Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture – 31
Applications of Definite Integral – III

So, we are in the 7th week of our talks of our lecture. So, we are almost running towards the end of our session with calculus. So, more we are going to learn about applicability in this few sections. And then one or 2 lectures and then we go on to the more deeper idea as I said that I would likes at least some students here to jump from calculus to real analysis. Those who would do it because these are the strengths one needs to gain specifically your knowledge of real analysis or advanced calculus would be extremely important when you are going to study engineering when you going to study economics when you are going to study physics. So, here we keep on our applications of definite integrals. This is I think the third lecture; I am not wrong.

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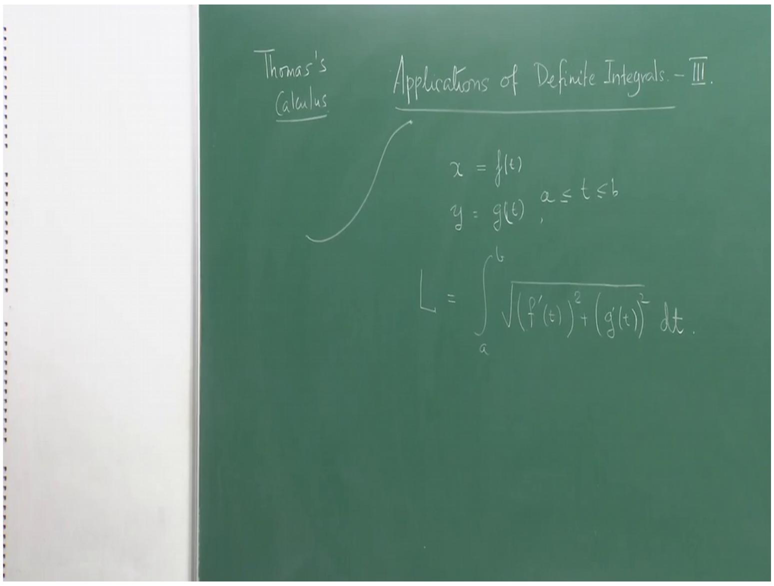

So, here if I recall what I did in the last class, at the very end we were telling that if there is a curve and the curve is described parametrically every point has a coordinate xy and it is described parametrically in the following form. Where t is restricted from b to a or something it could be no restrictions at all.

Now, the le now the length of the curve the length of the curve is actually given as an integral from a to b, root over f dash t whole square plus g dash t whole square. This is what we had also tried to prove. So, our aim here is now to show you some applications how do I actually use this whether by using this. We get results which you already know. So, that verifies the usefulness of this formula.

So, first you have to start with the most simple object of all, but the most intriguing object of all the circle. So, the circumference of a circle.

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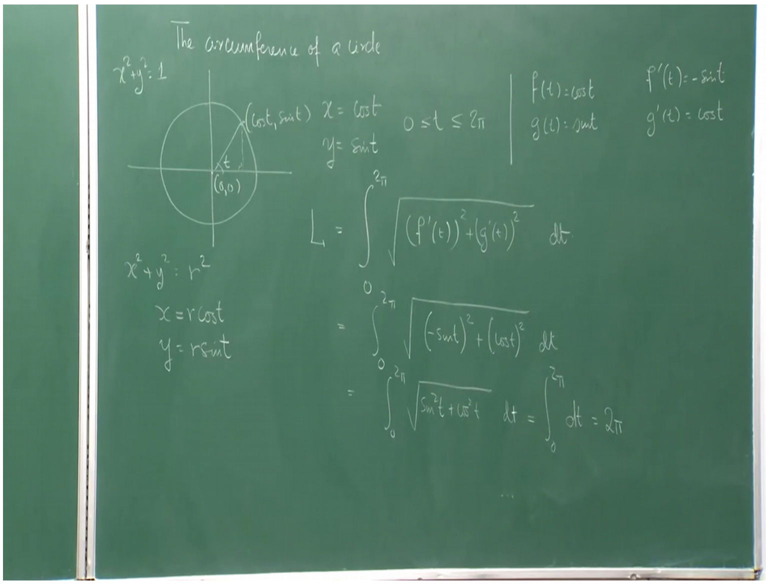

Whatever you want to say we say it is of radius r. So how can you write every x can be written as cos t y can be written as sin t rather I am taking this to be of radius one. So, I am taking a circle of radius, just 1 x square plus y square is equal to 1. So, if you take it and t 4 is varies from either pi to minus pi or 0 to 2 pi, whichever you want to whichever way you want. So, any point here on the unit circle with 0, 0 as a center can be written as cos t sin t what is this t is actually presenting this angle that, you will get when you join it with the point. Because here if you drop a perpendicular these are x coordinate and these are radius the hypotenuse of this right angle triangle.

