Game Theory

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Lecture 15
Zero-Sum Games:
Properties of Saddle Point Equilibria

In the previous lecture we have proved the Minmax theorem for finite games. Now, let us explore some properties of the saddle points.

Proposition: *Suppose* (x^*, y^*) *and* $(\hat{x}, \hat{y})$ *are two saddle point equilibria of the matrix game* A*. Then,*

$$x^{*T} A y^* = \hat{x}^T A \hat{y}$$

Proof. This follows from the fact that

$$x^{*T} A y^* = \min_{y \in \Delta_2} \max_{x \in \Delta_1} x^T A y = \max_{x \in \Delta_1} \min_{y \in \Delta_2} x^T A y$$

Let us look at the definition of (x^*, y^*). As we have seen earlier, $\pi(x, y) = x^T A y$. Then, the above equation becomes,

$$\pi(x, y^*) \leq \pi(x^*, y^*) \leq \pi(x^*, y)$$

for all $x \in \Delta_1$, $y \in \Delta_2$. This immediately implies the following. Consider the first part of the above equation:

$$\pi(x, y^*) \leq \pi(x^*, y^*) \quad \forall \; x \in \Delta_1$$
$$\Rightarrow \max_{x \in \Delta_1} \; \pi(x, y^*) \leq \pi(x^*, y^*)$$

Also,

$$\pi(x^*, y^*) \leq \pi(x^*, y) \quad \forall \; y \in \Delta_2$$
$$\Rightarrow \pi(x^*, y^*) \leq \min_{y \in \Delta_2} \; \pi(x^*, y)$$

Hence, we have

$$\max_{x \in \Delta_1} \; \pi(x, y^*) \leq \pi(x^*, y^*) \leq \min_{y \in \Delta_2} \; \pi(x^*, y)$$

Now, we have already seen that the other side of inequalities also hold, so, we have:

$$\max_{x \in \Delta_1} \; \pi(x, y^*) = \pi(x^*, y^*) = \min_{y \in \Delta_2} \; \pi(x^*, y)$$

Therefore,

$$\min_{y \in \Delta_2} \max_{x \in \Delta_1} \pi(x, y) \leq \max_{x \in \Delta_1} \pi(x, y^*)$$

and,

$$\min_{y\in\Delta_2} \pi(x^*, y) \leq \max_{x\in\Delta_1} \min_{y\in\Delta_2} \pi(x, y)$$

This implies,

$$\min_{y\in\Delta_2} \max_{x\in\Delta_1} \pi(x, y) \leq \pi(x^*, y^*) \leq \max_{x\in\Delta_1} \min_{y\in\Delta_2} \pi(x, y)$$

But, the following always holds:

$$\max_{x\in\Delta_1} \min_{y\in\Delta_2} \pi(x, y) \leq \min_{y\in\Delta_2} \max_{x\in\Delta_1} \pi(x, y)$$

where,

$$V^-(A) = \max_{x\in\Delta_1} \min_{y\in\Delta_2} \pi(x, y) V^+(A) = \min_{y\in\Delta_2} \max_{x\in\Delta_1} \pi(x, y)$$

This gives us,

$$V^-(A) \leq V^+(A)$$

But, we proved that,

$$V^+(A) \leq \pi(x^*, y^*) \leq V^-(A)$$

Hence, we have

$$V^+(A) = \pi(x^*, y^*) = V^-(A)$$

This means that if x^* and y^* are two saddle point equilibria, then the value corresponding to that x^* and y^* is the same as the lower security value which is equal to the upper security value. This proves that any two Saddle Point Equilibria will give the same value.

Proposition. *If $(\hat{x}, \hat{y})$ and (x^*, y^*) are two Saddle Point Equilibria, then $(\hat{x}, y^*)$ is also a Saddle point Equilibrium. This is known as the exchangeability property.*

Proof. Consider the following:

$$\begin{aligned}
\pi(x^*, y^*) &= \max_{y\in\Delta_2} \pi(x^*, y) \\
&= \min_{x\in\Delta_1} \max_{y\in\Delta_2} \pi(x, y) \\
&\leq \max_{y\in\Delta_2} \pi(\hat{x}, y) = \pi(\hat{x}, \hat{y})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi(x^*, y^*) &= \max_{y\in\Delta_2} \pi(x^*, y) \\
&= \max_{y\in\Delta_2} \pi(\hat{x}, y) \\
&= \min_{x\in\Delta_1} \max_{y\in\Delta_2} \pi(x, y)
\end{aligned}$$

Therefore, both $\hat{x}$ and x^* are outer minimizers here. Therefore, $(\hat{x}, y^*)$ is a Saddle Point Equilibrium. Few details need to be added here to make the proof concrete. This is left for the reader as an exercise.

