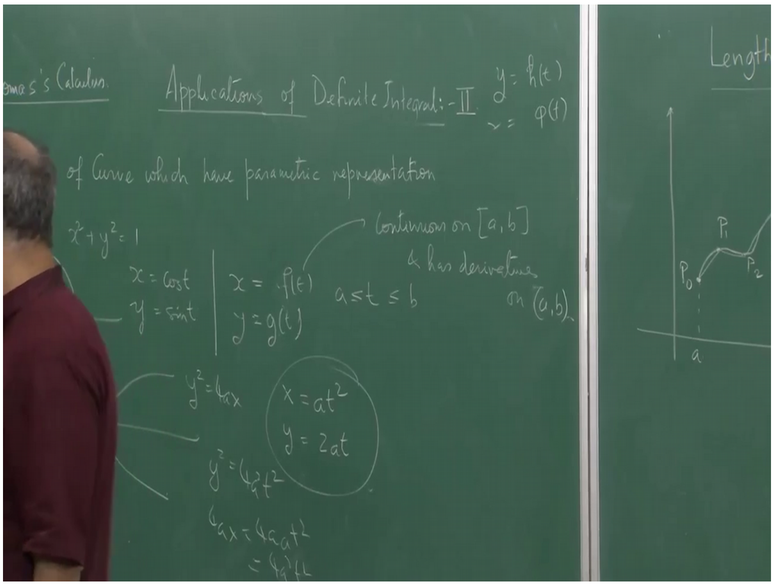

So, assume continuity a continuous on a b and has derivatives on ab adding a bit fast already a overtime. So, what does this give me this give me some t lying between tk and tk minus 1 which I do not bother f dash t star tk minus tk minus 1 whole square plus these also whole square. So, this I can write as delta tk. So, g dash t double star whole square tk minus tk minus 1 whole square, if I write tk minus tk minus whole square as delta k then lk is nothing, but root over f dash t dash whole square plus g dash t dash whole square into delta tk now what is t dash it is lying between tk and tk minus 1 t and what is t double star t double dash. Here t double star what is t double star it is also lying between tk and tk minus on this is just these are their stags. So, these are function value you are integrating over. So, you are actually if you sum them if you sum the lk.

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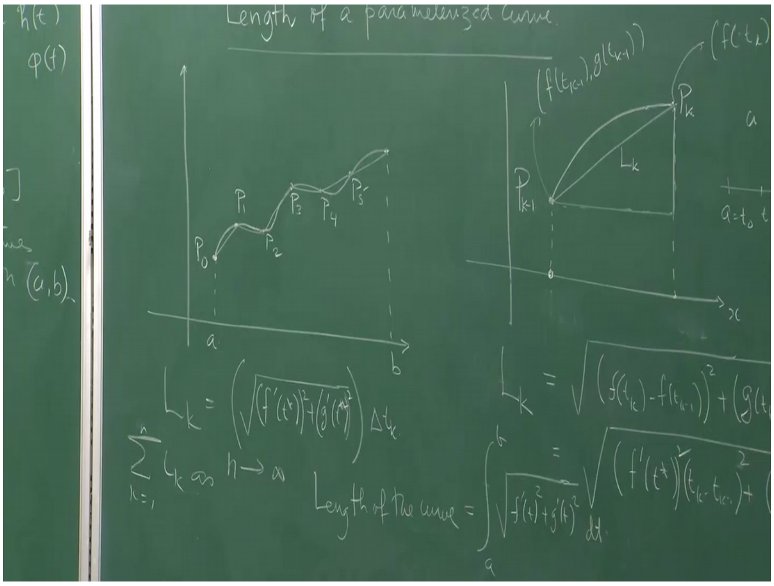

So, you get the length of the piecewise curve piecewise straight line right. The curve form by the piecewise straight lines , but that is, but that length will become more and more near to the length of their actual curve as you go in tending to infinity. So, these t star or t double star are nothing, but tags on the interval delta tk right. As k goes towards infinity that the position of the difference between t and t star t star and t double star in every interval actually come down. So, basically they will come to a tag point. So, if you look at it as n tends to infinity the length of the curve length of the curve.

So, this is now integration where this is the function of t right this is nothing with 2 tack points does not matter. I have 2 different curves I have I have written in terms of 2 different tack points. Ultimately when they will become smaller and smaller the tack points will becoming near and nearer and. So, ultimately what I do you will get. So, this is nothing, but length of the curve is integral a to b root over f dash t whole square plus g dash t whole square dt. That is exactly this a length and how we will apply to find it is the plane curves would be our discussion, we will start with the circumference of a circle finding it for some other curves maybe you can try for parabola whatever you want.

So, we will use some of them in our next lecture. So, our sixth week ends here and 7th week will start which will have some 2 more classes of these things and then applications. And then we going to you know more involves sees we are now moving from just calculus in to more advance task like errors flow and power series (Refer Time: 34:15) and all these which takes you which now allows you to cross the boundary from calculus for reanalysis only. So, those who would love some mathematics and then would you should take note of these 2 these 2 weeks. Because especially these things on Taylor theorem. And all there those who want you can cross the boundary and go to reanalysis.

Thank you very much.