Calculus of One Real Variable
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Lecture – 33
Numerical Integration - I (Trapezoidal Rule)

Welcome once again. We are essentially finishing our discussion on integration on which we are a huge one. We after this we are going to discuss something which is very, very fundamental and where we are going to make a transition to higher calculus of analysis. So, here we are trying to answer this very important question. You have seen that there are many, many functions which are not the derivative of some other function. So, you cannot use elementary methods of integration to really speak about them.

So, in such cases what are we going to do? Or how or is it that we can not find the value of the area of such functions, may be they are non negative functions can not we find the value of the area under such functions, can not we do that? Or can we not even try to get an estimate or approximation of what is the area? So, for such cases for example, if you talk about c to the power minus x squared dx as we told. So, it is a perfectly nice continuous function. So, if I want to know which and it is positive. And if I want to know that what is it is area between 0 to one what is area under the curve. I just can not do it by elementary methods because either it is not a derivative of any integral a or any function. So, it is not antiderivative you can not find it is antiderivative.

So, in such cases what is the recourse? What recourse we take? And the recourse that we take is numerical integration. That is we will use some technique this is the same technique by which integration is defined to build sort of rectangles, and then try to find the area and we say this is a good estimate of the area.

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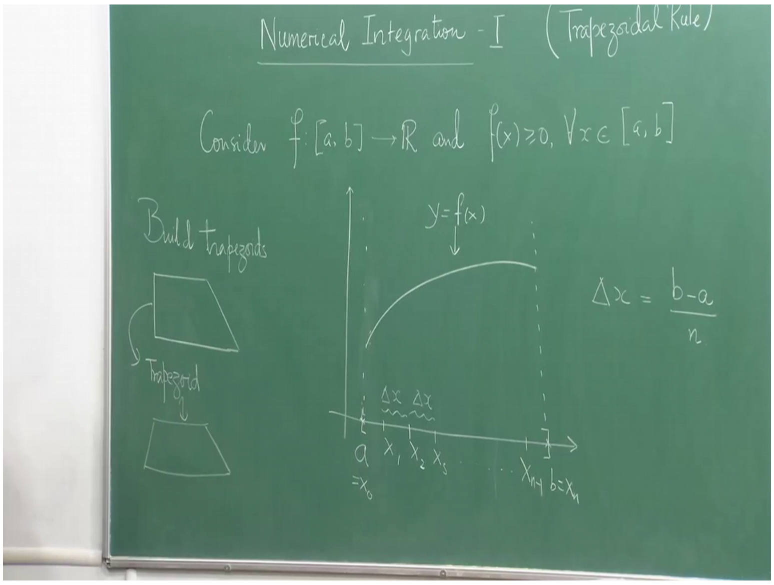

So, first is a trapezoidal rule where we view things in the following way. So, we consider a function f from a b to R and f of x is greater than equal to 0, for all x this is a sign of for all this inverted a. So, maybe the function is something which we are not. So, for such functions we were where we do not know the ant derivative these are very useful tool. And in certain cases the function would not be given to you. For example, in physics it is well known that temperature is well known that temperature is a continuous function of time right.

So, as time varies say you take the time now T equal to 0 and you record the temperature till midnight. Say from 12 (Refer Time: 03:21) in a known and then you record the temperature, you would see it varies. So, if you as you see if you make your 2 recording time difference very small the temperature variation is very small. So, from that from those sort of experimentation physicists gathered the fact that time is a con sorry, temperature is a continuous function of time, and obviously, it is non negative, but suppose you just record the temperatures you do not know, you do not know exactly what is the relationship right. Then you want to know what the average temperature in the day, how do you do So? And is exactly in those sort of situations where numerical integration actually comes into play.

