Calculus of One Real Variable

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Lecture – 36
Sequence (continued)

So let me begin sequences again when a rather is a continuation of what we have done. We would have to learn more properties of sequences as I told you, they are extremely fundamental, but before I continue doing. So, I just want to make some remarks I am going to give you some notes, which are detailed regarding the first very conceptual things about number countability uncountability, all those things. These are very important things; those things have to be kept in mind every time you do anything.

So, you will get a full detailed notes on that before your exam; however, certain things like they did differentiation, on integration, Riemann integration fundamental theorem of calculus all those things which have already been done in the class in the lectures. They would not be given in the notes, extra things should be given in the notes, you have to keep in mind this simple fact that extra things would be given in the notes.

Because notes are there for you to learn that other you apply what you have learned in these lectures, and try to understand the things in the notes. See if you are not looking at that lecture, and if you are just going to look at the notes then this whole exercise is not fruitful, then we could just give you the notes and put the notes on the site and say you learn from the notes; the fact that you can listen to the lecture.

Because as I speak on, there were a lot of things I will add on while speaking as things come in my mind I will never spell them out in the notes, because you know I will just be more precise in the notes. When people write notes or books, people are more precise, but when you are, you know giving a lecture which is almost extempore in most of the cases. So, you just tell what is there in your mind. So, you tell many extra things.

So, going back to sequences, you have learnt about what is the sequence three. The nature of the sequences one, is convergent and the divergent series as two nature; one is oscillating and one another one is blowing up or blowing down, whatever you want to say. So, we will now talk about something called a subsequence which is a key notion in our study subsequence.

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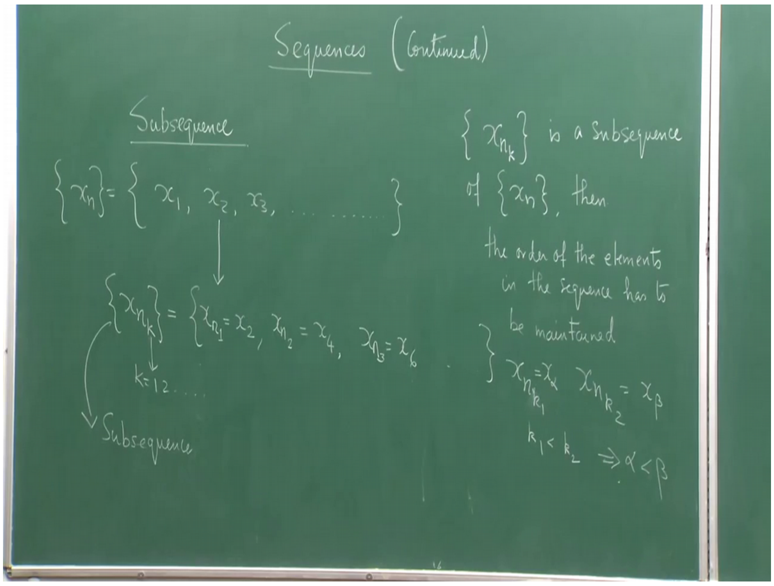

So, what is the subsequence? So, let us take a sequence. Now let me construct a sequence like this a new sequence. So, new sequence u 1, which I am calling as, whose first element is x 2, u 2 whose second element is x 4, u 3 whose x 6 and so on just take the ones with the even indices.

So, what I have done. So, I can write this thing actually u. So, sometimes u 1 actually can be written as instead of writing u 1, the general technique is to write this as x. So, this is a new sequence. So, you are generating a new sequence out of the sequence xn, and the new sequence that is generated it is first term is x 2, and the new sequence; that is generated from xn, it is second term is x 4 and it is third term is x 6 and so on.

So, this sequence is denoted as xnk. So, the index of the sequence is not n, but index of this new sequence this is k, k is the index. So, here k is going from 1 to whatever 1 2 dot dot dot dot dot. So, this sequence is called a sub sequence this new sequence is called the subsequence of xn. There can many sub sequences when a countably many sub sequences.

Now, what is the nature of the sub sequence? For example, I have x 1 n 1 is x 2. If I have xn 2 as x 6, I cannot write xn 3 as x 4, the order in which the points appear the elements appear in the sequence. The same order has to be maintained also in the sub sequence. If you do not maintain that order it is not a subsequence.

