Game Theory
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Lecture - 14
Zero-Sum Games: Proof of Minimax Theorem

Theorem 1 (Minmax Theorem, von Neumann)**.** *Every finite zero-sum game admits value.*

Before proceeding with the proof we recall two results.

Proposition 1. *Let C be a compact convex subset of a euclidean space $\mathcal{R}^m$ and $0 \notin C$. Then there exists a vector $z \in \mathcal{R}^m$ such that*

$$z \cdot x > 0 \, \textit{for} \; x \in C.$$

Proof. Since C is convex, there exists a unique point $z \in C$ such that

$$|z|^2 \leq |x|^2$$

for every $x \in C$.

Now consider the hyperplane for which z is normal and pick any point in this hyper plane. Note that for any $x \in C$,

$$\|z\|^2 \leq \|(1-\alpha)z + \alpha x\|^2 = (1-\alpha)^2\|z\|^2 + 2\alpha(1-\alpha)z \cdot x + \alpha^2\|x\|^2$$

Therefore,

$$0 \leq \alpha(\alpha-2)\|z\|^2 + 2\alpha(1-\alpha)z \cdot x + \alpha^2\|x\|^2$$

Dividing by α, we have

$$0 \leq (\alpha-2)\|z\|^2 + 2(1-\alpha)z \cdot x + \alpha\|x\|^2$$

Letting $\alpha \to 0$, we have

$$0 \leq -2\|z\|^2 + 2z \cdot x$$

which gives the required inequality

$$\|z\|^2 \leq z \cdot x$$

□

Proposition 2. *Let A be any matrix of order $m \times n$. Then either*

1. *there exists $x \in \mathcal{R}^m$, $x \neq 0$, $x \geq 0$ such that $x'A \geq 0$; or*
2. *there exists $y \in \mathcal{R}^n$, $y \neq 0$, $y \geq 0$ such that $Ay \leq 0$.*

Proof. Let $e_1, e_2, \cdots, e_n$ be the unit vectors in $\mathcal{R}^n$. Let the rows of A be denoted by $a_1, a_2, \cdots, a_m \in \mathcal{R}^n$. Let C be the convex hull of $-e_1, -e_2, \cdots, -e_n$ and $a_1, a_2, \cdots, a_m$, then C is a compact convex subset of $\mathcal{R}^n$. Now two cases arise: $0 \in C$ or $0 \notin C$.

Case $0 \in C$ **:** In this case, there exists non-negative real numbers $x_1, x_2, \cdots, x_m, \eta_1, \eta_2, \cdots, \eta_n$ such that

$$x_1 a_1 + x_2 a_2 + \cdots + x_m a_m - \eta_1 e_1 - \eta_2 e_2 - \cdots - \eta_n e_n = 0,$$

and $x_1 + x_2 + \cdots + x_m + \eta_1 + \eta_2 + \cdots + \eta_n = 1$. Clearly all of $x_1, x_2, \cdots, x_m$ can be zero. Indeed, if $x_1 = x_2 = \cdots = x_m = 0$, then we must have

$$\eta_1 e_1 + \eta_2 e_2 + \cdots + \eta_n e_n = 0, \eta_1 + \eta_2 + \cdots + \eta_n = 1$$

which contradicts the liner independence of the vectors $e_1, e_2, \cdots, e_n$. Thus we have non-negative real numbers $x_1, x_2, \cdots, x_m \in \mathcal{R}$, not all of them zero, such that

$$x_1 a_1 + x_2 a_2 + \cdots + x_m a_m = \eta$$

where $\eta = (\eta_1, \eta_2, \cdots \eta_n) \in \mathcal{R}^n$. Note that $\eta \geq 0$. In other words,

$$x' A = \eta \geq 0$$

where $x = (x_1, x_2, \cdots, x_m)' \in \mathcal{R}^m$, $x \neq 0$ and $x \geq 0$. This proves (i).

Case $0 \notin C$ **:** Since $0 \notin C$, there is a hyperplane separating 0 and C. In other words there must exist $z \in \mathcal{R}^n$ such that

$$x \cdot z > 0 \text{ for every } x \in C.$$

Since $-e_i \in C$, we must have $z_i < 0$ and hence $z \neq 0$, $z \leq 0$. Now $a_i \in C$ and hence $a_i \cdot z > 0$ for every $i = 1, 2, \cdots, m$. Thus $Az > 0$. Now taking $z = -y$ we obtain $Ay < 0$ which proves (ii). $\square$

With these two lemmas in hand, we are now ready to prove the minmax theorem.

Proof. **(Minmax Theorem)**

From the previous result either we have two cases: there exists $x \geq 0 \in \mathcal{R}^m$, $x' \neq 0$ such that $x'A \geq 0$ or there exists $y \geq 0 \in \mathcal{R}^n$, $y \neq 0$ such that $Ay \leq 0$. Letting $\bar{x} = \frac{c}{\sum x_i}$ and $\bar{y} = \frac{y}{\sum y_j}$, we note that $\bar{x} \in \Delta_m$ and $\bar{y} \in \Delta_n$ and either $\bar{x}'A \geq 0$ or $A\bar{y} \leq 0$.

The first case means that $\bar{x}'Ay \geq 0$ for ever $y \in \Delta_n$ which means that the lower value of the game

$$V^-(A) = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x'Ay \geq 0.$$

The second case means that $xA\bar{y} \leq 0$ for every $x \in \Delta_m$, which gives that the upper value of the game

$$V^+(A) = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x'Ay \leq 0.$$

Thus we have either $V^-(A) \geq 0$ or $V^+(A) \leq 0$. Let $B = ((a_{ij} - c))$, where $c \in \mathcal{R}$. Note that $V^-(B) = V^-(A) - c$ and $V^+(B) = V^+(A) - c$. Thus we must have

$$V^-(A) \geq c \text{ or } V^+(A) \leq c$$

for any $c \in \mathcal{R}$. This can happen only if both $V^-(A)$ and $V^+(A)$ are equal. This completes the proof of the minmax theorem. $\square$