So, this by the radius this distance by the radius. So, x by the radius is a cos of t. So, x equal to r cos t. So, if it is x square plus y square equal to r square. Then the parameterization is x equal to r cos t and y is equal to r sin t right, I will do it. So, if you add up and if you square and add you will get this. So, how does how do I compute the circumference of this. So, compute the circumference of this you now have to use the same formula. I am just doing this particular situation this case one, for r you can know what is answer 0 to 2 pi root over this I call as ft. So, f dash t whole square plus g dash t whole square dt.

Now, if I look at if I look at this situation what is my f dash t. So, in this particular case my ft is equal to cos of t. And gt is equal to sin of t. So, here f dash t is equal to minus sin of t and g dash t is equal to cos of t. So, what do you get? So, here you will get 0 to 2 pi root over minus sin t square plus cos t square dt. And that will become 0 to 2 pi root over sin square t plus cos square t dt. Sin square t plus cos square t is one and the root of one is one. So, this is equal to 0 to 2 pi dt which is finally, 2 pi.

Now, if there was an r here they that r would remain here it will come here, and this will become r square sin squ r square into sin square t plus cos square per t. So, root or r square will just give you r. So, it will become r into 2 pi. So, 2 pi r which you the formula which you are familiar with. So, redi if when the radius is one the length is nothing, but 2 of pi if you put also minus pi to plus pi then also you will have the same answer then also you will have pi minus pi it will get you get will get 2 pi.

So, another example from we will take an example from the famous book Thomas calculus, I just want to again we assert that we are using Thomas calculus. So, gb Thomas was a very famous teaching professor at the Massachusetts institute of technology in the us and he was. So, famous because of his takes that when he died his death was announced on the first page of the New York times. So, you see a great teacher also have a great value to our society.

And writing a book like Thomas infinite like when I was a young undergraduate student then we had it this book Thomas infinite calculus analytic geometry, but Thomas has diedly. After that now you have Thomas calculus which basically means the same book, but some other people are adding on to it and doing this stuff. So, it is just you just google it you have their authors keep on changing. So, this book is one of the most important most important book in the history of modern calculus teaching, like a modern calculus education it has so many things inside it, lot of things to learn and good fun problems to work on.

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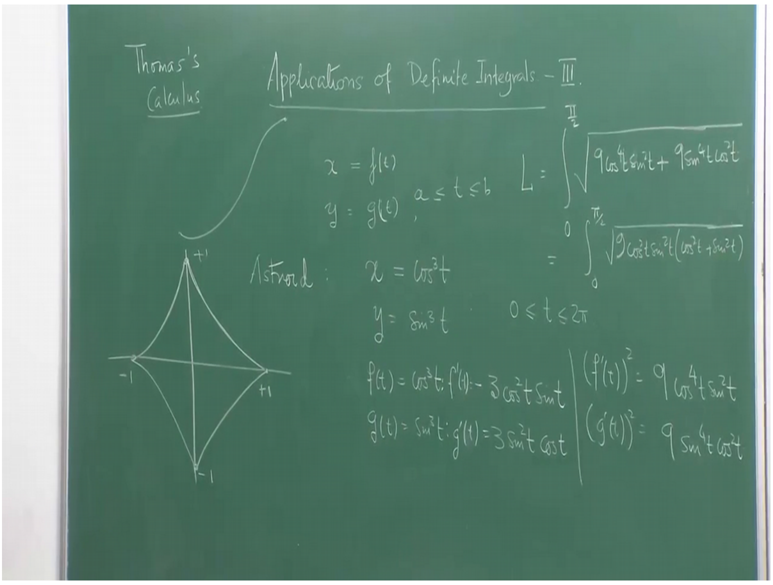

So, now we will look at a curve which is called an asteroid and it is parametrically given as x equal to cos cube t and y is equal to sin cube t. So, let us look at this curve looks like this. And these end points you can very well understand these are nothing, but the x axis is a plus one-point minus 1 point on the x axis plus one point on the x y axis minus 1 point is on the x axis. This curve called an asteroid has a following representation x equal to cos cube t and y equal to sin cube t with t lying between 0 to 2 pi. So, you have cubed it you have maintained that same story like the circle with cos and sin has the component function and then you have cubed it.