See again I am asserting that this little fact which I am trying to make clear to you, It will not be clear in the notes if I just write down the normal definition of a subsequence, people will get confused and from my own experience in research, I can tell you has seen research papers where people have made mistake by confusion in the definition of subsequence.

So, it is a very crucial definition, it might look very simple that you are just, it is not just a subset of the sequence that you have written down, it is something more, it is something where you have to maintain the order of elements that you choose right. Once you have x 6 you cannot. Now write xn 4 is x 1 it, this will not be allowed, it has to be something bigger than x 6.

So, that has to be; that is something you have to be very careful about. So, I have given one example. So, how do I put this in a mathematical language? So, xnk is a subsequence of xn, then the order of the elements of the sequence has to be maintained so the element in xnk 1. Suppose if I have write xnk 1 and xnk 2. oh sorry xnk 1 and k 2, where k 1 is strictly less than k 2.

If I write that then the corresponding element, suppose this is corresponding to some x alpha in the original sequence, and this is corresponding to some x beta in the original sequence, then alpha might must also be strictly less than beta. This has to happen; otherwise you do not get a subsequence. So, these are key fact, subsequence has a very important link with bounded sequences. So, what is the link between subsequence and bounded sequences.

So, let us explore that link which is expressed through a very important theorem called the Bolzano Weierstrass theorem, whose proof. I will ask my tas to actually put it up on the board, put it up on the notice board, what is that called the. Yeah I showed, because we are not going to prove here, because this. No much time to do things.

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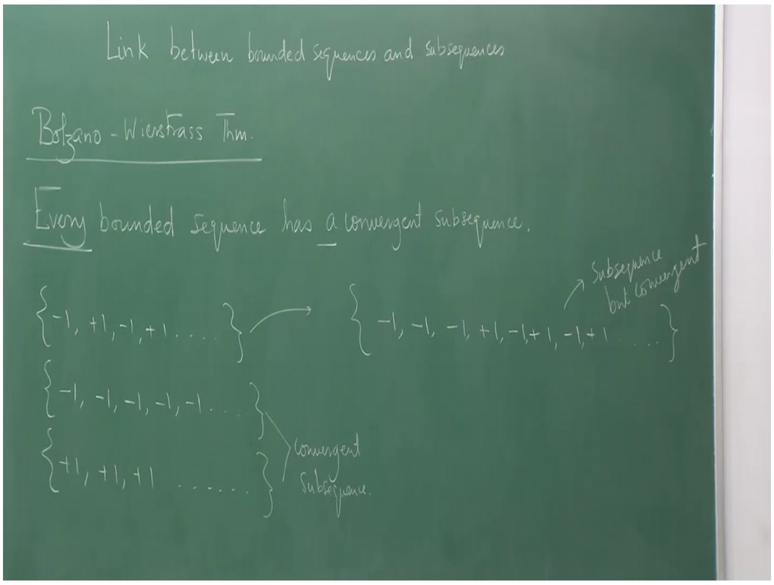

So, link between bounded sequences and subsequences one, is one of the key facts that we have learned about bounded sequences. In the last class was that every bounded sequence need not be convergent the example was the sequence minus 1 plus 1 minus 1 plus 1 and so on. So, the idea that we should have now, is that how is sub sequences link to the bounded sequences.

So, if we choose some sub sequence in a bounded sequence what happens does convergence has something to do with subsequence. So, here is one of the most important theorems in all our mathematics, is the Bolzano Weierstrass theorem and this thing, it is repeatedly applied, it is also available in higher dimensions. So, it is this thing, it is repeatedly applied.

There is a key, a key fact which says that every bounded sequence has a convergent subsequence, every bounded sequence has a convergent subsequence, remember that two words here, every and a. This is something you have to focus on when you look at a mathematical result, your first idea would be to focus on what it says, you have to get the logic, very clear before you do anything right.

So, every bounded sequence every in mathematics, using the term every means is a very powerful thing, every bounded sequence, it does not matter whatever, what is the sequence has a convergence sequence, means there exists at least one sequence which is convergent. It does not tell me that every sequence is convergent; it does not mean that every subsequence that I choose is convergent.

For example if I take the sequence like this, now I take a sequence only this subsequence, then it is of course, converging minus one; nothing else or I take the. So, this plus 1 and, these are convergent subsequences. So, I have more than one convergent subsequence.

