Calculus of One Real Variable

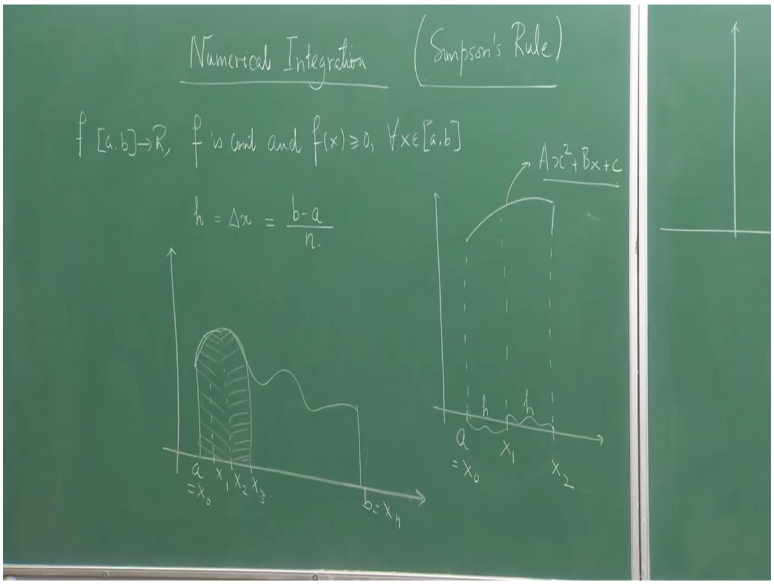
Prof. Joydeep Dutta
Department of Fconomic Sciences
Indian Institute of Technology, Kanpur

Lecture – 34
Numerical Integration - II (Simpson's Rule)

Thomas Simpson who lived only 41 years on earth made this phenomenal contribution to numerical integration why his contribution is important because it improves upon the accuracy that the trapezoidal rule gives you and makes it much more workable. So, instead of so there is a curve right the function is you know when you draw a function y equal to f x something like this or this so these are curves. So, these are not straight lines. So, straight line approximation is not good. So, approximating this by a straight line for example, here to here is worse than approximation that I can make by a curve like this for example.

So, a curve would approximate a curve better than a straight line just simple dictum. So, what he did, so this is how mathematics progresses from the linear to the quadratic term. So, straight line represents the linear term and then you go to the quadratic term. So, from the straight line, you go to the parabola. So, now, instead of having an approximation done here through a straight line the approximation is done using a parabola. So, that was what Simpson's contribution was about and that is what we are going to really work out. So, let me start.

(Refer Slide Time: 02:00)



So, again we will start with a f is continuous. So, f is from a to b; a, b to R, f is continuous and f of x is greater than equal to 0 for all x in a, b. So, again the delta x for reasons of economy in the writing, we will write delta x as h here a step length, which is again b minus a by n. But here our way of looking at the things would be very different. In the trapezoidal rule, we walked with two points y 0, y 1, y 1, y 2 at a time here we will work with three points right and that is exactly what I am trying to show you here that this was the idea.