Now, I want to find the length of this curve. So, should I bother about finding the length of this whole curve I would rather first concentrate or bother I will be bother about finding the length of this curve. Because you know there is a symmetry in the diagram. So, once I know the length of this curve all of these curves or parts are of equal length I just have to multiply by 4 So, what I want to do is just I want to figure out the length only in the first part. I am not going to figure out the length in the other part. So, let me just fi first find out what is ft is equal to cos cube t gt is equal to sin cube t. So, what is the. So, it is 3 cos square t and then derivative of cos t which is minus sin t here it is 3 sin squared t and into the derivative of sin t which is cosine of t and the etcetera.

Now, if you take the square of these. So, sorry I am making mistake. So, here I am having f dash t is this and g dash t is this not this it is not this is not the this a func function. So, you will have f dash t whole square is equal to 9 cos 4 t sin square t and g dash t whole square is equal to 9 sin 4 t cos square t. So, my length which I just only want to find from where the angle is in the first part, but the angle varies from 0 to pi by 2. So, 0 to pi by 2 that is 90 degrees then I will have root over 9 cos 4 t sin cos 4 t sin square t plus 9 sign 4 t cos square t. So, that is equal to 0 to pi by 2 root over 9 times cos square t sin square t into cos square t plus sin square t.

So, what I will have l is equal to 0 to pi by 2 root over 9 cos square t sin square t.

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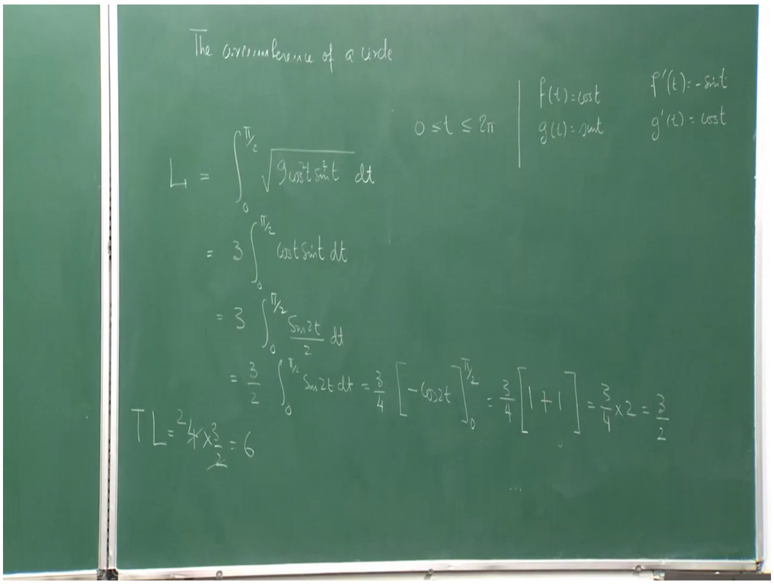

So, if I take the limit out because cos t into sin t is positive, I will have 0 to pi by 2 3 cos t sin t dt. So, this can be replaced by 3 0 to pi by 2 sometimes your minds mind wanders because you want to you think as you are teaching also. So, this is cos sin 2 t sin 2 t is equal to 2 cos t sin t. So, it is sin 2 t by 2 and then of course, I leave it leave into you to find the answer which is nothing, but 3 by 2 at the end. So, this will be this. So, in by put 0 that cos 2 it will become 0 it will become one and it will become and t equal to pi by 2 it will become minus cos of pi. So, 3 by. So, there will be 3 by fourth of course,, cos of pi is minus 1. So, it is 1. So, it is 3 by fourth cos of because cos 2 t by 2m 2 as come out 3 by fourth. So, it is 1 minus plus cos 0 is 1, it is 2. So, it is 3 by 4 into 2 which is equal to 3 by 2. So, 3 by 2 is the answer for this part only. So, for the length of the whole curve. So, total length tl total length is 4 into 3 by 2 which is 6, 6 units whatever units you want.