Now, there can be a subsequence which is not convergent. For example, I can create a subsequence from here, for example, let the first term be minus 1 second term be minus 1, third term is minus 1, and the fourth term is plus 1. Again fourth term you see then there be a just after fourth term, it again the same repetition starts, you take the same elements. So, then again it all starts oscillating.

See in a sequence when you look at a sequence, it is head; that is the first finite sum, finite number of terms if you give away does not make any difference on the actual nature of the sequence, this is the tail of the sequence that plays a fundamental role, and that is something you should get into your mind. So, this is a subsequence, but not convergent, it does not mean it. See the whole idea is to tell you that this theorem does not tell is that every subsequent that we have is convergent, there will be a hm.

Student: Sir.

Oh there is a small correction here; this sorry this should be alpha less than equal to beta; that is all right, I just skipped. So, you see, means I can repeat the terms here; that is what I means alpha is equal to beta here, because I am repeating the terms minus 1 plus 1 plus 1, but here if I repeat up to some term, and again then go back to the actual form of the sequence, then I am getting a subsequence which is not convergent.

So, the whole point is that, every subsequent is need subsequent, need not be converging, but some subsequence are convergent; that is the clear message of this result. Now what we are going to. Now study the following is that there are certain kinds of sequences which has this very interesting nature of increasing or decreasing; that is the function values keep on either remain same or goes down, or either remain same or goes up the corresponding values.

So, now how do I talk about an increasing or non decreasing sequence, if you want to say what non decreasing.

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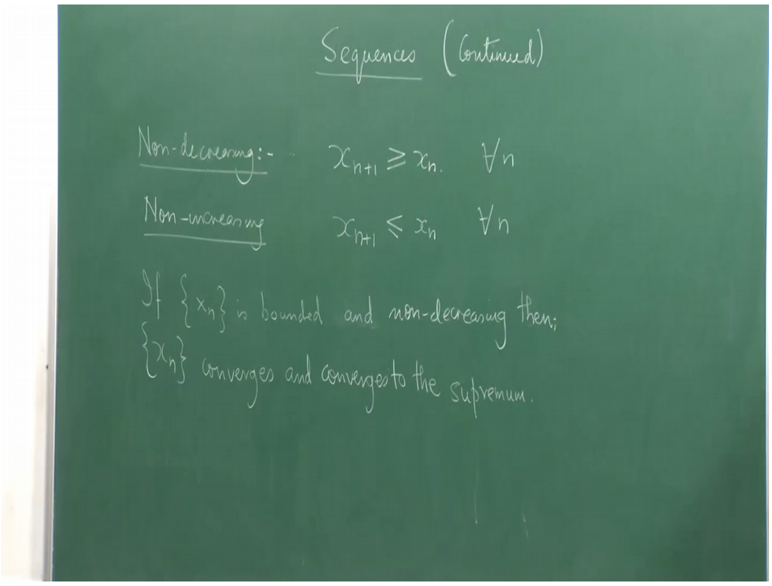

So, either increases or remains same. So, how to talk about this non decreasing sequence, what does it mean? It means that x n plus 1 is either strictly bigger than xn or it is equal to xn.

So, similarly non increasing is the opposite. So, non increasing, it means is either decreases or remains same, it does not increases. So, here it is just opposite xn plus 1, either decreases or remains same, it is true for all n example here. Of course, you know the decreasing sequence, the example is the harmonic sequence which we wrote in the beginning one half, one that you see the sequence is decreasing.

Sequence is increasing, the very natural number set n is an increasing sequence 1 2 3 4 5 6. So, what is so important about this? Here what will tell you that we will link the increasingness that may this nature of the sequence to it is boundedness. Now the question is, if every bound how, how do we reach this thing.

So, let us see how mathematicians would argue. So, every bounded sequence need not have a, need not be converging. So, what we figured out there the convergent subsequence can there be a bounded sequence, which is convergent, can there be some additional behavior of the sequence, a bounded sequence which makes it convergent, and these additional behaviors or sufficient condition, it does not mean that if a bounded sequence is convergent it has to have this nature.

But if a bounded sequence is convergent right for example, this is neither increasing, for example, this is not an increasing or decreasing sequence right this one which I have given example. So, this is a sufficient condition which tells you that if I add this to this thing of bounded sequence, this notion of bounded sequence then such a bounded sequence will always be converging.

See when a nice looking thing like a bounded sequence fails to be converging, we will try to see if there is something we are missing, is there some extra condition which will make it bounded. So, these are the conditions. So, what does it says, it says if xn is bounded above right, not just write bounded, do not bother about a.