So, here let us look at this function. So, here is my a and here is my b. Do not bother too much about the drawing, I am just trying to make it way. So, let this be the function right. So, this is my y equal to f x, this is a graph. So, what I do in numerical optimization sorry in numerical integration techniques, what I do is that I divide a and b into equal into like put partition points in such a way that this difference between this a x 1 length is same as x to x 1 length. And so and so forth. And I will say that though it might not look in the picture each of them has a length delta x. So, in this partitioning what I have done is So, delta x So, suppose I have n partition. So, delta x is nothing but. So, this is what delta x So, is hole b minus a is divided equally into n partitions. So, each partition of length is delta x. So, there n partitions n into delta x is u minus a, because there are n minus 1 points taking a and b into consideration.

Now, what should be my next step what, what I want to do? What is the meaning of trapezoidal rule? So, you know of this term called a trapezoid from Euclidean geometry. So, we are going to build trapezoid. So, we are going to build trapezoids, instead of building rectangles what we had done while we are trying to explain what integration is all about, we build rectangles build trapezoids. So, trapezoids are if you if you I we want to recall our quadrilaterals whose 2 sides are parallel well other 2 sides need not be parallel. So, every parallelogram is a trapezoid, but while every trapezoid need not be a parallelogram. For example, this is a trapezoid, but not a parallelogram. So, this is an example of a trapezoid, these are example of a trapezoid, these are trapezoids.

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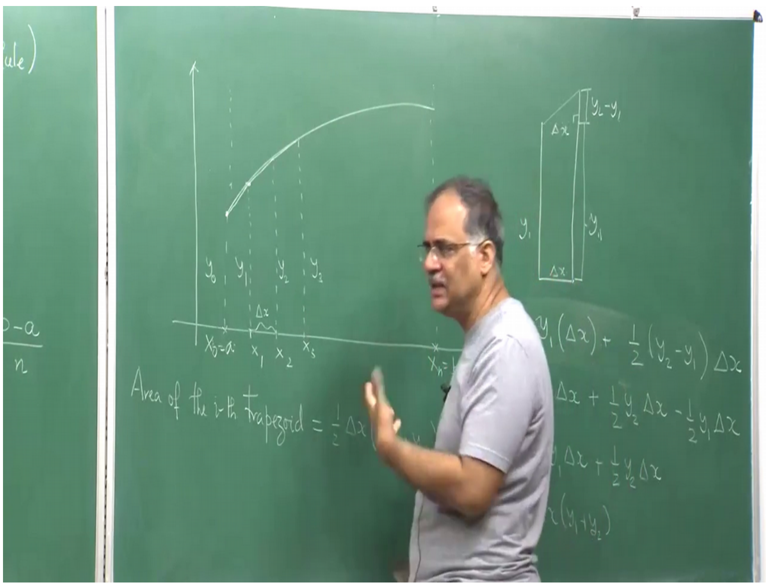

So, let us see what we do here, I am sorry. So, here is the function, now what is the idea? So, here is my x 1, here is my x 2 and so on. So, what I do? At every point I draw the function values I go up to the function values on the graph, and in x 2 I go up to the function value on the graph from x 3 for example, I go up to the function value in I go up drop to the function value drop to the graph. Now you see I have 2 sides parallel, but I start joining these 2 points on the graph. So, I joined a and my x 0 and f x 0 I have joined x 1 with x 1 f x 1. So, I have joined So, this is x 0 f x 0 and with x 1 f x 1. Here I have joined, here I have joined, you see when I am doing this a very, very little area left I have some under the curve very little amount of area gets left. So, there will be always an error because you are trying to estimate there will be always an error.

So, you have a trapezoid here which is of this type. So, each of this trapezoid for example, this trapezoid has a base. So, for example, I will call this as y 0, this length is y 1 this is y 2 and this is y 3 and so on, is a function value is basically f of x 1 f of x 2. So, basically I am calling the y 1 y 2 y 3 just to keep in mind that every time I need not have a function given to me it could be just some x and y s. And so that is why we are calling it as this one.

