Game Theory
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Lecture - 13
Zero-Sum Games: Existence of Saddle Point Equilibria

Let S_1 and S_2 be the action spaces of player 1 and player 2 respectively. Let $f : S_1 \times S_2 \to \mathcal{R}$ be a continuous function. Player 1 chooses an action x from S_1 and player 2 chooses y from S_2. As a result player 1 gets $f(x, y)$ from player 2. The objective of player 1 is to choose his action so that f will be maximized and at the same time player 2 chooses his action to minimize f.

If player 1 chooses an action $x \in S_1$, then the best he can get is $\min_y f(x, y)$. And hence he chooses an x which maximizes this minimum value i.e., he choose $x^\star$ such that

$$\max_x \min_y f(x, y) = \min_y f(x^\star, y)$$

This maxmin value is called the security level of player 1. In a similar way, the security value for player 2 can be introduced and is given by minmax value $\min_y \max_x f(x, y)$. A pair of strategies $(x^\star, y^\star)$ given as above are called minmax (or maxmin) strategies of the players.

Proposition 1. *Security level for player 1 is always less than or equal to the security level of player 2 i.e.,*

$$\max_x \min_y f(x, y) \leq \min_y \max_x f(x, y)$$

One can find numerous examples to show that the above inequality can be strict. When the security levels for both the players are same, then the pair of minmax strategies play a crucial rule in the study of zero-sum games. These can be shown to be equivalent to saddle point equilibrium. we now introduce the definition of saddle point equilibrium.

[Saddle Point Equilibrium] A pair of strategies $(x^\star, y^\star) \in S_1 \times S_2$ is called a saddle point equilibrium if

$$f(x, y^\star) \leq f(x^\star, y^\star) \leq f(x^\star, y)$$

for all $(x, y) \in S_1 \times S_2$.

We now list some properties of saddle point equilibrium.

Proposition 2. *If $(x^\star, y^\star)$ is a pair of saddle point equilibrium, then*

$$f(x^\star, y^\star) = \max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

Proposition 3. *If*

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

and the outer maximizer $x^\star$ and outer minimizer $y^\star$ exists, then $(x^\star, y^\star)$ is saddle point equilibrium. In other ways, minmax strategies and saddle point equilibrium coincide.

Theorem 1 (Minmax Theorem)**.** *Let S_i be a compact convex subset of some euclidean space and f be concave-convex function (i.e., f is concave in x-variable when y is fixed and it is convex in y-variable when x is fixed). Then a saddle point equilibrium exists.*

Proof. We first assume that f is strict concave in x-variable and strict convex in y-variable when the other variable is fixed.

By the strictness, for each x, there is a unique $y(x)$ such that

$$f(x, y(x)) = \min_y f(x, y) := m(x)$$

Since f is uniformly continuous (as it is continuous on a compact set), $y(x)$ is continuous as a function of x. Also $m(x)$ is concave (being minimum of a family of concave functions) function. Note that minimum of a family of concave functions need not be continuous in general. However, our $m(x)$ is continuous (Verify!). Let $x^\star$ be such that

$$m(x^\star) = \max_x m(x) = \max_x \min_y f(x, y)$$

Now our aim is to show that $(x^\star, y(x^\star))$ is a saddle point equilibrium. Note that

$$m(x^\star) = f(x^\star, y(x^\star)) \leq f(x^\star, y) \text{ for all } y \in S_2 \tag{1}$$

from the definition of $y(x)$. Also,

$$f(x^\star, y(x^\star)) = m(x^\star) \geq m(x) \text{ for all } x \in S_1 \tag{2}$$

from the definition of $x^\star$. Let $\tilde{y} = y((1-t)x^\star + tx)$, then

$$m((1-t)x^\star + tx^\star) = f((1-t)x^\star + tx^\star, \tilde{y}) \geq (1-t)f(x^\star, \tilde{y}) + tf(x, \tilde{y}) \tag{3}$$

Using (1), (2) and (3), we now get

$$m(x^\star) \geq m((1-t)x^\star + tx^\star) \geq (1-t)f(x^\star, \tilde{y}) + tf(x, \tilde{y}) \geq (1-t)m(x^\star) + tf(x, \tilde{y})$$

Now letting $t \to 1$, we obtain

$$m(x^\star) \geq f(x, y(x^\star))$$

for all $x \in S_1$. Combing this with (1), we get

$$f(x, y(x^\star)) \leq f(x^\star, y(x^\star)) \leq f(x^\star, y)$$

for all $(x, y) \in S_1 \times S_2$. Thus $(x^\star, y(x^\star))$ is a saddle point equilibrium. This proves the theorem for the strict concave/convex case. We now prove for general case.

Let

$$f^\epsilon(x, y) = f(x, y) - \epsilon|x|^2 + \epsilon|y|^2$$

for $\epsilon > 0$. Then f^ϵ is strict concave in x and strict convex in y when the other variable is fixed. Thus from above there is a pair $(x^\epsilon, y^\epsilon) \in S_1 \times S_2$ such that

$$f^\epsilon(x, y^\epsilon) \leq f^\epsilon(x^\epsilon, y^\epsilon) \leq f^\epsilon(x^\epsilon, y) \tag{4}$$

Since S_1, S_2 are compact, we can extract a subsequence which is convergent. With an abuse of notation, we denote this subsequence by $\{(x^\epsilon, y^\epsilon)\}$ itself. Let $(x^\epsilon, y^\epsilon) \to (x^\star, y^\star)$. Then letting $\epsilon \to 0$ in (4), we obtain

$$f(x, y(x^\star)) \leq f(x^\star, y(x^\star)) \leq f(x^\star, y)$$

for all $(x, y) \in S_1 \times S_2$, which completes the proof of the theorem. $\square$

The proof above is due to Karlin. The general way to prove this theorem is to apply fixed point theorems. But the above proof avoids the use of fixed point theorem relying on strict convexity/concavity.

In the above proof, we haven't use the fact that space is euclidean. We have only used the fact that the strategy sets are compact and convex. So therefore it goes beyond the euclidean space, even though we stated it only for euclidean space. So, this theorem is valid even in an infinite dimensions as long as $S1$ and S_2 are compact and convex sets. In next session, we will study Von Neumann minmax theorem.