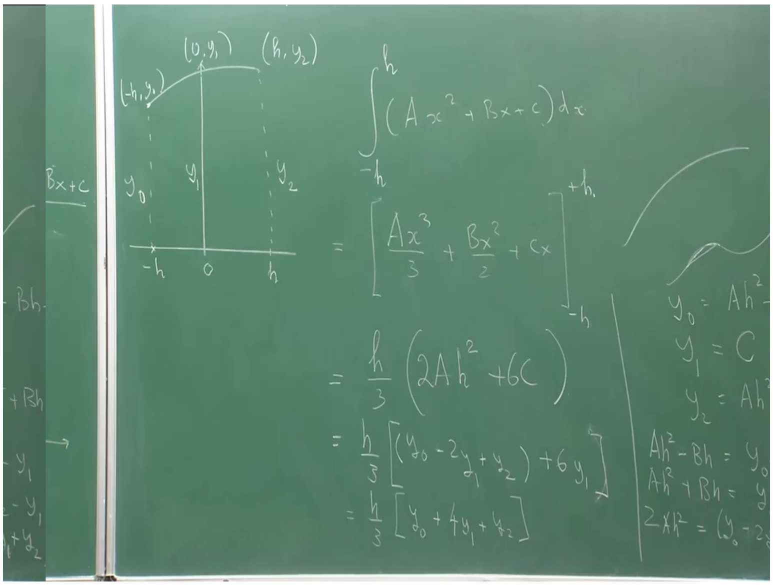
This was Simpson's idea. So, here is this curve. So, for example, no matter what whichever way into draw it you put it like this and just arbitrarily drawing it a and b. So, here is my x 0, here is my x n. So, here is my maybe is this amount, this is my x 1, and then this is my x 2, this is my x 3 and so on. So, what I do, I consider these three points and I approximate the curve here by a parabola, because for a parabola of the form a x square plus b x plus c, I can immediately evaluate the integral, because for which my anti derivative is known. This thing cannot be done, if I take some arbitrary curve, so that is why we have to we had approximated by a parabola, so that is so here is x 0 of course. So, what we have done we have made an approximation here. So, this is this is our approximating machinery.

So, let us look at take this thing up and make a more closer look. So, here is my a equal to x 0, and this is my x 2 with x 1 here. So, this length is h and this length is h. So, here is our function, here is a quadratic function, this is actually of the form A x square plus B x plus C. And essentially we want to find the area of this area; we want to find this area.

Again you take go to the next part, and again you fit by some that curve and again you compute this area might thing is a complicated way it is slightly complicated way, but it has lesser error that is what we would find, one would find.

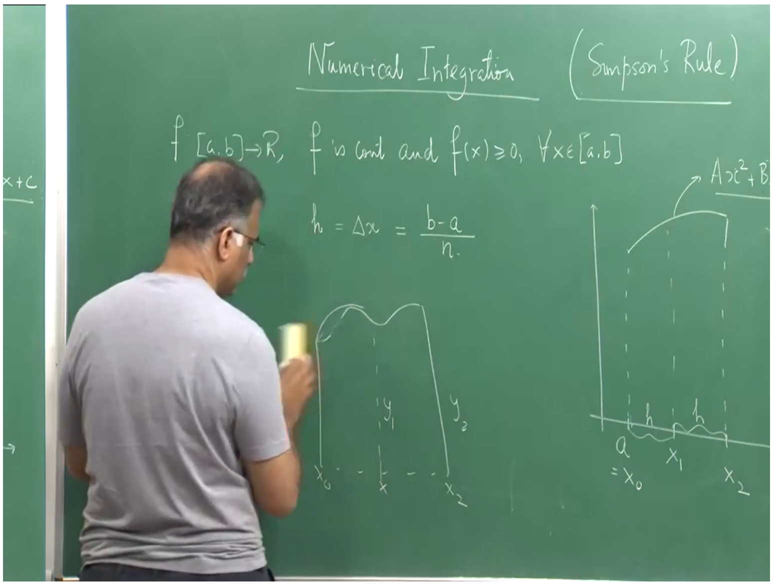
If I want to do actually find out such an area rather than doing it from x 0 to x 1, it is much more easier if we look it in it in this way. Basically x 1 can be considered as a symmetrical point this could be thought of the negative point, this could be thought of the positive points is up to you. Basically, we are making a translation basically we are now trying to compute the area; we are trying to compute the area of this curve. So, I do not know. So, I am fitting it by A x square plus B x plus C, I do not know what is A, B and C, you can choose A, B and C, but I need not. So, without choosing A, B and C, how can I do it that is the key idea some curve A, B and C are not known to me.

