Calculus of One Real Variable
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Lecture - 05
Limits of a Functions - Part I

So, welcome once again. We are taking gradual steps to come in to the main themes of calculus. Today we are speaking about limits. Now many of you are more ambitious younger students possibly waiting urgently to get some knowledge about limits about derivatives integrals and all those things. I would suggest that you keep this in mind that you would understand calculus far better if you know at the very outset that. These derivatives and integrals are names given to some special kind of limits.

So, limit is actually the most central and possibly sometimes viewed as a very slightly difficult concept in calculus. And you will understand calculus much better if you know at the very outset that every continuous function did not have a derivative, we will come to all those things later, but first of all let me tell you what is calculus actually what does calculus study mean it is just to is it just write some derivative something from the air. And then just make some marks in the examination is that for the reason for which you study calculus of course, many of you would say I have to do it in my I d j I have to give my engineering exam math one course.

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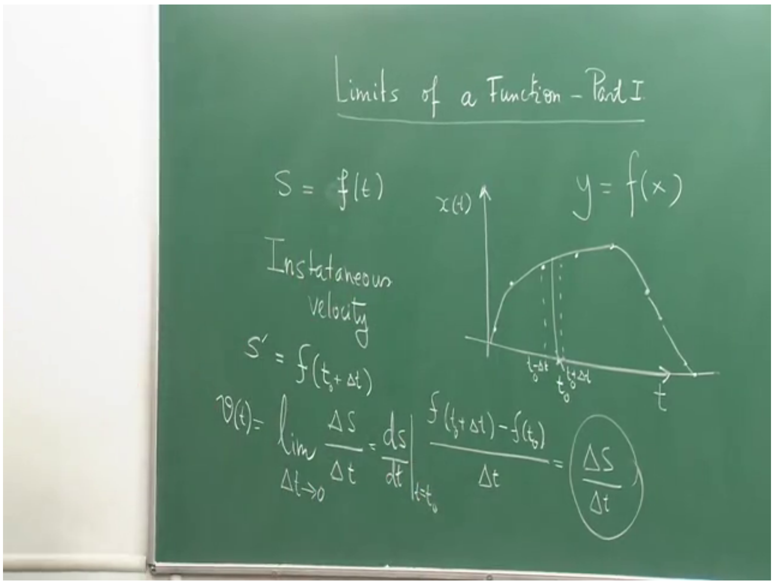

So, I will just do something, I have to know the tools I have to apply it blah stuff, but the real issue which is hardly seldom told in our classes that calculus studies change. It tells you that if I have a function written in this form y equal to f of x, then what happens to how does y changes when x changes. How does the function value change with the change of the independent variable, this takes variable which we call the independent variable and the dependent variable? Calculus actually deals with the rate of change per unit time. So, time also gets linked with calculus this variable largely has been time talking about time. So, calculus arose because people wanted to understand motion.

And if you read the or if you look at the history of calculus, it was essentially by trying to understand motion of bodies motion of heavenly bodies rather that calculus first took it is root. And it first came in when people started talking about the velocity of object at a given time at a given time instant. So, what people like newton and others realized that, if I want to look at the velocity the how a particle is moving I can draw a graph and I can put this x t as function of t as the position. Because you can understand that you can always consider time when you are standing still at time t equal to 0 then you start moving. So, as you move the distance changes whereas, time flows you cover more distance. For example, a train starts at 3 o clock from a say the Howrah station and has to go towards say the bandel station. So, then for example, it starts moving at 3, at 3 15 it is in the next station liluah and 3 30 it is at some other place 3 40 it is some other place and so and so forth. So, distance is essentially a function of time though.

So, the graph could be straight like this, or graph could be something like this so even at even at discrete times you can do like this, you can plot it like this if you in observe it say after thirty minutes time interval assuming that time given minutes or even seconds. So, every 60 second interval. So, it could be like that. So, as a train starts it is speed increases and you see the speed is increasing and so you basically have some sort of an increasing function if you look at in continuous time. So, now, the question is what was it is velocity at 3 40. 3 40 pm what would be the trains velocity, if velocity would come down for example, it starts start to decelerate and stop at a station is velocity could come down like this. And it could be 0 after certain time means it has come down and stopped at a station. So, this could be the graph.