Now, how what is the area of this say this one? This rectangle is y which has parallel sides y 1 and y 2 how do I find? So, let me look at it in this way. So, this is my delta x, this is my y 1, this is my y 2. So, what is the area? This is draw perpendicular here. So, that will be this. So, this will part will be same as y 1, this is because now it becomes a rectangle and this part becomes same as delta x. So, here the area of So, this part is now y 1 and this part is y 2 minus y 1. How I find the area? So, first I find area of the rectangle then area of the triangle and sum them up that is the idea.

So, area of the rectangle is delta x into y 1, plus or y 1 into delta x if your more come I think that is more comfortable way of writing if you might just get confused it is y 1 into delta x plus area of this triangle. So, it is half of delta x into oh, sorry or y 2 minus y 1 into delta x 1 how base into altitude. I am just writing it you can write like that I am writing it. So, that it looks much more cleaner.

Oh shit, now what does this give me? It gives me y 1 into delta x plus half of y 2 into delta x minus half of y 1 into delta x. So, it simplifies into half of y 1 into delta x plus half of y 2 into delta x. And So, beautiful formula comes out which says that this is nothing but half of delta x into y 1 plus y 2 (Refer Time: 11:52). So, you once you know about one of them, you know about the other. So, the ith trapezoid. So, these are a year of the second trapezoid. So, area of the ith trapezoid, say what is area of the ith trapezoid? The area of the ith trapezoid is given in a very simple way. It is just the same thing half delta x and you know what to do y I minus 1 plus y I, as you replace too with I n one is of course, 2 minus 1. So, it is I minus 1.

So now how I am going to estimate this integral right. So, basically what I am going to do? I am going to add up the same areas in trapezoidal areas, and say that that is my estimate. So, here we have in robbed and got the picture done again.

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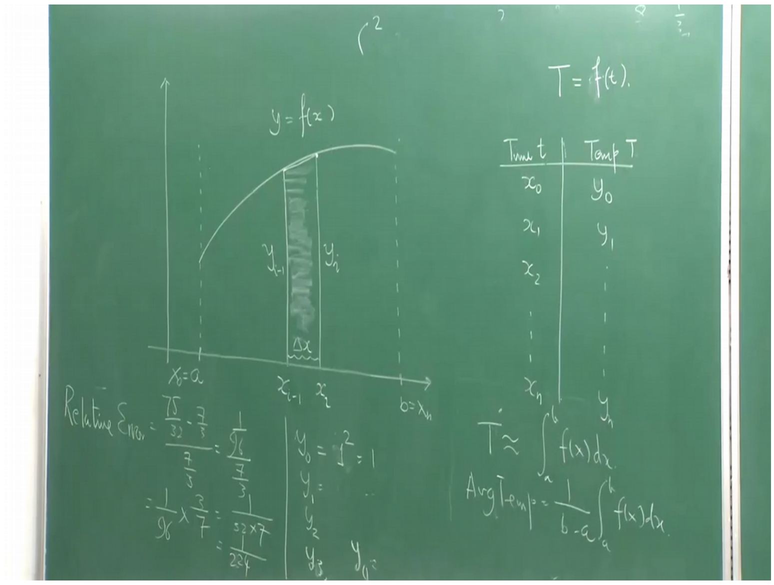