(Refer Slide Time: 07:00)



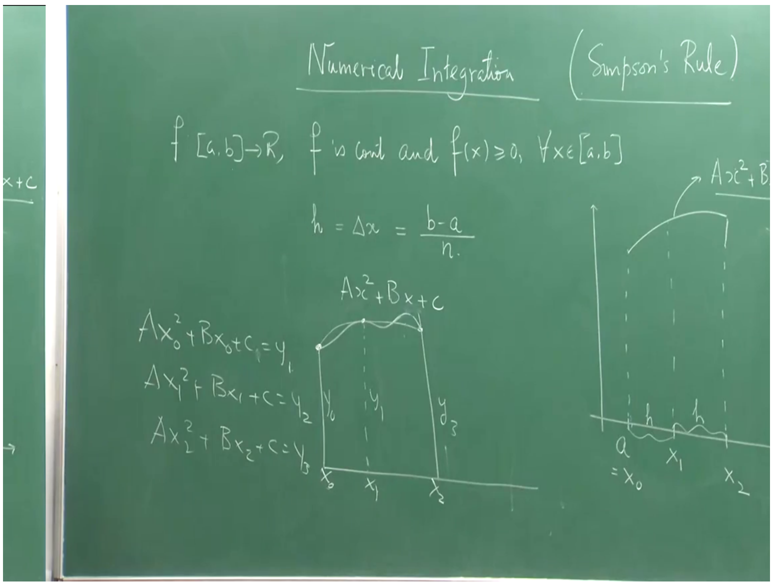
So, here is my curve. So, this is the point minus h this is the point zero and this is the point h. So, this is. So, what is this area. So, call this as y 1, call y 1 or y 0, this is y 0, this should be y 1, this should be y 2. So, this is the point minus h, y, 0, y 1, h, y 2. So, let me just try to understand because every time we are taking a pair of three points here we have x 0, x 1, x 2. So, it would be minus h, 0, h. So, we are tuff trying to understand how to evaluate. So, basically if I evaluate these sort of area under this particular curve right and I am going to add up those areas. So, how do I do knowing what is y 0, y 1, y 2 that is the idea. So, the idea here is the following that I think I should the curve has to bit clearly done wait, wait, maybe I have not explained here properly, I will just do explanation bit clearly.

(Refer Slide Time: 08:50)



See what happens this is the function between x 0 and x 2, and this is your x 1. So, this is your y 0, this is your y 1, this is your y 2 the idea is that you followed draw a curve which is quadratic and which takes these values.

(Refer Slide Time: 09:22)



Just a moment. So, what you divided in such a way. So, the idea is that the curve A x square, what Simpson said that if you want to estimate it. So, you take the curve A x square plus B x plus c in such a way that the function value at x zero function value at x one that is that the partition points coincides with the function value of that quadratic function. So, the value of x 0, so A x 0 square plus B x 0 plus C is actually y 1. So, A x 0 square plus B x 0 plus C is equal to y 1, while A x 1 square plus B x 1 plus C is equal to y 2 plus A x 2 square plus B x 2 plus C is equal to y 3 so that is the requirement of. So, this is the requirement that Simpson process.

So, earlier picture was not very clear. So, this is this is the idea that this is the way I have to fit in a curve you may not always be able to do so that is the hulk. So, wherever you can do so when we assume that I can do. So, then what should be this coefficients a b and c how they are related to y 0, y 1, y 3 that is the key idea. That I want to draw the curve in such a way that the given point data points x 0 y 1, x 1 y 2, x 3 y 3 are also the points on the curve A x square plus B x plus C they also lie on the graph of the curve x square plus B x plus C. So, what should be such a curve what is your A, and B, and C, how are the related to y 0 y 1 or y 1, y 2, y 3, so that this fitting makes sense this fitting actually happens.

So, basically once I have this curve I am now computing this area under this curve right. So, this is y 0, y 1, y 2 they are the same because they are also lying in this curve. So, here we are now going to evaluate this integral. So, what is this? So, I want due to compute it out, I do not really do not want to waste my time, not waste my time I can never say that, but rather waste my energy doing so which is anyway if you want again I just do it. This two elementary not to do it, and then of course, you know what the answer is and you can if I am writing down the final answer, the final answer of is I am writing it down from the Thomas's calculus is so sorry 2 A h square. So, you put h and then you put minus h and do all sorts of things.

Student: Sir, there will be 6 (Refer Time: 13:16)

Hum

Student: Three also.

Yeah all right sir right here.