So, now what is exactly the velocity at a given time t. So, a distance s is usually written as a function of time. So, if I do not if you do not mind x t is the standard way physicists right. So, I will just put it in a mathematical way s is function of t s is a distance and f t is the function we go by which it is related with the time, is the train speeding with the distance or it is slowing in the distance everything can be captured. So, you can. So time. So, at every time we see the nature of the motion of the object.

Now if I say exactly what is the speed at this time. You might say oh come on it is so difficult to find exactly whatever because our instant of time merely passes suddenly you are you are in the past and suddenly you are in the future you hardly understand what an instant is, but this notion of instantaneous velocity has severe important ramifications in physics and also for mathematics. So, how does one do it. One says let me see what happens if I see the, if after a very small amount of times say delta t just delta is to talk about very small time.

So, what is my need I have s dash I come to s one I was at time t, I was at s at time s dash at time t plus delta t at the very small time, I am at s dash. So, this is my distance. So, assuming that over that period I have moved in a given direction. That is I have not changed my directions much. So, for a very small instant of time. So, we can compute the average speed over that that distance which would be f of. So, this thing is also written by the physicists as a change in the distance very small change though. So, you can make the change on thus for example, at this point you want it say at the point t you want it, but you can make the change either here, t plus delta t, t could be positive or it could be t minus delta t whichever way you want you can look at what was the velocity. So, you are looking at the velocity you are looking at the distance it had covered just few minutes few seconds before or just you look at the distances you have covered few seconds after you can do either.

Of course you can say this though instantaneous velocity is a very strange thing to think about, but at time 3 40 it did the train did have a velocity right. Then the question is if I know the velocity of this or the time that it about the distance that it has covered in this small time period, and then if I make the time period much smaller and smaller and smaller. So, small the time period goes toward 0 then possibly what I have at the end is the instantaneous velocity. So, this is what in the concept of now this whole thing is a function of t because ultimately everything is depending on the choice of t the delta t.

So, if I fix the t then it is a choice of delta t. Delta t can be now changed I am making the intervals around the point t smaller and smaller if you want to be more comfortable you can put t instead of t t0. So, this is delta s by delta t which is a function of t. So, delta s is a function of delta t. The change in velocity the change in distance is a function of the change in time. Now I want to know what would happen to this change this this rate of change or this velocity.

So, how much has the velocity changed. So, what I am measuring here is the distance covered by that time. So, average velocity over this small interval of time period the small time period. So, I want to know if this one time period we made as small as I like, I can make it as small as I like basically I want to in my mind shrink it and bring it to the point t will this ratio move to any anywhere. So, taking this process was understood and called by newton the notion of a limit.

So, as I am making the time go towards 0, the change in time go towards 0 this average velocity or this very small time period is going to what is known as the instantaneous velocity v at the time t. It is sometimes also written by this fashionable symptom a symbol called d, d t of s which is called the derivative of the function s with respect to the time t, computed at the time t if you are more if you are slightly confusing you can put this as t naught if you want. So, I am exactly looking at the time t naught. So, then you can write this at time t equal to t naught. So, this is nothing. So, when lord kelvin we whose name you must have known for the function of absolute 0 minus 273 degrees Celsius.

Or stake your class in Cambridge and people were getting rather first about this notion of derivative using a calculus class he simply told do not get so much worried about it this is nothing, but the velocity. So, here the first idea of this intuitive idea of the limit comes from trying to understand the motion of a body. Now this was just one category of function where that the we are looking at distance moved by a body as a function of the time elapsed. Now as a mathematician we take the question further. The mathematician who does all this work by newton was largely for physics for motion of bodies. It was Leibniz who got in to more asking more general question and later on Euler who took it up in much more detail is that, when I have a general function y equal to f x.