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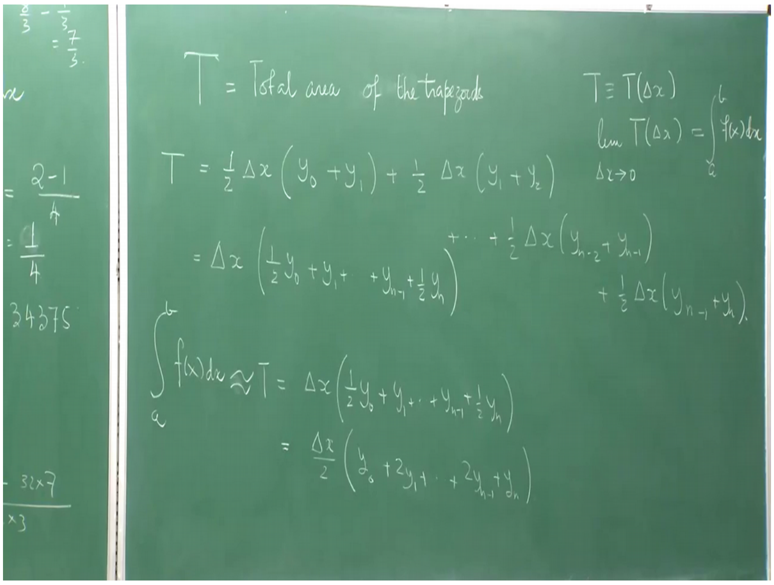

So, let us write T as a total trapezoidal area t, total area of the trapezoids total area of the trapezoids, under the trap made by the trapezoids or this total area of the trapezoids.

So, T of course, is like this half of delta x which is of course, common in everything y 0 plus y 1 first rectangle plus half of delta x y 1 plus y 2 plus so and so, plus half of delta x y n minus 2 plus y n minus 1 plus half of delta x y n minus 1 plus y n. So, that is clear they just summed up all the n rectangles.

Now, if I sum them up you see all of them is half and half adds except the endpoints y n and y 1. So, y n and y 0. So, I have delta x into half of y 0 plus y 1 is a neat formula plus y n minus 1 plus half of y n. So, this is the trapezoidal rule, that this integral a to b f x d x is it is very easy to implement, is approximately given by T which is nothing but delta x into half of y 0 plus y 1 plus y n minus 1 plus half of y n.

So, let us take an example from Thomas feeny and sorry I keep on telling Thomas feeny now all Thomas has calculus, and I show you how this can actually be done. So, we will discuss first of an integral which is very well known you can you know how to compute that integral it is very simple it has an anti derivative, or just to check with you that how effective this method is. So, for that we are going to consider this integral say from, and you know what the answer is to this. So, use one is. So, I will compute this by using the trapezoidal rule.

So, you see the whole point is that even for such functions you do not bother about anti derivatives. Let us see this of course; you know what the antiderivative is. So, will make a check of what is the error. Now let me take n equal to 4 points. So, divide this interval between 1 to 2 with n equal to 4 points. So, my n is 4 say as you could make it 5 or 6 or 7 does not matter n is 4 and this would imply that delta x delta x would very much small. So, it will be 2 minus 1 by 4. So, it will be delta x would be 3 by sorry, 1 by 4. So, 1 by 4 is quite a small length. You can make it smaller of course, that is a different issue. And So now, if I go by the trapezoidal rule for this I use this formula.

So, in n is divided into. So, what are what have I done how I in this particular case is 1 to 2 it is partition into by 4 points. So, 1 1 plus 1 4th right. So, it is a 5 by 4^{th}, 5 by 4 plus 1 4th it is 6 by 4. And 6 by 4 plus 1 4th.

Student: 7.

7 by 4. And 7 by 4 plus 1 4th. So, the points we have given is basically we have divided into n is the 4 4 intervals that we have divided into 1, 2, 3, 4. So, we need 5 points.

So, basically what you do? You have to know what is y 0, what is y 1? So, what is y 0? It is x 0 that is 1 square 1 y 1 and also y 2. So, you have to know y 1 y 2 y 0 y 1 y 2 y 3 y 4. So, 4 points up to y 4 y 3 y 4 y 3 and y 4 these points have to be calculated. So, I am not calculating it I am assuming that such a calculation can be done many you can easily do it yourself I like to tell you to do it yourself other than a relying on my calculations.

