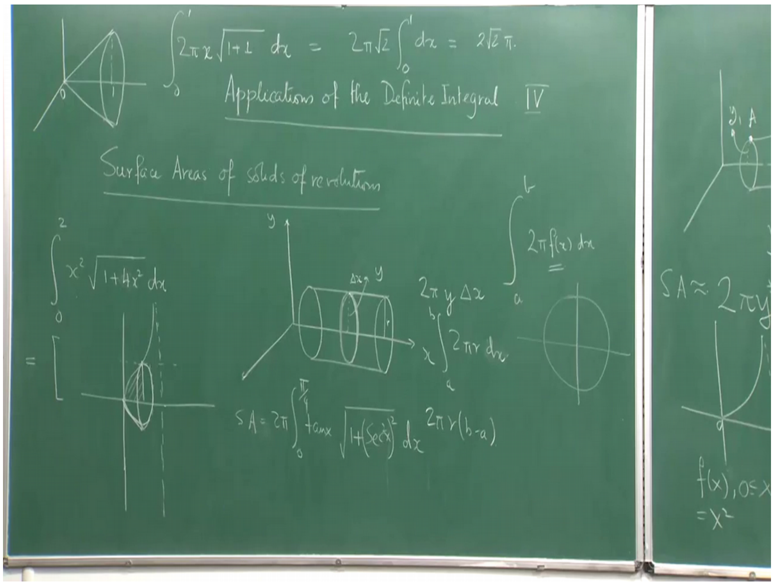
Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture - 32
Applications of Definite Integrals- IV

So, now we are going to find surface areas. We have found length we have found areas under curve. We have found volumes of solids of revolution, but we are now going to find surface areas of solids of revolution.

(Refer Slide Time: 00:27)



So, if you look at a cylinder. And I am supposed to find the surface area. So, what is a cylinder? Cylinder you would take the cylinder is like this, but you cut it along vertically along a line and you open it will be just being a rectangular paper. So, we take a paper cylinder and I will just get a piece of paper which I see here.

So, here I am trying to make a cylinder. So, here is a cylinder of some radius right. Here is a cylinder of some radius. Now what is the length or what is the length of this circumference at the base. These 2 pi into the radius of this circle and that is exactly the length of this line which has which we have used to make this cylinder. So, that is it. So, basically if I cut along this line I get back a rectangle the cylinder becomes a rectangle. So, that is a key idea when you are working with surface areas is that that, suppose you have this situation that is x this is y. And basically I have rotated at a semi a rectangle around this x axis.

So, what I generate is a cylinder. So, this this is a radius r. This is a length of one side of the rectangle or half of half of it. So, then you take any cross sectional area of a cross section of this basically is chopping it off. Then what is the length of this? Basically this is for rectangle this is same as 2 pi r. So, basically. So, what is the surface area? You basically move this small area over this whole domain basically you have you chopped of the whole thing into small pieces and each surface area if you take the all the surface area of the small pieces small chopped pieces and some them up to give you the surface area of the whole thing.