Student: 6

Huh 6 C (Refer Time: 13:24) because h minus h C h minus c h

Student: plus

C h no no minus [FL] tell you minus

Student: then minus of minus (Refer Time: 13:39)

Minus such a 2 C h.

Student: (Refer Time: 13:42)

Two [FL] three is a 6, I can write.

So, now how do I relate it to the points y 0, y 1, y 2, see it does not matter whether I am integrating it from minus x naught to minus a, I am integrating from x naught to x 2 are does not matter. See this y 0, y 1, y 2 these values matter [FL]. So, ultimately this whole integral can be represented in terms of y 0, y 1, y 2. So, even if we integrate it from x 1 to x 2, it does not matter, they are ultimately this is the representation.

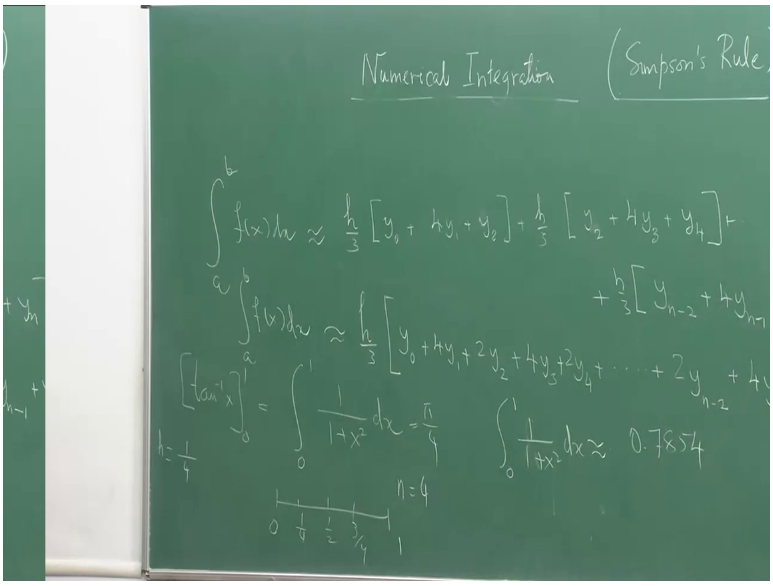
So, how do I do it? So, I know that it passes through the points minus h y 0. So, it passing through this curve is passing through these points 0 y 1 and h y 2. Now, when you know that, so you know that it passes through minus h zero. So, y 0 is equal to A h square minus B h plus C. And then know you know it passes through 0 y 1. So, y 1 is equal to say put 0 y 1 is equal to C. Then you know it passes through h y 2. So, y 2 equal to A h square plus B h plus C.

Now, knowing that c is equal to y 1. So, from here what I will get from here I will get A h square plus B h, what is that A h square plus from here sorry A h square minus b h is equal to y 0 minus y 1 because C is y 1. And from here I will get A h square plus B h, if c ultimately have to find A h square and C that is my goal depreciated in terms of y 0 and y 1. So, A h square plus B h it is becoming y two minus y 1. So, if I add these two I will get simply 2 A h square with B h, B h getting cancelled is equal to what y 0 minus y 1 plus y 2 minus y 1. So, it is becoming y 0 plus 2 y 1 sorry my y 0 minus 2 y 1 plus y 2 this is what is what is. So, this is my 2 A h square.

So, let me see. So, of my this integral finally, it is written in terms of h 3. So, I have converted it does not matter what is your end of the integration, even you take x 0 to x two still the same story will come through, ultimately the same story will come through that is the key clear idea that is why we did it in this. So, basically these are thought experiments you translate this whole thing and put it here, here we are your x naught coincides with minus h one and x 2 coincides with h a minus x 2 coincides with h and x naught coincides minus h x 1 coincides with 0. So, it makes it look easier. So, you have y 0 minus 2 y 1 plus y 2 minus 6 sorry plus 6 y 1 oh it is dangerous. So, what is my aim h by 3 to y 0 minus 4 y 1 sorry plus 4 y 1 plus y 2, so that is the integration.