So, if xn is bounded which means it has a infimum and supremum; that is what it means, it is upper bound and a lower bound and hence in an infimum supremum, and non increasing, non decreasing, increasing basically as you (Refer time: 18:18), just think increasing in your minds adding this term non decreasing is very difficult to keep in your mind, you have the feeling that it is decreasing. So, just keep increasing in your mind, if that this term makes you uncomfortable, it usually makes me uncomfortable, but. So, I always think it numb. So, increasing function, you can just think it is increasing sequences. So, you can think it now. So, increasing sequences.

So, if xn is bounded and non decreasing then. So, xn is bounded and non decreasing then xn converges and converges to the supremum xn converges, and converges to the supremum. You might be wondering that a why I should now speak about this particular result, maybe I should tell, try to tell you how to prove it right.

So, xn is bounded. So, this is one aspect. So, let us see how can we prove such a result

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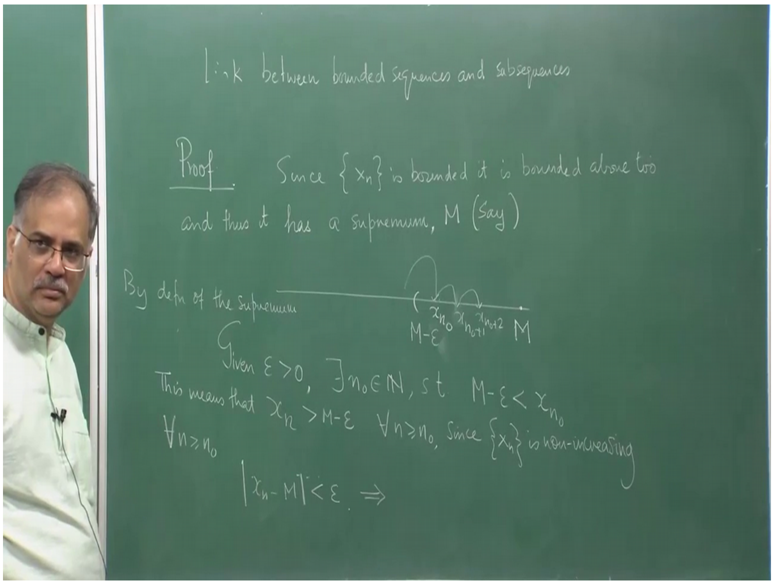

Now here we will first start with the idea that the sequence is bounded for this result. So, since xn is bounded it is bounded above bounded above too and thus it has is xn is bounded it is bounded above too and thus it has a supremum and let us call the supremum to be m say right supremum m say.

Now, you see the idea of the proof just depends on the definition of the supremum and let us see how we do it will not go into too much of writing. So, idea is this. So, here is your m. So, then take an epsilon 1 and make it m minus epsilon 1. So, m is the supremum then there is some n naught for which xn naught is here. So, given epsilon epsilon one what I have done. So, given this epsilon 1 say. So, I have got this xn naugh here. Now what I will do, I will choose an epsilon 2 in such a way that, now what I what would happen.

. So, given that epsilon sorry just given that epsilon greater than zero, there will be an xn naught which would be bigger than m minus epsilon. Since m is a m is the supremum. So, will do define now anything xn naught plus 1 xn naught plus 2, everything lies here, because everything is bigger than xn naught. So, everything should be inside this interval.

So xn naught is bigger than m minus epsilon. So, xn naught plus 1 x. So, any n for which n is bigger than n naught must be bigger than m minus epsilon. So, this. So, what is happening? So, the distance between for all xn the distance between xn minus m right, because m is the supremum, the distance between x and minus m that is always less than epsilon, it is obvious.

So, you see how geometric one is just by drawing you can do the thing. So, what it means given epsilon greater than 0. There exists n naught element of n just by the definition of the supremum. So, maybe if you want to be more clear by definition of the supremum given epsilon greater than zero, there exist an n naught such that.

So, what happens is that m minus epsilon is strictly less than xn naught equal to less than equal to m so, but this is true. Now this means that xn is also strictly bigger than m minus epsilon for all n bigger than n naught since xn is non increasing right this is what we can write. So, once we can do this fact once we have written such things.