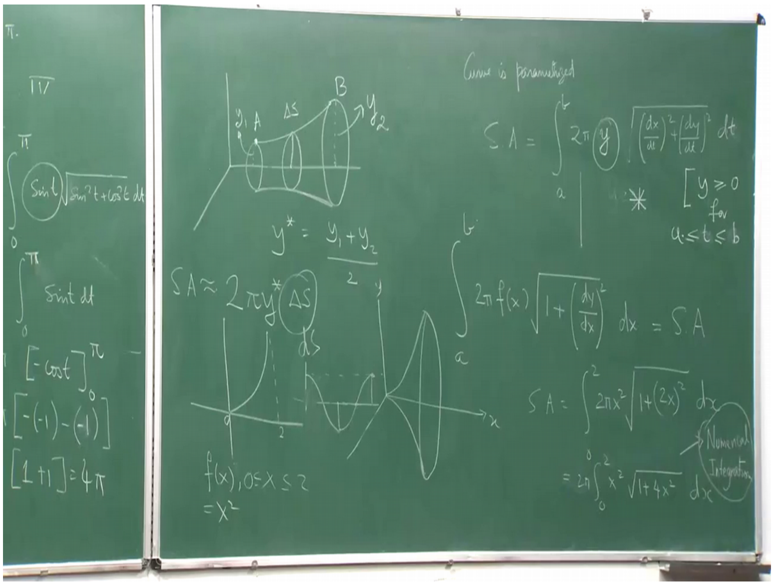
So, that is the key idea basically you have. So, this is your y basically y or r whatever this is your y and this is your say change in x. So, you know how to. So, it is basically 2 pi y that is exactly in do this. So, how do you. So, 2 pi I y is the surface area. When I make the change of del x right. So, 2 pi I into del x when I make a unit change just for one. So, so 2 pi y is my surface area. So, those 2 pi y surface areas, if I can some them up over this whole length, that is integrate them hum. So, basically it looks like it should be something like this something like 2 pi i fx integrate them from x varying from say a to b that should possibly give me the surface area. Because 2 pi r is the circular surface area that I am anniversary in the body. I am not going to get into the details of how our surface area is done.

But a surface area could be like this for example. So, in this case how do we come to this sort of formula see what happens. So, this is. So, if I move over delta x zone they essentially you are thinking that delta x as slices each having surface area to pi x. So, when I am moving over a length delta x my total surface area is 2 pi into delta x that is the intuition. And hence if you sum them and take the limit as delta if y is nothing, but fixed y in this case some or f x if you want only f x or y. Or r 2 pi r delta x if you sum them up for the cylinder and take the limit as n tends to infinity as delta x goes to 0 this is nothing, but the integral say is a to b.

So, that is and this becomes dx. So, it is 2 pi r b minus a for the cylinder, but it is not that is the only thing and this is the cylinder is the only one whose surface area we are trying to find. Surface area could be like this also where you know this thing is not straight.

And you know these do not have equal surface areas in the surface areas changes where this fx would player start playing a role, right.

(Refer Slide Time: 06:26)



And in this case suppose you have a scenario like this I will generate the ideas intuitively for you.

So, 2 points A and B, I take the 2 points A and B. So, these are the solid of revolution right x axis is actually going through the middle in drawing might not be fine. So, I have chosen 1. So, this radius might be y 1 this circular radius must be y 1 and this circular radius is y 2 drawing is very bad I accept it this is y 2.

Now, you see this area of this sort of this or length of this circumference is changing over this length. So, this is a part of this whole object set the whole object may be very big. This length is del s. And what I do I look at into something some of them something of an average right. I look into what could be. So, this is varying right this is this to this is varying the shape of the circle. So, the circumference is varying.

But here again comes idea of tagging. So, you taga point. So, the idea here is that here is the curve is changing there is a del a del s the change in the arc length. So, here there is a small. So, here when I make the change from this to this little big normally something which is in between average which more most of the lengths are near to that length. So, we find some y star, which is average of these 2 and take that y star and that area as a representative one and we say that the area of this frustum. So, the surface area of this frustum surface area of this part is more or less same as 2 pi y star into del s.

So, I assume that more or more all of them are almost of this area of the length of these are on the average that is this. So, it is on the average the total surface area is this it is approximation it is a good approximation by the way you cannot say this is exactly hum exactly the surface area, but hum now, how to handle this situation from here. What can you say how to make intuitive jump? The intuitive jump would be that essentially what you do is you integrate 2 pi. So, you just cannot take on average point you take fx as a function value changes which is y into d of s, but what is d of s it is root over d s this this as is become smaller and smaller. And it becomes d s is represented as d s this is nothing, but dy by dx whole square dx and where x is moving from wherever x 1 to x 2 or a to b whatever you want to say.