Another way to see this is the following. For any $y \in \Delta_2$,

$$BR_1(y) = \{x^* \in \Delta_1 : \pi(x^*, y) = \max_{x \in \Delta_1} \pi(x, y)\}$$

This is called the *Best Response* of Player 1 when Player 2 plays $y \in \Delta_2$. Similarly,

$$BR_2(x) = \{y^* \in \Delta_2 : \pi(x, y^*) = \max_{y \in \Delta_2} \pi(x, y)\}$$

is the Best Response of Player 2 when Player 1 plays $x \in \Delta_1$.

Result: (x^*, y^*) is a Saddle Point Equilibrium iff

$$x^* \in BR_1(y^*) \quad \& \quad y^* \in BR_2(x^*)$$

The proof is left for the reader as an exercise.

Also, note that $BR_1(y)$ and $BR_2(x)$ are convex and compact sets. This is true as x^* is the maximizer of $\pi(x, y)$ which is a linear function in x. Hence, as we are maximizing a linear function over a convex and compact set, the set of maximizers is also convex and compact.

Also, even though we have proved the above results for matrix games, these results hold for general games as well. Of course, we need to make sure that the payoff function satisfies the concave-convex property which we have assumed in the minmax theorem in a general setup, that is the payoff function is concave in the maximizing variable and convex in the minimizing variable. Proving them in a general setup is not hard and is left for the reader to check.

Next, we will see the linear programming connection of matrix games. Recall that, $\pi(x, y) = x^T AY$. Now, note that for $x \in \Delta_1$,

$$\min_{y \in \Delta_2} x^T Ay = \min_{1 \leq j \leq n} x^T Ae^j$$

This is intuitive. Given Player 1's choice $x \in \Delta_1$, there is a pure strategy j such that $y = e^j$ minimizes $x^T Ay$. Here, e^j signifies the pure strategy in the mixed strategy form: action j is played with probability 1, the rest have probability 0 of being played.

Finding the optimal strategy for Player 1 amounts to the following optimization problem:

$$\begin{aligned} &\max_{x \in \Delta_1} \min_{1 \leq j \leq n} x^T Ae^j \\ &s.t. \quad x_i \geq 0 \\ &\qquad \sum_{i=1}^{m} x_i = 1 \end{aligned}$$

Now, we know that

$$\min_{1\le j\le n} x^T A e^j \le x^T A e^j \qquad \forall\ j = 1,2,..,n$$

Let $t = \min_{1\le j\le n} x^T A e^j$. Then, the above inequality becomes,

$$t \le x^T A e^j \qquad \forall\ j = 1,2,..,n$$

Now, the problem becomes,

$$\begin{aligned}
&\max t \\
&s.t.\quad t \le x^T A e^j && \forall\ j = 1,2,..,n \\
&x_i \ge 0 && \forall\ i = 1,2,..,m \\
&\sum_{i=1}^{m} x_i = 1
\end{aligned}$$

This is a linear programming problem. Similarly, Player 2's linear programming problem is given by the following. For $y \in \Delta_2$,

$$\max_{x\in\Delta_1} x^T A y = \max_{1\le i\le m} e_i^T A y$$

This gives us,

$$\min_{y\in\Delta_2}\max_{x\in\Delta_1} x^T A y = \min_{y\in\Delta_2}\max_{1\le i\le m} e_i^T A y$$

Now, let $s = \max_{1\le i\le m} e_i^T A y$. Then, Player 2's problem is given by:

$$\begin{aligned}
&\min s \\
&s.t.\quad s e_i^T A y && \forall\ i = 1,2,..,m \\
&y_j \ge 0 && \forall\ j = 1,2,..,n \\
&\sum_{j=1}^{n} y_j = 1
\end{aligned}$$

Here is an interesting exercise. Show that these two linear programming problems are dual to each other. This is an important and interesting exercise left for the reader. In the next lecture, we will see some more interesting properties of zero-sum games.