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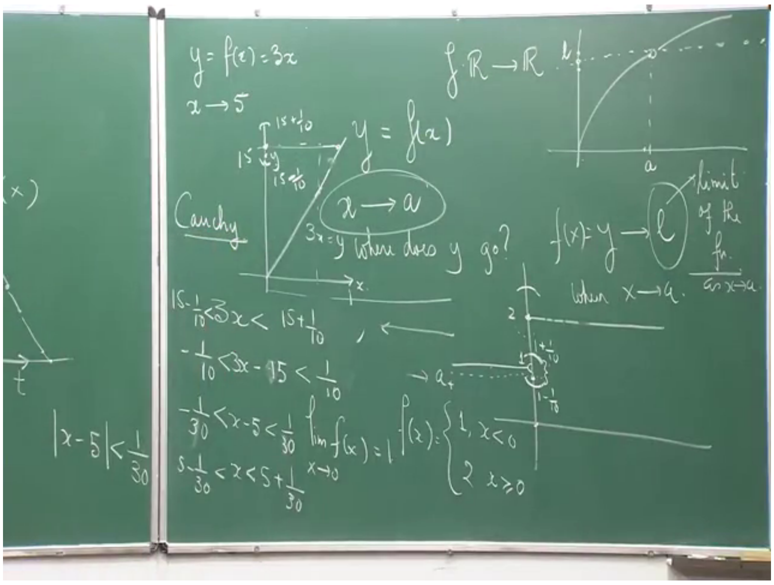

And when x move towards a real number a of course, f is a function on a real number line mean f is just a function say from r to r. Or from some interval a b to r does not matter you can that is your choice.

Now, when s goes to a, where does y go that is the question. Now the interesting part of this whole issue is that first of all what is the meaning of s going to a. The whole question is what is the meaning of x going to s. We are here in this course not to show you very basic how calculation of limits are doing you know very simple calculations and showing what is done in a class. This is a course for the ambitious people I have mentioned in my introduction and hence we are essentially looking at concepts at the first place.

The concepts are of central importance in this course rather than just showing examples by which you can pass exams, because somewhere the subject must come first and other things has to take a back seat, where does y go now you might ask me what is the meaning of x going to a. Or what is the meaning of delta t going to 0 and then when I was in my eleventh standard this was the equation our teacher wrote on the board and we were quite just finding it very strange. Because we hardly understood we do you had no teaching knowledge of the calculus. So, we asked our physics teacher what do you mean by this what is meaning of delta t goes to 0 he says oh the delta t tends to 0 in the limit we ask him what is the meaning of limit they just (Refer Time: 14:34) I tell too many of my students he said you see limit means coming, but still not coming which might appear that is a is a joke, but you know it is somewhere it is essentially the true fact. So, what does x goes to a means. So, here on the number line is your real number a. What is the meaning of x going to a?

Now, x can approach a from either side, from both side. Either from the right or from the left. So, x can come approach a from this side x can approach a from this side, but x need not be exactly equal to a may not jump on to a. So, here, if I if x comes from this so we say x goes to a from the left side a minus and x goes to a from the right side a plus. So, it this is called a right hand approach and the left hand approach.

So, when I am meaning x going to a, I am trying to mean that you know I can take the very x variable move from any of the sides. Now what I what do I mean by that where does y go I want to mean by where does y go, is that if I make x approach from either this side or that side what is the value of y well what is the value of y where is the value of y going we say that the y value goes if y value goes to a particular value l. So, if y goes to l or the function value goes to l, that is goes to l when x goes to a means it does not matter whether x is moving to a from right side or left side the effect values goes to l. Then only we will say in that case we will only say that l is the limit of the function limit of the function as x goes to a.

So, what it means by effects going to l. It tells you that f x is somewhere near l the f x value where that f x value could be on the right side of one and could be on the left side of l see x need not be a because the function itself need not be defined at a. So, here is my may be not defined at a and then it is like this so function. So, now, so, this is suppose this is your l those who know some calculus you might understand what I am trying to do, but; however, this is this is the l that you say that it approaches. So, what it means. So, whenever I find an x such that f x is very near l either this side or that side, I should be able to tell you a corresponding x near a. That if I say x is near l f x is near l you have to show that f x is also near a. Then only it means that f x is going to l when x is going to a.