So, this can also be written in a slightly different way you can take for example, del x by 2 out. And make a more effective writing and write this as y 0 plus 2 y 1 plus 2 y and minus 1 plus y n. So, you also you can write in a same way. So, you put del x is del or x is 1 4th. So, 1 4th by 2. So, 1 4th divided by 2 is 1 8 and that one 8 into 1, who this is 1 plus 2 into y 1 minus y 1 means 5 by 4 whole square which is nothing but 20 5 by 16 and so and so forth. So, if you do that in this case T turns out to be, as given here is 75 by 32, which is a rational number which is now expressed in decimal is 2.34375. So, it is a terminating decibel.

Now, what this is a number that we have got, but what is the that integral? So, if we compute the integral, if I compute the integral, then this integral will turn out to be this, x cube by 3. And that you raise down from 1 to 2. So, it will become 2 cube by 3 minus 1 cube by 3, which is 8 by 3 minus 1 by 3 is 7 by 3. 7 by 3 also around the same sort of number. So, what you can now do basically you can. So, there is something called relative error relative error means, what is the error per unit? So, basically 75 by 32 minus 7 by 3 is absolute error.

So, what is the error? So, if I calculate it out what would what would it come? So, I am just writing, writing it down I am just not getting into it will become 75 just maybe I should do the computation, 75 into 3 minus 32 into 7 divided by 32 into 3. I think there are much more expert computation is on.

Student: 1 by 96.

1 by 96. So, it is 1 by 96 and good, it is 1 by 96. So, basically you know this is my error, you see the error is pretty, pretty small. Not very, very large. Maybe you can improve it by taking more points. And you see that this error or there something called relative error this is like a unitary method relative error means that nothing but 75 by 32, minus 7 by 3 divided by 7 by 3. So, it is equal to 1 by 96 divided by 7 by 3 which is nothing but 1 by 96 into 3 by 7. And that turns out to be 3 I can cut. So, 1 by 32, 32 into 7.

Student: 20 one (Refer Time: 23:01).

1 upon.

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7 to the 2 to 4.

Student: 2 to 4.

2 to 4 right 2 to 4 is 1 upon 2 to 4. So, relative error it is 1 upon 2 to 4 which is you see pretty small. So, this method is robust in some sense that it is easy to implement it gives you. Of course, there is a chance of improving this method, but this is pretty decent.

So, let us just show that if I take as delta x as n goes to infinity it has means which I mean delta x is length would go to 0. You can just look at the definition of delta x is b minus a by n. So, as n goes into b minus is fixed. So, as n goes to infinity delta x goes to 0. So, as delta x goes to 0 that this T this T is n. So, T obviously, T can be written on the function of delta x T is actually function of delta x. So, what we want to show is that limit of T delta x is delta x tending to 0, is actually the integral and that would tell you that how effective this method is.

So, once we do it we will show you what happens if I do not have an exact function, but just have a list of values y 1 y 2 y n of what can we do about it how we handle it. So, let us just prove this (Refer Time: 24:28). So, what can I do? How do I write it?

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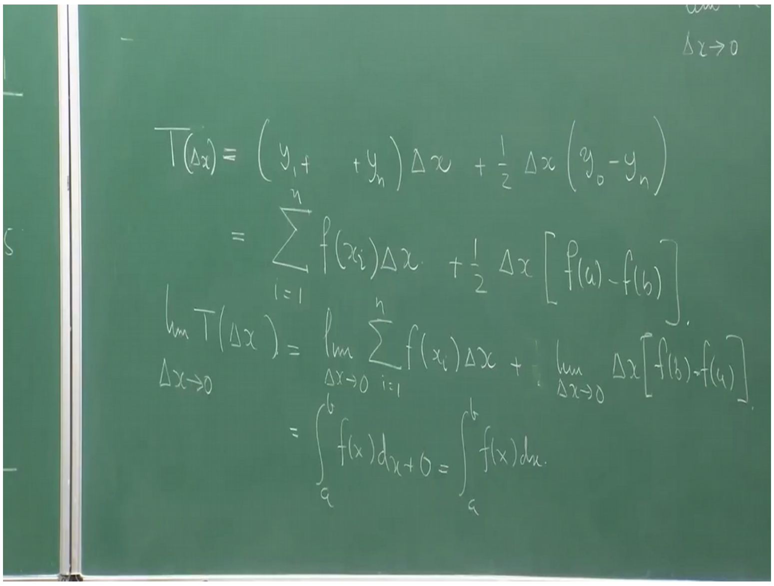