So, this is the surface area. So, just by using our intuition how much can be achieved. So, here we have obtained the formula for a surface area by just by intuition. So, I am not getting into the nitty gritty every details of proofs. Of course, the time is less if you just do the proofs, then we do not know examples, but again the idea is you have to see how does the idea comes from. There is a lot of approximation built into it which and the approximation vanishes when we take the limit that idea. Approximations hold in when you are looking at a very small piece of the whole structure.

But then in the limit the approximations vanish and they give an exact answer right. Now in an example that I want you to find. Now this is what you have for a curve y equal to fx now suppose I have a curve which is parameterized. So, if the curve is parameterized like a circle and the t is the parameter and what would be the formula. The formula is simply this, that the total surface area s a is integral a to b 2 pi f of x, into their length is given by what in the elemental length is dx dt square plus dy dt square y fx is y that is all as simple sorry this not s is dt.

but what is f x you do not write fx, fx is y. So, y is some g of t basically. So, this is how you write the case when the curve is parameterized. Here where the curve is non parameterized. Let us take a simple example and see. So, you take the curve fx equal to x square between x lying between 0 and 2. So, fx equal to x square, x lying between 0 and 2 here is an exercise problem in the Thomas calculus. So, when I am writing. So, I am trying to find when x is between 0 and 2. Basically now what I am doing, I am rotating about the x axis. So, if I rotate about the x axis the surface area finally, So, this is x this is y the surface area would look like some funnel something like this.

So, again what is the area for. So, I am taking the function fx equal to x square x is between 2 and 0 and I am trying to find the area fx sorry fx, fx equal to x square and x is between 0 and 2 and I am trying to find this area it looks like a strange a funnel type of thing or a trumpet type of thing. So, in that case what I what will be my surface area a say would be 0 to 2, 2 pi x square fx root over 1 plus what is dy dx dy dx is 2 x whole square into dx. So, immediately see it is 0 to 2 2 pi is out x square in the root over 1 plus 4 x square dx.

Now you need to integrate this, I do not see a very simple way how you can integrate this. At least it is not coming to me it is a. So, here integral might be complex or may not be. So, maybe I will first try to do a little bit of pi parts and see what happens. Still it would be very complex because here where root and if I yeah if I do not integrate this. I then I am in a very bad situation if I integrate this out this is all right then some cube will come and then. So, if I in try to integrate this root over 1 plus 4 x squared dx I do not know a function whose derivative is this.

So, you cannot do it in terms elementary terms. So, you see a very simple looking function it gives rise to a surface area which cannot be computed directly in elementary terms. So, what are you going to do in such a situation? So, such a situation you have to talk about numerical integration. So, here I am bringing you to the reason why we need to talk about numerical integrations

So, you numerically solve this problem. You partition the x axis y axis do something let us say for several rules Simpsons rule trapezoidal rule, we will come to all these do not worry, but this is what it is it looks to me if I am there I would like to go for a numeric integration. I think just symbolic integration just using anti derivatives is not possible I do not know a function whose derivative would do play x square into root over one minus 4 x squared, it looks quite fearful to me. Let me try with another exercise will end with the circle today will end with the circle today, where I saw either my surface area of a sphere today, but let us let us. So, you let me let me still try this problem out a bit.

And then try out some problem maybe I would go for numeric immediate, but I think I have lost the patience, but let me just go ahead 0 to 2 x square root over 1 plus 4 x square d x. So, if I try this out I do not know because I keep x square I do not integrate I take this as a first function and try to integrate root over 1 plus x square, what would be the integral. I do not think there is any way because the problem here is this if here this was a higher this was a higher power than this it was x cube. Then I could possibly use something because I could make the derivative of x a cube equal to this.

so, but here the powers are same. So, I do not know what we need to do with this, but to numerically integrate. So, I do not know of a scenario which is good for this. So, some other little looking stuff, for example, tan x between x lying between 0 to pi by 4 this is an exercise problem. So, I am again just trying. So, if you have tan x how does tan x look like tan 0 is 0 goes like this is pi by 2. And tan pi by 2 and pi by 4 is 1. So, if you rotate this around the x axis we will have some strange looking stuff. So, if you rotate this. So, this is this this is you this is your object this sort of sort of some strange looking stuff strange looking it is like a bowl. So, here what I will do is at tan x business.