So, what does it simply means, it is simply means that for all n greater than n naught corresponding to this epsilon greater than 0, we have xn minus m the distance between this that is basically m minus xn basically is less than epsilon. So, this distance m to m minus epsilon this this distance is epsilon right. So, this is; obviously, strictly less than epsilon.

So, hence what we have done, we have proved that xn which implies that xn converges. So, all the xns are now lying between m and m minus epsilon. So, it is lying between. So, m minus epsilon m this closed boundary, it is lying between this.

Student: Sir all we can write from this it is done that xn minus m is greater than minus epsilon and since m is m plus epsilon is, we take that epsilon is always less. So, what is it is obvious just see you can.

So, this simply we just looking at the picture you simply conclude this this implies that xn goes to m I remaining the with an on increasing part what the on increasing part will say that xn is a bounded xn is bounded and non increasing, then xn converges and converges to the infimum. So, you can prove the same thing, it is just a matter of now there is a this is you know this calculus way things were done during the time of newton Leibniz even in the time of Eulers Laplace and all these people where hardly can be considered to be rigorous by outstandards.

If you read Richard Dedekinds book on numbers essays on the theory of numbers, you will for some of the people who are very well conversed in mathematics would laugh at the proofs given, there some of Eulers proofs are absolutely not rigorous by our terms, but all these results are correct.

So, in mathematics sometimes insight is more important than rigor, because just bothering about rigor as Sanjoy Mahajan says in his book street fighting mathematics essentially that rigor can essentially cause you rigor mortis. So, you basically lose the insight. So, proofs what is called a proof today I do not know whether it would be proof by any standard after 200 years.

So, for us it is a proof, because we assume a very important thing called axiom of choice, which I do not want to get you into that there are issues in the foundations of mathematics. Then we all our mathematics that you see is based on certain beliefs certain assumptions that this is their and on which I am building the castle trade, but it is work, that is fine, it has worked fine. So, it has been working fine for a long time. So, maybe we go through it.

But many things which was there during the time of Newton Leibniz etcetera these things are completely non rigorous by our standards. See this Cauchy who first came and tried to inbuilt rigor into calculus and that he gave the name of analysis. So, he also built analysis for complex numbers called complex analysis.

So, and this what you see is essentially Cauchy idea. I mean lot of people like Delaunay Posso and many other people had actually built up this whole edifice, what is now known as real analysis, and is one of the most important branches of mathematics and now of course, one of the most possibly, the most. Now all these people who think that I just want to know the techniques of the calculus, you need not bother about, this can; obviously, close your books and go away. You have learn techniques of the calculus, the basic things you do not have to really bother.

But those who want to see how mathematical thinking has progressed. So, this last week is essentially about some mathematical thinking, but here we will also come into something called Taylors Theorem. In the last three classes you will see how important, it is actually gives you a tremendous power to approximate things, because I want to remind you what Bertrand Russell has said; it is very interesting to note that in exact sciences we are always relying on approximations.

Now, Cauchy figured out that a sequence what kind of sequence converges, it is a very natural question what kind of sequence converges, any arbitrary sequence you can look, and say I do not know whether it will converge or not, I can give a very strange looking sequence for example, which you may not be able to figure out. So, why so many people asked, cannot just look at it, and figure out that why it is, whether it is convergent or not, what kind of sequence actually converges.

So, Cauchy defined a category he found a sufficient condition, he found first a sufficient condition. If this is the nature of the sequence it will converge, but it was not shown later on that, any sequence that converges must have that property. So, here is the power of mathematical thinking which tells you a sequence, which converges has to have some property, and if that property is satisfied the sequence converges.

So, this is a complete characterization of a convergent sequence, and that is what is called a Cauchy sequence. So, one of my ts told me that many of you are asking about Cauchy sequences. So, let me tell you. So, the idea behind Cauchy sequence is to really characterize what essentially is the nature of a convergent sequence. So, we will talk about Cauchy sequence first.