So, my T I can rearrange the T as follows, right? I can rearrange the T as follows as I am not half of an half. I have I can take this y 1 y 2 all these things which are. So, what I do I put y 1 y 2 y n into delta x, as rearranging what I have known for T. Delta x plus half of delta x into y 0 minus y n because y n was half. So, y n delta x minus half y n delta x is half y n delta x.

So, that is what you do, but this what is y 1 what y 1 is f of x 1 y 2 is f of x 2 y n is f of x n. So, I can write this as summation f of x I delta x. So, this is nothing but the this is nothing but the Riemann sum, plus half of delta x into f of a minus f of b delta x ok.

So now as delta x goes to 0. So, T you can see whatever we say and do, T is essentially a function of delta x. Because these are all fixed numbers delta x is under your choice, it is because of how you choose the end. So now, as limit this happens, then what happens is the following? This is nothing but the Riemann sum. So, as delta x tends to 0 this limit. So, maybe I should write it down this is nothing but the integral or even you do not bother out the integral is just the Newtonian limit, because we were looking at a function which is continuous and non negative continuity of the function is guaranteed we are not going to write continuous function continuous function everywhere. So, what does this show? This shows that this is nothing but integral a to b, f x d x plus 0 is 0 at limit. And that is exactly what we intended limit of this is this which is nothing but; obviously, at the end integral a to b.

now I want to tell you that, suppose you want to record the temperatures of say today. This is an example gives the specific numerical example given it there in the book of Thomas as calculus, but you do not need to really bother about too much numerix because you can you can put in the numbers yourself. See for example, if I get something like this, that and that is what would you have for example, any engineering design and many engineering problems you get this x forces y. For example, your mood row looking at the time and the speed of a train then you want and you want to find average speed of that train over given over the given period.

So, were all or this is this is how do how do methodological of his decide temperatures. So, they decide they never give you they record the temperatures, but the overall temperature of the day the average temperature is done through this method. So, here is your time T and this is a temp temperature with capital T. And the temperature is a function of the time T that is a continuous function let us ready the there sorry, and these are continuous function of the time. So, what do you have let me call it to the values of T is are given T at time T equal to x 0 x 1 x 2 x n. So, you have recorded.

So, each hour you have recorded, say n hours for n hours. So, for n, n gaps each n hour you have recorded. Starting at time T equal to 0 ending at set an n it is moment. So, this is your time x 0 this time x 1 time x 2 time x n. So, it will be temperature would be given as y 0 y 1 y n. Then it is the same procedure, it is just the same procedure right. T equal to this So one So, basically once you know the T. So, you know that it is a to b f x d x. And if you want to find the average temperature is nothing but divided by b, b 1 by b minus a that is it you can try this out. So, that is our.

Let us let us give you interesting experiment to do at home those who have a thermometer at home fine, those not take go to the net I am assuming that a lot of you have an access to the net. Those you do not have try to try to do something else maybe you have innovate something else which I can not tell you right now, but those are the net the idea is following. Just Google suppose you live in a city forever here it is Kanpur which could be in Delhi whatever you want. So, you say at every time see the time temperature at time you look at this time. So, they will give you temperature at this time now current time you see record after one year you can see the Google record to up to midnight, and then you can use the same procedure yourself and tell people what is the average time. And sorry, what is what is the orders average temperature. You see how much fun it would be thank you, and next we are in the next class we are going to speak about another rule, which is a simpsons rule and end our discussion of integration.

Thank you very much.