So, why I want to know find the length of a curve given by the function y equal to fx I do not I have not told you anything about the parametric representation how can then this idea be used there. So, that is the important issue. So, let us have a curve y equal to fx which is continuously differentiable de derivative is very nice and then how would you try to find the equation of such a curve using the ideas that we already have built up here.

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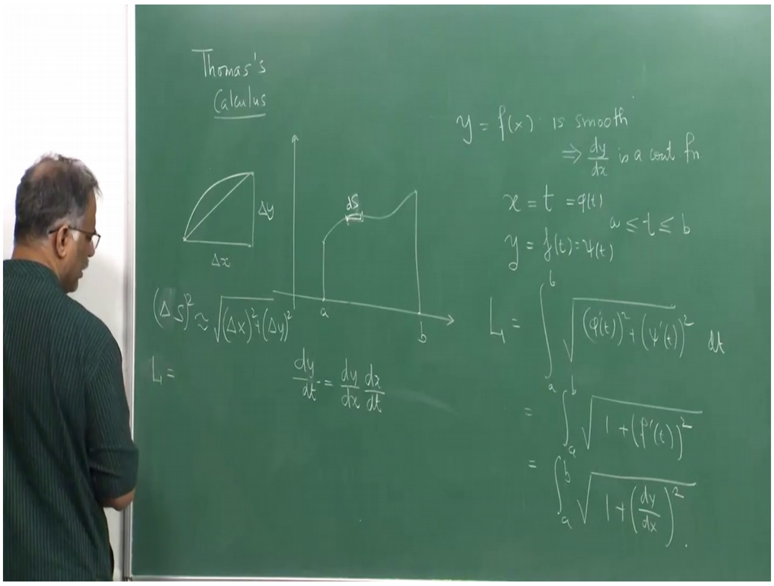

So, let us see what we can do there I have problem, would be that I am given a smooth curve y equal to fx smooth curve means, that it has a derivative is smooth means the derivative is continuous this implies dy dx is a continuous map continuous function. So, now, find the length of this curve say from 2 points a to b. So, it is defined say it in the a to b. So, you find the length of this curve between a to b. I am trying to tell you now that this is my y equal to fx between a and b and you try to find the length of this, whatever knowledge you have gained in finding the lengths of parametric curves.

So, now how do I parameterize this curve? So, what I do I put x equal to t and y itself as f of t where t of course, has to vary between a and b. So, as the t is same as x as x varies between a and b. So, now, you can ask me what about the derivatives at this point one could be that the function is also known over the whole r. So, the relatives are there at a and b and they are continuous one could be that I take the derivative at the endpoints to be only derivatives in the left no right hand derivative sense and left hand derivative sense. And right hand derivative sense when you come to end when left hand derivative sense when you go to b and as a functions in that state these are continuous you can look at it in either way.

So, here what would I what would one do, that is very in interesting and important the fact, that we have now parameterize like this my length is nothing, but integral a to b root over. So, this is my say phi t and this is my shy t. So, I will have phi t dash whole square plus I dash t whole square, but then dt then a to b root over what is phi dash t phi dash t is one what is shy dash t shy dash t is f dash t whole square. So, what is dy.? So, basically df dt or other dy dt, dy dt is dy dx into dx dt. So, dx dt is one. So, dy dx is equal to dx dt because dx dt is because x is same as the function phi basically. So, dx dt is just one. So, it is nothing, but. So, this is nothing, but dy dx. So, what I get is integral a to b root over. So, here I have the use a chain rule please observe this one plus dy dx whole square. So, how simply we obtain once we have that knowledge we obtain this very simply.

There is another way of looking at this curve you know, there the lengths is that you take a very small elemental part here, these are called elemental part in calculus and call this length to be d of s where s is the curve right or d of l whatever you want to say d of l say right or the length or ss arch, small arch. So, s is a typical symbol of an arch. So, d s. So, assume that is almost straight basically. So, now, this point is some xy this point is some another xy. So, this is the change in x. So, basically what you are having if you magnify this. So, here is your x and here is y the same thing we did yesterday. So, change in x and change in y. We are approximating this length del x and del y. So, we are approximating you are writing that ds square, when this is very small is almost same as or rather I should write del x del x square plus del y square.