Whenever I repeat it again which is very crucial, that whenever x is going means if you take f x value here very near l, you have to tell me whether the x that you have chosen of that for that f x is actually near l actually near a. You cannot say if x is very near l and x is somewhere very far from a. Then this whole thing does not stand the chance.

This little fact that if you claim that you have a f x value near l you have to also show that the corresponding x possible also near a that is exactly the meaning that if f x would be near l whenever x is near a. So, or rather whenever x is near a f x must be near l. So, if you tell me effects if you give me an f x near l, you should be able to show me that x is near a if x is far from a does not matter right. For example, I will tell you give you an examples of a function like this. So let me define the function f x is equal to one when x is negative is equal to 2 when x is positive. So, when it is one is negative and it is 2 and it is positive now here the function is taking either 2 or 1.

So, whatever x will take either it will take 2 or 1. Now if you claim that f x takes the limit as 1. Now suppose I take x here a y here. Functional a functional value. So, suppose I say that in this case the limit of f x as x tends to 0. So, now, suppose I claim here in this case that limit of f x as x tends to 0 is actually one. So, yeah. So, suppose now here is a point I have chosen here. So, here is my 1. So, I have taken some length or very small length say 1 plus 1-tenth and this is 2 by 10. So, this is 1 plus 1 by tenth and 1 minus 1 by tenth. So, this is my interval that I have taken within that interval that open interval 1 plus 1 by 10 and 1 minus 1 by 10, I take any point. So, I have to show that the corresponding x here the corresponding x here the in this particular case. So, given this x this value is there any corresponding x for which if x is near 0 and it gives you the same value there is no corresponding x there is a function takes only the value 1.

So, all are 2. So, you might say fine. So, this is a very stupid thing to do to take a value near one because a function does not take the value and what should have. So, what are the values it is taking 2. So, I can now only consider the function value either 1 or 2. So, 2 is it can it be considered around. So, it is within say this particular neighborhood of one right now in this neighborhood the function values could be just 1 or 2. Now one does not corresponding to any x on this side, but if you shrink the neighborhood if you shrink the neighborhood then; obviously, you can not get any functional values. So, what it appears is the following, is that in this very special looking case you were your only 2 choices. And when you approach from the left you come towards 1 and you approach from the light you can come towards another you cannot talk about a limit.

You cannot show me up you cannot just keep on shrinking the neighborhood and every time you get a the interval every time you take a point in that interval there will be a corresponding point in a corresponding small interval around x you cannot do that. So, unless you can you do that you cannot show the limit exist that is your high school knowledge of having the left limit equal to right limit means that any interval around the chosen limit. If you take any point in a whatever be your interval size any point, that you choose in that interval must have a must have and function x value must have must have come from an f x, x value which is in a corresponding small interval around the points a, which means that if you give me an interval around the limit l I should be able to tell you an interval around the point a. So, this is this can be explain why every simple, example I will use the book of spivak where some very fascinating examples are in place.

Now, for example, I take the function y is equal to 3 of x. And let us look at x what happens when x goes to 5. You say oh does not matter it is just 15. I know it, but no I am not talking about just doing and getting the answer. I am talking here about the structure that will tell you that what you have done actually is making sense. See this will bring us to something called the epsilon delta argument. And people will put up their hand and say oh I do not like it is so horrible, but please understand this is just nothing, but a translation of in English language in to symbols. So, that is exactly what the epsilon delta thing does. So, now, I said these are all it. So, let us do an experiment. So, I give you an interval say of length 1 by 10 that is you say 15 is the limit. I agree I will test your 15 is your l. So, I will take 15. So, here is y is 3 x y equal to 3 x and so here say it is 5. And this is say 15. So, where here at 5 so.