So, now let us so if I go to other points, say if I go to the next set of points which is y 2, y 3, y 4, it will become h by 3 y 2 plus 4 y 3 plus y 4 so that is just this is a cycling process which goes through. Because the next set of points would be x 2 y 2, x 3 y 3, x 4 y 4. So, let me just now try to write down the estimate of the integral that this method gives me. So, now you know that if I go to the next one you know what would happen. So, I try to make an estimate by the Simpsons rule of this.

(Refer Slide Time: 18:51)



So, the first one gives me h 3 plus y 0 plus the first one 4 y 1 plus y 2. Second one will give me h by 3 into y 0 plus 4 sorry y 2 plus 4 y 3 plus y 4. So, first points are 0, 1, 2 then its 2, 3, 4 then 4, 5, 6 and so and so forth and. So, the last section the last part of this dot dot dot dot dot it continues the last one would give me h by 3 y n minus 2 plus 4 y n minus 1 plus y n that is what would happen the last part. So, if I sum them up it gives me a very beautiful symmetric thing because you see y 2 land up twice [FL] y 4 lands up will land up twice. So, all the even ones even indices will land up twice while the odd indices will land up four times excepting y 0 and y n. So, these are the endpoints.

So, it is so this I can now write integral a to b is y 0 plus 4 y 1 plus 2 y 2 plus 4 y 3 plus 2 y 4 plus if I go on, it will become 2 y n minus 2 plus 4 y n minus 1 plus y n it is not the odd and even type basically that the two endpoints were the one which has a common endpoint, the one the y 2 is a common endpoint between two such blocks. So, those the common endpoints one will be repeated twice and obviously, the inner the other ones which are in the middle will come from behalf of four as the coefficient because that is what it what we get do.

You see the beauty of this as the power of this is that sorry you [FL], you just do not need to know anything about the nature of the quadratic function do not need to know what is A, B, C that is the key idea that without knowing that. That is why Simpson's rule is so important that without knowing. So, I am fitting a curve which has to have some properties it has to pass through all the points which is given to me and it can be used for integration. But without really knowing the nature of the curve, the capital A, capital B, and capital C you can actually just use the y 0, y 1, y 2 these the functional values given to you the x naught y naught, these y value is given to you to actually evaluate the integral this is a very, very interesting point which one has to observe.

So, let us just take care of this thing for example, so y is delta x here. For example, let us look into this integral you know what the real answer is to this integral this is and what is tan inverse 1, it is pi by 4. So, basically this integral is pi by 4. So, let us see what is the difference between pi by 4 and this integral if you compute them numerically. So, let us use the Simpson's rules. If I use the Simpson's rule. So, in the it and I use n equal to four use n equal to 4, n equal to 4, then you have to evaluate this function at this 4 points. So, I am just taking out the result from by Simpson's rule. So, n equal to 4 point means what within the interval 0 and 1. So, n is 4 means 1 by 4 is again 1 is 0 1 minus 0 by 4. So, this is your h, say it 0 to one-fourth, one-fourth to half, half to three-fourth and this.

So, you have to know the functional values at these, these, these, these points 0 of course, you know is one at one you know its half. And then you know that of course, the function value is decreasing. So, it one-fourth you know what is the value, if A to half you know what and it compute those values. So, knowing that you can compute it this out and it comes turns out to be 0.7854. So, approximately this is 0.7854. So, let me use my calculator here in the machine and then let us see what is pi by 4. So, pi by 4, watch the 1.3 0.1415 let me just do a do a little division, just take pi to be 3.14. So, let me just see the calculator where the calculator is here is the calculator. So, here is a 3.14 divided by 4, it comes out to be 0.785. So, it is amazing match you see. So, it is almost roughly the match is quite amazing. So, you see Simpson's rule is giving you much doing a much better job for you use it that is why Simpson's rule is important.

So, life is important when you really contribute to understanding human knowledge, and this man just lived 41 years and that is what surprises me that he could give such so many things in. So, we now shut down this game integration game. So, in the next lecture on sequences, we are going to make a higher jump we are going to go into what is called advanced calculus or real analysis. We will start talking about sequences and that I will hardly take I can talk it out for many, many books, but I will just tell in the way I will just go on doing even an extreme for lecture on it.