So, it is again the surface area is 2 pi 0 to pi by 4 tan x into root over 1 plus dy dx the derivative of tan x. So, it is cotx. So, dy dx I forward derivative of tan x. What a strange guy I have become that forgotten the derivative of tan x. So, maybe just tan x it is derivative is cotx sec square x. So, root over 1 plus sec square x sec square x whole square d dx of tan x is sec square x sec square x whole square dx. So, you see here you again have these sort of strange formulas and so sec to the power 4. Now if you look at the derivative of secs where sec to the power 4 x, then it is for sec x into sec x tan x. So, here immediately you do not have the derivative.

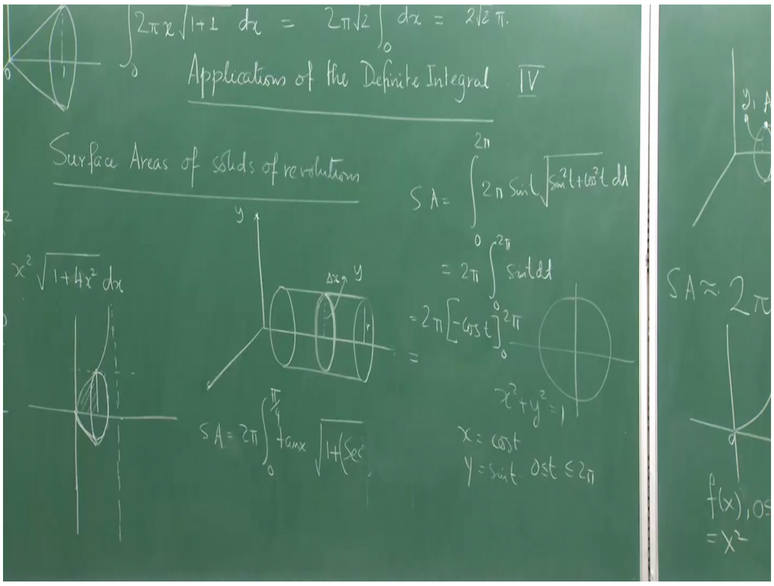
So, again it is all a numerical setup that you really have to go into. So, take up very simple situations. Let me see where you can do something like y equal to x is the line y equal to x. So, if you rotate it you form a conical structure and you are asking it surface area. So, it is 2 pi 0 to 1, 2 pi f x is x root over 1 plus fx square is one dx 0 to one. So, this is handle level. So, it is 2 pi into root 2 0 to 1 dx. So, it is 2 root 2 pi, is on is the surface area of the cone on the conical structure.

So, you see several examples where you can solve several examples where you cannot solve the things are very bad. So, we end our discussion today. So, here something is missing numerical integration. So, we have to now get into our hands in what is called numerical integration. We should be able to do some numeric somewhere. So, I will try to use a mathematical software called maple to show you how you can do symbolic integration or you can do numeric integration, when I start numerical integration which would be the end of our really integration part and going to more advanced stuff.

So, let me now talk about the sort the spherical the sphere all right. So, what is the sphere how is it obtained. So, you take it on the x axis you take the circle and rotate it you get the sphere. So, you take a circle and rotate it, around the x axis you get the sphere this is exactly what you get or when you have is exactly what you get. When you have all right when you this is exactly what will happen when you want to get a sphere you rotate that circle you get the sphere that is exactly what we discussed yesterday that was that is what I was supposed to tell you.

now here we will use the parametric business again. So, we will use the simple situation where x square plus y square equal to 1. Because I do not want to get in this our business is just an additional thing.