So, here is 15 may be my drawing is very bad. So, now, let me take an interval of length 2 by 10 totally. So, it is 15 plus 1 by 10 and 15 minus 1 by 10. And now once we have done this, so what have we done. So, we have done the following. So, I am telling. So, you say that your limit l right, is such that you take any point here any point let me take any point y. So, I take a point y equal to 3 x such that 3 x value is lying between 15 plus 1 by 10 and may be I should rub the board and do it better 15 minus 1 by 10. So, now, I am just getting a value. So, for me I made a small interval it could be lesser it could be bigger also, but lesser is what is required you always understand this whole story comes from the instantaneous velocity business. So, you have to shrink the intervals.

So, I will take a function for. So, take a functional value 3 x for some x where it is lying between 15 plus 1 by 10 and 15 minus 1 by 10. So, subtracting 15 from both sides I have one by 10 x minus 5 s x minus 15. I am subtracting 15 from all the sides. Now what sorry 3 x minus fifteen. So, what I will now do I will divide all the sides by 3. It is it can be done because we are doing division of a positive number. So, what do I get I get minus 1 by 10 less than x minus 5 less than 1 by 10 sorry I divided by 3. So, I have to have one by thirty here sorry for the mistake now what does this mean. So, x must now lie, x where should x lie. So, x should lie in the following interval.

So, this whole thing can also be written as x minus 5 is strictly lying by. If you look at this argument very carefully you replace your 10 by any epsilon that you want. Epsilon y I am talking about epsilon because it is a standard in mathematics and who has made it standard in mathematics very famous French mathematician called Cauchy. So, is all these definitions that you see here was bought in by Cauchy.

We are in to the late eighteenth century. So, it is Cauchy, Cauchy's book which gave modern calculus this flavor. So, I am being able to talk about the structure. So, any x which is lying within this zone, 1 plus thirty from 5 either from this side or that side would give me this. So if you give me a functional value which is this close to the limit, I will give you an x which is that this close to a on x. So, I give you I will show that the corresponding xs must lie within this particular band, band around this point a.

Your 5 is your a extending to 5. So, x is (Refer Time: 29:56). So, what does it mean of course, I am not taking x equal to 5. Now what does it mean. So, give me any epsilon I will give you some other number delta. So, here if my epsilon was one by 10 my delta is actually one third of epsilon. So, if you can just replace it this with epsilon because anyway you will divide by 3. And you will simply get this answer 5 minus f one third epsilon 5 plus 1 third epsilon. So, from mathematical point of view what is now the meaning of a limit.

So, with this definition we will end the discussion and we will give one more example how to show for very curious kind of example to show that 2 examples how to really find out this epsilon delta. And an example in the next class showing that why when what situations even by this epsilon delta argument you can show that limits do not exist right. So, these are the conceptual framework that we have to really get in to our mind. So, do not get too much pissed off by this epsilon delta thing that I am going to write now you might just forget it, but you just have to remember this fact. If x is if f x is very near y is x very near a that is the question. If you can show it every time then you have actually showed that f x goes to l, when x goes to a. So, if f x is very near l is x very near a that is what you have to show. So, this epsilon delta are quantification of these words nothing else.

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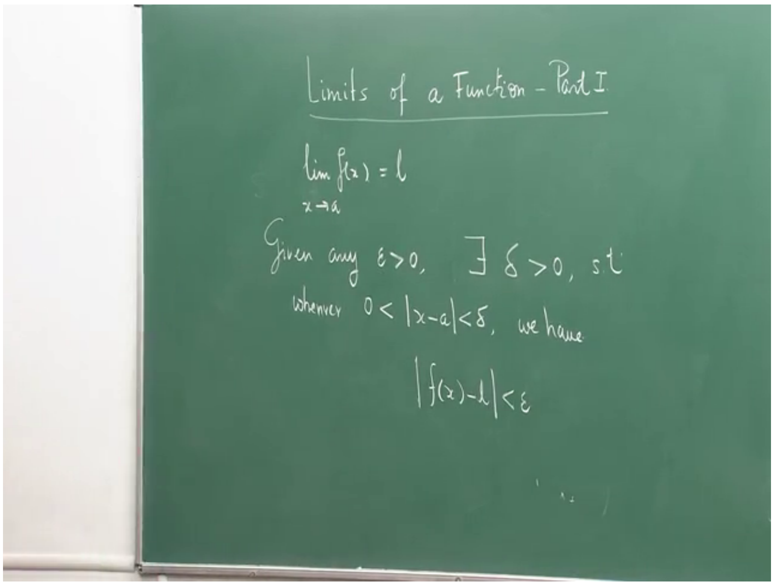

