Game Theory

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Lecture 10
Zero-Sum Games: Introduction and examples

In the previous lectures, we have seen combinatorial games. Now, we will see classical game theory, developed by Von Neuman. We have already seen in combinatorial games, that there are two players who make decisions alternatively. In classical game theory, which is sometimes also called economic game theory, two players, instead of choosing their actions alternatively, they choose them simultaneously.

In fact, in this classical game theory, games where the players choose their decisions or actions alternatively are known as *extensive form games*. We will come back to that at a later stage. To start with, let us consider the following example, known as '*Matching Pennies*'.

In this game, there are two players and both the players have two strategies H and T. Now, if both the players choose same strategy then Player 1 gets an unit of money from Player 2. If they differ then Player 1 has to pay Player 2 an unit of money. The following matrix gives a pictorial representation of the game:

		P_2	
		H	T
P_1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

We call this game a *zero-sum* game as in each outcome, the sum of utilities of the two players is 0. Whatever one player wins, the other player loses. This can be seen in the 'Matching Pennies' example.

The next example we look at is a well known childhood game known as 'Rock-Paper-Scissors'. Again, it is a two player game and each player has three strategies. The choices are 'Rock', 'Paper' and 'Scissors'.

		P_2		
		R	P	S
	R	$0,0$	$-1,1$	$1,-1$
P_1	P	$1,-1$	$0,0$	$-1,1$
	S	$-1,1$	$1,-1$	$0,0$

As we have mentioned earlier, both players have 3 choices each, given by R, P, S. Now, it is interpreted as follows: Rock can be covered by Paper, Paper can be cut by Scissors and Scissors can be broken by Rock.

This means that if Player 1 chooses Rock, Player 2 chooses Paper, Player 2 can hide Player 1's Rock and therefore, gets 1 unit from Player 1. This is shown by the outcome $(-1,1)$, where -1 is Player1's payoff and 1 is Player 2's payoff. Similarly, if Player 1 chooses Rock and Player 2 chooses Scissors, Scissors can be broken by Rock and therefore, Player 2 must pay 1 unit to Player 1. Finally, if Player 1 chooses Scissors and Player 2 chooses Paper then Player 2 has to pay that 1 unit to Player 1. We assume that when both the players choose the same action, it is a draw and they get nothing. This is another classic example of a zero-sum game.

Now, we look at another game. This is known as a Coordination game, with two players. The story behind this is as follows: There are two friends, one of whom likes to go to the movies and the other likes to go to watch sports. But more importantly, where ever they go, if they are together, it gives both of them a strictly positive payoff. If they go to different places, then both of them get 0 payoff. The game matrix is given by,

		P_2	
		$C1$	$C2$
P_1	$C1$	$4,2$	$0,0$
	$C2$	$0,0$	$2,4$

For the sake of simplicity, we take 'C1' and 'C2' as their strategies, denoting Choice 1 and Choice 2, instead of 'Movies' and 'Sports'. Note that, the sum of these entries is not equal to zero. This is known as a *non-zero-sum game*. Let us formally define these games. A two player game is given by,

- Players: $N = \{1,2\}$
- Player 1's set of choices: X
- Player 2's set of choices: Y
- Player 1's utility: $\pi_1 : X \times Y \to \mathbb{R}$
- Player 1's utility: $\pi_2 : X \times Y \to \mathbb{R}$

Each player's objective is to maximize their utility. The important question here is: What are the optimal choices of the players? But first, we need to understand that they are different from optimization problems. In an optimization problem, there is a single player: the decision maker. She has some set of choices and accordingly, a utility function available and she has to choose the best choice among them. If it is a maximization problem, there will be a utility and if it is a minimization problem, there is a cost. But for now, we consider everything to be a utility maximization problem. But here, Player 1 cannot simply optimize over his choices. The reason is that his utility depends on the choices made by the second player. In the same way, the second player cannot simply maximize over her choices because her payoff depends on the first player's choices. So, game theory is basically deals with such situations where there are multiple people and each player has their own utility which depends not only on their choice but on the choices of other players.

Let us extend our notation to multiple players:

- Players: $N = \{1, 2, .., n\}$
- For $i \in \{1, 2, .., n\}$, Player i's set of choices: X_i
- For $i \in \{1, 2, .., n\}$, Player i's utility: $\pi_1 : X_1 \times X_2 \times .. \times X_n \to \mathbb{R}$

Recall that, in a combinatorial game like Tic-Tac-Toe, Player 2 observes what Player 1 does and then makes her move. In the case of Matching Pennies however, the players do not know what the other player would choose. Both players make their decisions independently and simultaneously. For example, in the Matching Pennies game, if Player 1 lets his decision be known to Player 2, then Player 2 will make the move which assures her a utility of 1 and hence, Player 2 will get -1. But, both players are utility maximizing agents.

This leads to the concept of '*Rationality*'. This means that players are selfish. Every player wants to make decisions which would be the best for themselves irrespective of the other players' utilities.

There is another concept known as 'Intelligent'. This says that the players are intelligent enough to understand what decisions they should make. We assume that both players are equally intelligent and rational. Moreover, every player knows that every player is equally rational and intelligent.

Now let us say, Player 1 somehow knows what Player 2 is playing, say $y \in X$. Because we have assumed that he is intelligent, he will choose $x \in X$ which maximizes his utility. His utility is given by $\pi_1(x, y)$ which he maximizes over $x \in X$. Hence, he solves,

$$max_{x \in X} \pi_1(x, y)$$

Similarly, if Player 2 knows what Player 1 is playing, say $x \in X$, then she fixes x and chooses $y \in Y$ which maximizes her utility. Hence, she solves,

$$max_{y \in Y} \pi_2(x, y)$$

Whichever y maximizes the above, Player 2 plays that action y. This introduces the following notion:

$$(x^*, y^*) \in X \times Y \tag{1}$$

where Given y^*, x^* maximizes $\pi_1(x, y^*)$ and given x^*, y^* maximizes $\pi_2(x^*, y)$. This is the equilibrium notion and is known as the *Nash Equilibrium.*