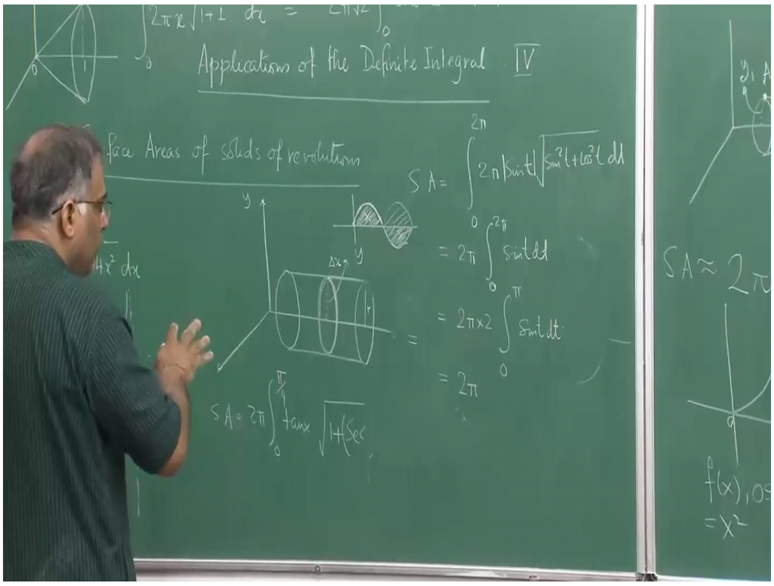
(Refer Slide Time: 24:19)



So, x square plus y square equal to 1 this is been. So, let me write x equal to cos t y equal to sin t. So, I will go by that formula s of a is equal to 0, 2. Now 2 pi plus t varies from 0 to 2 pi that you already know and then I have 2 pi into y is sin t. And the length as you know we have already done it in the last thing which is sine squared t plus cos square t we have done it in the last class when you are talking about length.

So, what do you have these 2 pi 0 to 2 pi sin t dt. So, it is 2 pi minus cos t 0 to 2 pi. So, cos of. So, I am finding the surface area. So, now, you see if you do this you get surface area to be 0, looking on cos this is 0 minus plus 0 zero 1. So, it will become minus 1 cos 2 pi is cos 0 which is same as one minus 1 minus 1 plus 1 into become 0. That is very strange the idea is this the idea. Then how do I get the surface area the idea is that if you are doing that idea a is not to look into this because surface area has to be positive surface area cannot be 0 or anything because really finding an area, you have to observe that this curve sin x when you integrating from 0 to 2 pi is like this.

(Refer Slide Time: 26:22)



So, one area is positive one area is negative in both the areas are actually equal one is negative area one is positive as I said you know in counter clockwise and then immediately you will know that they cancel each other. So, we do not want this. So, what we want is the modulus of sin t basically. So, essentially what we want is we just have calculated one of these areas a sin t is positive. And then you take the double of these things that is what we want. So, basically we want modulus of sine t. So, modulus of sin t would give you like this. These 2 areas are what we want. So, otherwise surface the surface area would have meaning. So, this is 0 to pi by 2 because you have to understand that were here sin t only.

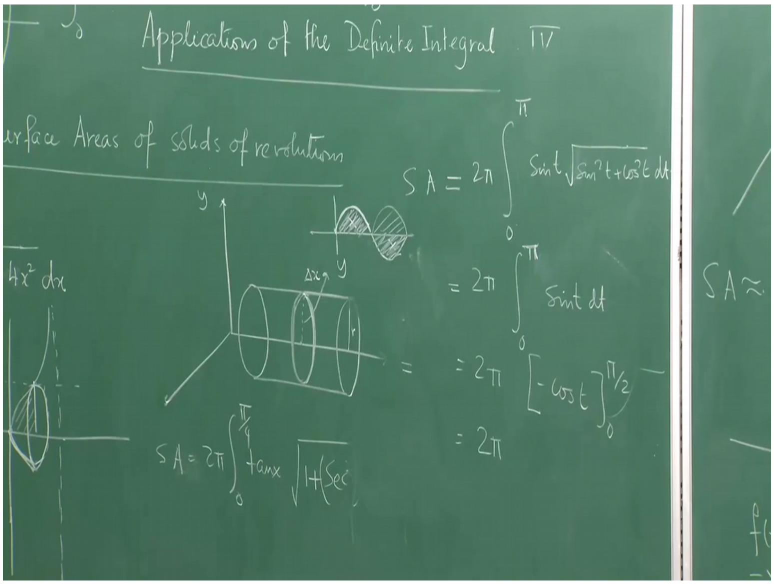
So, 0 to 2 pi sorry 2, 0, 2 pi 2, 0, 2 pi. And this is something important. So, instead of a 0 to 2 pi you have 2 0 2 pi. So, if you do this. So, let us look at the just 0 to pi what is the integral. So, basically I am taking the absolute values right. This is something very important that we should observe that why when take it has to be positive. This y has got to be positive. So, without positiveness you cannot get the things see this. So, so what we are going now learn, that if I do not have my y value positive throughout my domain of integration, then I will get 0 answer in these sort of situations various symmetry.