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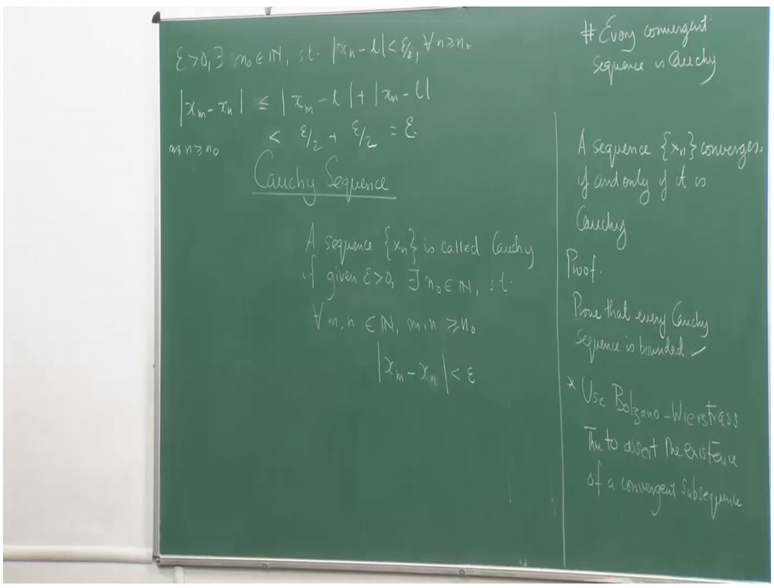

Obviously Cauchy did not tell that it is a Cauchy sequence, it is people, who later named it as a Cauchy sequence. I do not know the historical; I have never seen Cauchys book Cours De Analyser. I think if I am not very wrong, I could be wrong, writing in French Cours De Analyser, Cours De Analyser, something like this, I do not exactly recollect the name of the book, forgive me for that, but that part it is Cauchys result.

Cauchy was the very rich man, science again I want to tell you, was done by people who had money. It was their hobby and that is why they did. So, well now it has become a profession. So, all professional issues come into it, and sometimes it merge the joy of doing it. So, here sequence xn is called Cauchy. If given epsilon greater than zero; no matter how small there exists n naught element of n; such that for all, any pair m and n which is bigger, which is bigger than n naught.

Any pair m and n in n of course, and mn bigger than, sorry bigger than n naught, where the n not equal to n naught the distance between xm and xn can be made smaller than epsilon. So, one of the major results of real analysis, or when you are just looking at one variable functions is the following, but just numerical sequences.

So, it is following a sequence xn converges if and only if it is Cauchy. So, how do you prove this factor, what are the steps to prove? You see here even Cauchy see is sequence is Cauchy then xn plus 1 minus xn if n is bigger than n naught is also can be made less than epsilon; that is the neighboring terms the distance between them becomes smaller and smaller.

But that are, but such a sequence I will just ask you to experiment with it, and I just try to figure out. This is a very interesting example that you just have xn plus 1 minus xn is less than epsilon that does not mean that sequence will become Cauchy; that is something to you I want to warn you at the beginning . So, now, what are the steps by which you will prove I will ask request my ts to just write down the proof is a very simple proof, but I am going to I am not going to write too much about it.

So, let me just tell you what are the steps in the proof. So, you want to prove this the first step, is prove that every Cauchy sequence is bounded, this is the first step I will leave you, because you are not learning to, does lots of proofs. Here we have done. Yes in the last class also. So, you can pertain you my ts would write down the proof, prove that every Cauchy sequence is bounded this is the first step.

Once you do that the second step is use the Bolzano Weierstrass theorem. Bolzano Weierstrass theorem to say that there exists a subsequence of that Cauchy sequence, which converges, and once you know then, and then you do the proof use Bolzano Weierstrass, once you figure this out then it is alright.

Use the Bolzano Weierstrass theorem to assert the existence of a convergence sub sequence; that is it actually this convergence of subsequence itself will be a Cauchy sequence, because I would like you to first prove this fact every convergence sequence is Cauchy, sequence is Cauchy, this is very simple to prove convergence sequence is Cauchy.

So, how will you prove convergence sequence is Cauchy, just a two line matter, I will just give you a very brief thing, you can write down in detail see. So, what happens for all. So, l is the limit then given epsilon greater than 0 there exists n naught element of n such that for; such that module xn minus l is strictly less than epsilon for all n bigger than n naught; this is what is happening.

So, once I know this fact I will simply write this thing. So, take m and n both bigger than n naught. So, m and n both greater than n naught and for m simply write this, this is less than equal to that is you add put plus m and do also add and subtract plus of n ln minus l. So, add and subtract l. So, xm minus l plus xn minus l.

I would not go into the intermediate steps which you should be able to do my ts will do that intermediate step for you and just what does this mean both are bigger than n naught. It simply means this is less than. So, I can write this two epsilon by 2, if you do not mind. So, given an epsilon I have epsilon by 2 and for which there exists n naught. So, this is happen.