So, when I am writing this statement a statement which many of you must be knowing very well statement like this it means given any epsilon greater than 0 no matter how small does not matter there exists. So, this meaning is this there exists delta greater than 0 such that whenever 0 is strictly greater than x minus a, modulus less than delta means whenever x is in the neighborhood of a plus delta a minus delta with x not equal to a that is why we have kept this symbol.

We have the difference between f x minus l to be less than epsilon. So, whenever x is near a if x is near l. So, whatever be the nearness you want of f x from l I will give you a corresponding nearness. You might say it looks very funny because you sometimes say whenever if x is near l show me that x is near a, but why can not I think about the reverse thing that I tell you x is very near a. I tell you if x is I give x very near a, but show that f x f x is near l, mean what bound effects should be to how much this is a very tricky question.

Because sometimes what would happen is that that for some intervals some small intervals you might show this that there is a corresponding one. Or from for some other intervals you might not be just make it smaller you might not be able to show the function could be that bad. So, that is why it is always important to say that I already know that f x is I am choosing f x from the intervals very near l if I know that x is very near a my job is much more clear. Rather than looking in to the reverse direction that if I give us x near a is f x near l. Of course, in most cases that would be true, but in some cases things might be slight it not.

So, easy to get to that is why the reverse thing that what would happen is not easy to guess. Now one interesting thing is that one another reason why this approach is much more useful because if you give me the epsilon the neighborhood in which you want f x to lie. Then from that I can tell you which x should correspond to these f x whether that x is actually in a neighborhood small neighborhood around a.

So, this calculation is actually much more simpler. That is why we try to define limit now it does not become a definition, that whenever you tell me the distance I want effects to be from l I should corresponding to this epsilon there must be a delta such that every x for which f x value if f takes this value f x must be in that neighborhood the delta neighborhood. That is the simple idea. So, again I am repeating and ending this lecture that whenever x is near a whenever x is near a we should have effects near l that is that is the standard way of looking at it. Because it says limit extends to a when f x is equal to l, but a more structural way of looking at it is that first ascertain, for example, you at second functions are you can of functions like this right.

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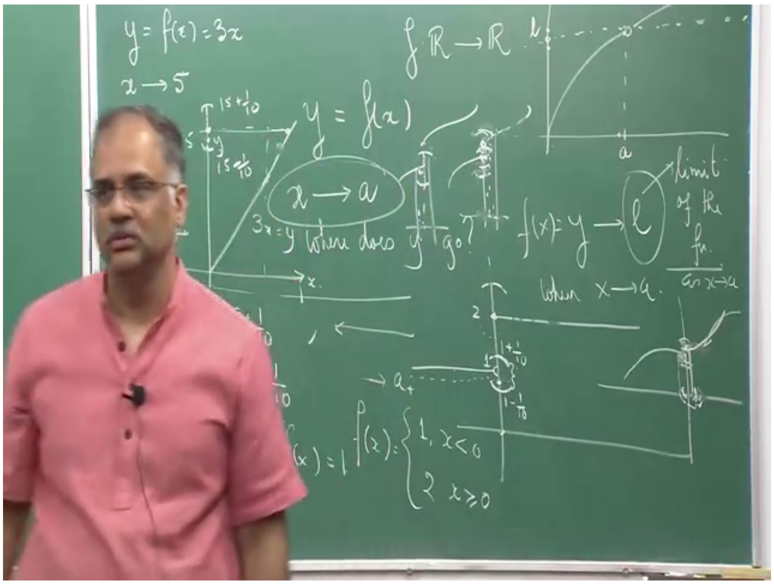