So, that is not a good idea. So, in this formula this formula does is valid if y is greater than equal to 0 for x belonging to for t belonging to b to a. So, this is something very important that you have to understand that, I have to only choose those wise that y should be poor non negative when t is belonging from b to a. So, you see getting stuck in this problem that you just directly integrate you get 0. So, you may come on man how can I surface are be 0 you understand that there must be something which we have missed here. So, now, if this is our mathematics is build. So, here I am getting something which is not matching with reality. So, there must be something which I need to put here assumption.

So, that the whole thing finally, matches with the real stuff. So, here y must be greater than equal to 0 you see here. I just want to look at sin t then between 0 to pi and 10 from 0 to pi. So, basically 0 to 2 I mod of sin t. So, both the positive parts. So, this is so, only if I write it in this way only after I take to into the positive part because these 2 areas are same, I get the I basically get the area under this curve basically I should write not sin t I should write mod sin t. So, basically how do I tense find the surface area. So, I just find the surface area of a little part where my angle varies from 0 to pi by 2.

So, between 0 to pi by 2 my stuff is positive and then I just multiply it by 4. So, this. So, you see what I now see that this sin t is positive.

(Refer Slide Time: 30:23).

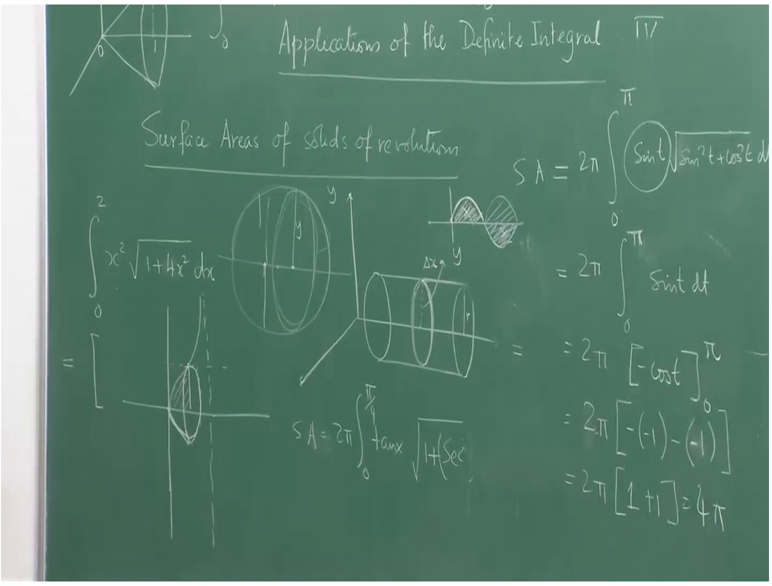


Because we have written yx equal to sin 2 y cos 2 y will sin t sin t is positive when x is between 0 and pi by 2 or even 0 and pi right. So, sin t remains positive and then it goes to negative. So, if you have just does to 0 to pi by 2 sin t s into root over of cos you know 2 pi the outside sin squared t plus cos see one single problem just the circle fear is teaching. So, many things.