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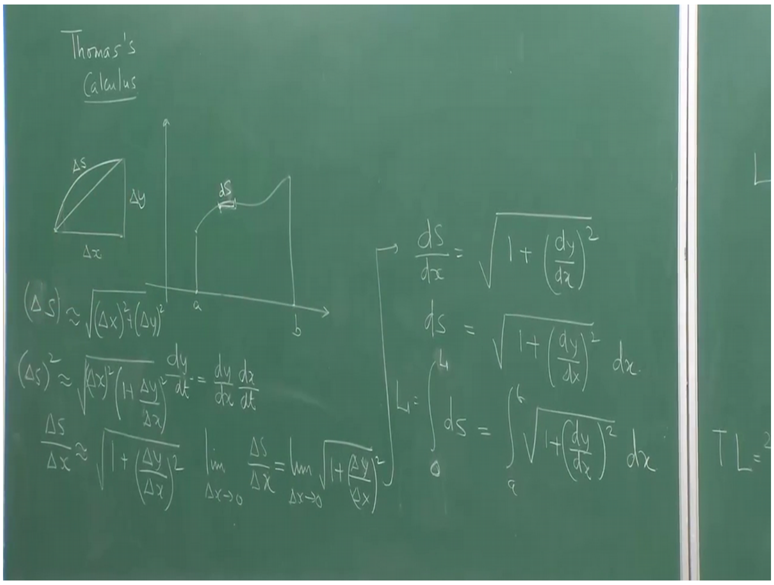

So, what is the length of s. So, length is basically when you take the limits is integral a to b root over dx square whole square plus dy the differential when I am not; obviously, I have not; obviously, I am not told you exactly what the differential is, but or you can do like this. So, this is another way if you do not know even about this standard this thing, this is another way. So, this is what you have done. So, what you do here is that del s square is root over del x square oh sorry root ove not dl s square del s square is this del s is. So, apply of Pythagoras theorem Pythia. So, this is this is actually what you are you are actually approximating the del s or del s by this hype hypotenuse. So, del s is approximately this. So, del x square into one plus del y by del x whole square. So, you bring del s del x, you take the square out is equal to root over not is equal to approximately one plus del y by del x whole square.

But you see as you make this smaller and smaller as the difference becomes smaller and smaller, that is when del x and del x tends to 0 and then del y also tends to 0 because the function is a continuous curve, because small change in x much give you a small change in y what is happening the curve almost approximates the straight line. So, in the limit as del x tends to 0 del s del x in the limit, these 2 ratio this the hypotenuse length almost approaches actual length of the ah del s. So, in the limit the lengths coincide. So, basically this is now equal to this in the limiting form they are same. So, when you are coming to the limit they are same. Because you can also just do it you can also do an epsilon delta argument, in the sense that if you take del s instead of del s you take the del h and del s minus del h that difference could be made less than epsilon. All these argument you can do, but here we are making intuitive jumps sometimes that is also fun.

So, this is nothing, but when you now, what is in this limit it says. What do I get I am rubbing this point? So, I am rubbing this point. So, what I am getting here. From here this side simply means ds dx and because this function is continuous the limit can be bought into it because root axis function is continuous in the positive or then d 1 plus some square, now any square of any number is positive. So, one plus when you take the limit it is nothing, but dy dx whole square. So, d s is equal to root over one plus dy dx whole square dx. So, total length is nothing, but sum up of all the small elemental parts. So, it is a to b d s which is your l total length.

So, it is a to be or rather, I should write 0 to l d s a is not a to b. Because when a x is a the I starting length is 0. And when x is b I have covered the total length 0 to l. So, I moved from 0 to l. So, l is this and this whole square dx. So, this is how you compute the length and that is exactly what we have done using the other ideas. So, it is the same sort of limiting ideas that is pervasive all of calculus and that is what I really want you to show. You cannot figure out any issues like that. I do not want to get into the situation where there could be a discontinuity in dy dx. So, once you have discontinuities has certain issues which ne need to be handled in a different way. Then you have to write y in terms of x you have to do dx dy. So, things are slightly configure it will not get into that.

So, you know how brief and a clear idea how to computed for a curve you can try out and do some calculations yourself taking things from the anywhere on the web. So, we end the lecture now. And then in the next lecture the second lecture of the 7th week. So, eighth week is the last week. So, we will talk about how to calculate surface areas. So, that is exactly what would be our discussion.

Thank you.