So, take all n bigger than n naught. So, that xm minus xn less than this, this, and each of them is less than epsilon by 2, because m is bigger than n naught n is also bigger than n naught. And hence this is equal to this epsilon. So, this is so strictly less than epsilon. So, xn xm minus n is strictly less than epsilon for all mn greater than n naught, and when epsilon n is given to be greater than zero.

So, it proves that every convergence sequence is Cauchy, and once you have proved that that is it. So, you have to use that again that fact, and then just use the basic things, basic manipulations that will be done. So, that is how you prove that every Cauchy sequence converges. So, one of the important learning is that, this xm minus xn is strictly less than epsilon is very important xn plus one minus xn is strictly less than epsilon is not going to give you a convergence. Sequence is not going to be a Cauchy; sequence an example would be written down, it is available on the web.

So, we will mention the particular website, and will write down the exams. So, you can check it up in the web also, because possibly that is the biggest library you know see the idea is that, why does not happen if I put xn plus 1 minus xn. See what happens, I will just give you the mathematical idea of constructing such examples.

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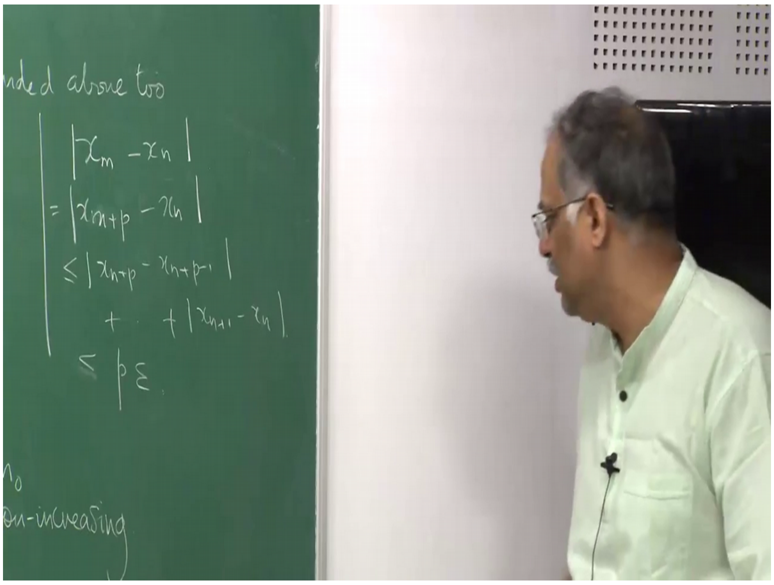

The idea is this. Suppose I have put xm minus xn. Now xm could be xm is n plus p right. So, this can be written as xn plus p minus xn plus p minus 1 and plus xn plus 1 minus xn [FL] all of them are less than epsilon. So, basically less than p epsilon strictly less than p epsilon but you know that p could be very large, but xm any mm n bigger than n naught.

I can choose a m very far from m, but bigger than that m naught given n naught right for which xn plus 1 minus xn is bigger than 0 is less than epsilon. So, what I have to what I do I choose a very p, when a p is very big, then this is not very small the difference between them is actually not very small.

So, then there beats the purpose of being a Cauchy sequence. So, Cauchy sequence means all the terms are actually huddled into a very small neighborhood, a very small two epsilon zone; that is what it says that all the terms are huddled into a very small two epsilon zone for a small epsilon. So, that is the key idea that when an epsilon is given I can show that after a particular term. All the term is all huddled in that in a two epsilon zone that is the key idea.

That every term if; however, for one term is from the other by the index number, the term is actually given though each of them are within a two epsilon zone of each other. So, that is not true, if you take these things, p could be just too large. So, the distance of them could be actually very large. So, this is the key idea behind which you construct such examples.

So, to do check Cauchy sequence you really have to choose for any mn there. These are not such a easy idea by the way, but it is very important. The conceptual idea is very difficult that you that thinking of actually clustering everything into a small zone; that is the key idea that just I look at terms individual minor term side by side, that they are very near, and I do not bother about terms which two terms, which are very far off, then that is not two.

Even the two terms which are very far off should be also very near, whose indexes are very far off should also be very near; that is the key idea of Cauchy sequence. Not just the neighboring terms within the area. So, this is very important to keep in mind.

Thank you very much and will talk sort with series in the next class.