So, these are mathematics is done by guess. And test to try to do something fails you see the it does not match with reality you go and do the corrections there and then try to figure it out how you will do it. So, 2 pi. So, this is our stuff (Refer Time: 31:14) in math I guess and test. So, so now what I do? Now I do the same integration now. I 0 2 pi by 2 sin t d t. Let me see what does it give me it gives me 2 pi minus cos t 0 to pi by 2 cos of pi by 2 is 0 and cos of 0 is 1. So, it is equal to 2 pi. So, that is one answer. So, you have got 2 pi. So, basically 2 pi is the surface area of what when sin is positive, that is if you also wanted to put pi here then it would have become minus 1 minus 1 2.

So, essentially you have to just bother about this part when. So, now, what I have to do I have to just see multiplied by 2 to get the full answer. So, I have got the half of the thing. So, here. So, what I have done. So, 2 pi is not the answer the answer should be 4 pi r square that is what I know. So, sin t you know is positive between 0 to pi. So, integrate from 0 to pi do not integrate from 0 to 2 pi. So, here you will be only restricted to the case where y is greater than equal to 0. So, if y is greater than equal to 0 between thus a interval a to b you integrate within that. So, y is greater than 0 between 0 to 2 pi 0 to pi. And that will give you the surface area do not take the whole range here right from 0 to 2 pi. So, we do this is simply means cos pi is minus 1, minus 1, minus 1, plus 1, minus 1, minus 1, plus 1, minus, minus 1. So, if you have this sin from 0 to pi. So, let us do it what happens surface area is 2 pi into minus cos of t 0 to pi.

(Refer Slide Time: 33:21)



So, this is 2 pi into cos of t is cos of pi is minus 1, minus 1, minus 1. So, it is again giving me 0 minus 1, minus 1, plus 1. This is minus 1 cos also minus cos of t. So, it is minus 1. So, it is minus cos of t minus 1. So, it is giving me 2 pi in to minus 1 or 1 plus 1 and minus 1 minus 1 plus 1. So, it is 4 pi. So, you see by choosing the range, where sin t is non negative I am getting the area sorry when I am say pi by 2 that is not correct it is pi. So, have to choose the range where sin t is. So, if I chosen pi by 2 you have to multiply by 2 because I have just broken the whole thing into 2 parts.

So, the surface area when you do this using when you use a parametric curve you only have to consider you this is only possible this integration, is possible if y is greater than equal to 0 in your range of integration. So, what you have to do you have to get the whole thing done where this is positive. You have to actually take the range of integration where sin t is positive. Only in that case you will get the surface area other otherwise you will not get it. If it is not positive because that is what we are doing because if you look at this what is this this is the positive part of the x axis not the negative part and then we are multiplying with the positive part.

So, I basically I have to look into the case, but sin t is positive basically surface area of sphere were taking it like this. Here this y is your sin t, but sin t here the y is positive here. So, that can be that is possible when you have varying between 0 and pi. That is when your angles are varying between 0 and pi you are only looking at those wise and then you are calculating the areas because you would take the y in the negative side you have to take the negative value because this is a we are in the coordinate system, but you will you do not want a radius to be negative you want to take only it is length which is just the positive part.

So, it is enough to integrate and between. So, you know enough to look at the whole thing where the whole thing is positive because they are just to be taking this the circumference of this and you are just moving this circular disk throughout; obviously, the y keeps on changing as the t changes, but you are moving it and that summing above those lengths each of those lengths if you small lengths if you gather them up they form an area and that is exactly the idea that is why needs to be positive this is something which is very important and has to be kept in mind with this.

We end our discussion of the second lecture of the 7th week. We have to do something more about numerical integration that is what (Refer Time: 36:50). Third one in last 3 lecture would be on sequence is this Taylors (Refer Time: 36:58) all this stuff. And the very last lecture, I hope, I have time you proved the rationality of pi which will show you the application of integration application of Taylors everything to prove one concept that the number pi is not rational. So, we are told you have no number pi has been generated showing that c by d some component by the diameter is constant.

Thank you.