I am just telling you here the there is a there is a small break here. See you take this neighborhood around say 0. And take any of any one of them as the limit, this value or this value. There is a small gap between them and take a small, say I take I say that corresponding to this so what are the f xs. So, f xs are all in these are the f xs corresponding to this right. So, the effects is all the f x values are within this zone right this particular zone right. So, f x value is within that. So, here if I magnify it here there is a small gap here. And from the x that the interval that you chosen around x if I say this is the limit it shows that f x values are within this.

So I can all have it shows here or maybe if we choose a limit like this again still shrink the interval and it might you might just find the limit values are here then any f x value that it that you have here any f x values that you have here right. So, whatever value that you have here in this space mean I can do the very small whatever f x value I. So, what is happening that here I have made this small intervals and corresponding to which you these x values are coming from this small interval.

So, I have some values around the small interval of x and immediately I can have the f x values lying within this interval, but this is this function does not have a limit. You see it does not have a limit. Because ultimately they would give me 2 different values if I come from both sides you look at it in the traditional sense. So, in that even if I look at it the traditional sense going from a distance around x around a to a distance around l is not the correct idea. The correct idea is to come from the distance around l.

So, once I am sure that f x is lying within that neighborhood for whatever be the neighborhood; however, small for example, if my neighborhood content this, any point here then for these zeros here you see what is happening for these xs here the function for this this particular xs here. For example, very small neighborhood for very particular xs here very small neighborhood, there are functional values at this point right. Because suppose this function is increasing whose like this so for very small if the neighborhood is very small at the upper part there are no functional values.

So, you might be tricked by the fact that because in this part there is no functional value corresponding to this spot corresponding to x equal to 0, this part the values are all here. So, in this part there is no x for which. So, you are telling whatever effects I am having here they are actually corresponding the xs here. So, you have given me a small x around I am showing they are around a small neighborhood of f x, but that is not true because in this part there is no functional value I have to pick up a point anywhere in this neighborhood.

So, I have to take the limit point and I can take any point in that neighborhood. Other than a if I take any other point there must be a corresponding x which is near a that is the meaning of limit here it is a very bad trick the function does not really have a limit, but I am taking a situation where I know that this upper part this upper part does not have any functional values it does not correspond to any point here does not corresponding to any x here, but I say all the values here is for these f x that I have taken x is that I have taken all the values of f x are here come from here.

So, because you have already given up very you have chosen your neighborhood around x in such a way that you get a small neighborhood around f x, where f x values a part of that on the top does not have this part does not have any x. Then there is no functional values here there is no x for which any point here is a functional value, but here everything is a functional value within that small neighborhood. This part, this is these are the functional values. So, what we will say I have taken such a small neighborhood. So, I am getting a small neighborhood for whatever mean all the functional values are lying within that small neighborhood from here. So, I have taken a or any functional values that I am getting on that neighborhood is actually corresponding to some x in that neighborhood means, you are actually here done the reverse thing you have taken some neighborhood very near x and you are showing that now I am taking only those xs for which these are corresponding to values of effects from here or in that neighborhood. So, I have x values which are in that neighborhood. So, I can forget about the rest.

So, this is a very tricky thing and this has to come in to your mind that this is not the correct approach to go from the x thing to the f x thing you have to always come from the f x thing to the x thing that, what it does not matter whatever be the interval any point that I choose, it must corresponding correspond to an effects this may not happen in this sort of situation when limits do not exist.

You may have an interval around f x where it where a point does not corresponding to a given interval you have already chosen around x. So, you cannot chose an interval around x a priori, it will choose around f x and you have to make sure whatever view point you choose in that interval mass corresponding correspond to and f of x, the y must be f of x where x is in some neighborhood which is which is near, which are small neighborhood around that point around the point. So, this is something extremely important which is hardly taken care of. So, we will give some more discussions in the next lecture.

Thank you.