Game Theory
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Lecture - 16
Zero-Sum Games: Computing Saddle Point Equilibria

Welcome to the NPTEL course on game theory. Today, we will start exploring computing the saddle point equilibrium for zero-sum games, matrix games. So, let us start with some simple ideas. We start exploring computing the saddle point equilibrium.

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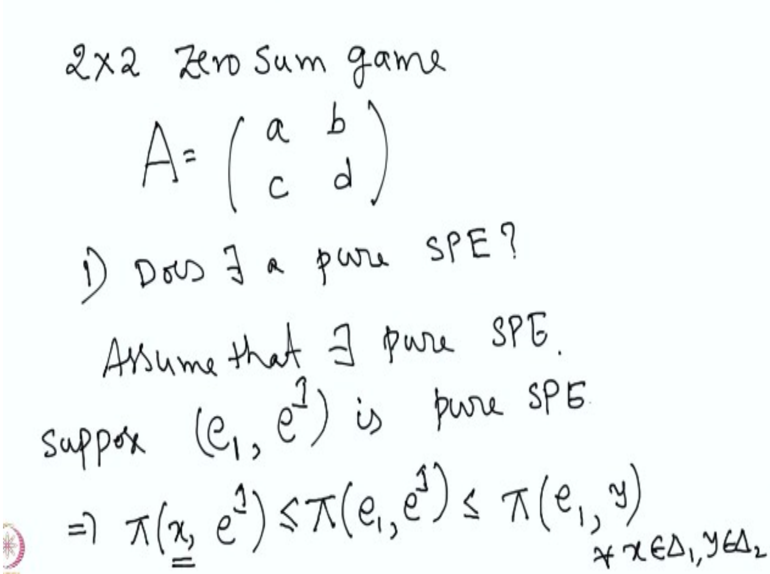

We start with a simple case. We construct the two by two zero-sum games. So you have a matrix A. There are only 2 pure strategies per both the players. How do we calculate the saddle point equilibrium? So two things, one is does there exist a pure saddle point equilibrium? This is the first question. So recall what is the pure saddle point equilibrium? The pure saddle point equilibrium is essentially Player 1 is selecting one of the 2 rows, Player 2 is selecting one of the 2 columns.

If there is a row and column pair which satisfies the saddle point equilibrium condition, then we say that that is a pure saddle point equilibrium. As we already pointed out earlier, the pure saddle point equilibrium need not exist all the time. So, one needs to go for mixed equilibrium. Right now, let us consider the case where the pure saddle point equilibrium may exist. For example, let us assume that there is a pure, let us assume that there exists pure saddle point equilibrium. So let us see how this looks like what if suppose the row 1 and column 1 is a pure equilibrium.

This means that the payoff that player one is getting satisfies this relation or this should be true for every x in delta 1, y in delta 2. In fact as we have seen in the previous lectures, this pi is a bilinear function. Once you fix one of the strategy verifying the pure strategy is sufficient, is a very important aspect, so let us check in particular here, in place of x we need to see if you take x to be e1 or e2, how does this work? So, let us check that conditions.

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$$\pi(e_1, e^1) \leq \pi(e_1, e^1) \leftarrow \text{obvious.}$$

$$\pi(e_2, e^1) \leq \pi(e_1, e^1) \qquad A: \begin{pmatrix} \boxed{a} & b \\ c & d \end{pmatrix}$$

$$c \leq a \leftarrow 1$$

$$\pi(e_1, e^1) \leq \pi(e_1, e^2)$$

$$a \leq b$$

$$\text{If } \underline{c \leq a \leq b} \Rightarrow (e_1, e^1) \text{ is pure SPE}$$

Illy we can work out for other cases.

So we need to check pi e1, of course e1 less than or equals to pi e1, of course we were interested in this, this is of course obvious statement. The next thing is we need to see this, what is this one, this is when Player 1 selects row 2, Player 1 selects column 1, this is nothing but our remember A is a, b, c, d, so therefore this is c and this is nothing but a, c less than or equals to a, this is the condition that we will get it here. This is one condition, okay?

Now, let us look at the second condition here when y is equals to e1, that is the obvious statement, so y is equals to e2 means pi e1, e1 less than or equals to pi e1, e2, so this means a less than or equals to b. That means if c is less than or equals to a less than or equals to b, then e1, e1 is pure equilibrium. So, this is a condition for these to be equilibrium. So, look at this one. If this has to equilibrium when Player 1 chooses this row, then Player 2 should not deviate from column 1 to column 2 that means a should be smaller than b, that is exactly this condition.

Now for Player 2 if we fixes this column, Player 2 should not deviate from row 1 to row 2, that means a should be bigger than c that is this condition, so this gives it. And similarly we can work out for other cases, so that means instead of row 1, column 1, we can also ask when is row 1 column 2 is a pure saddle point equilibrium and like that there are 4 possibilities in this.

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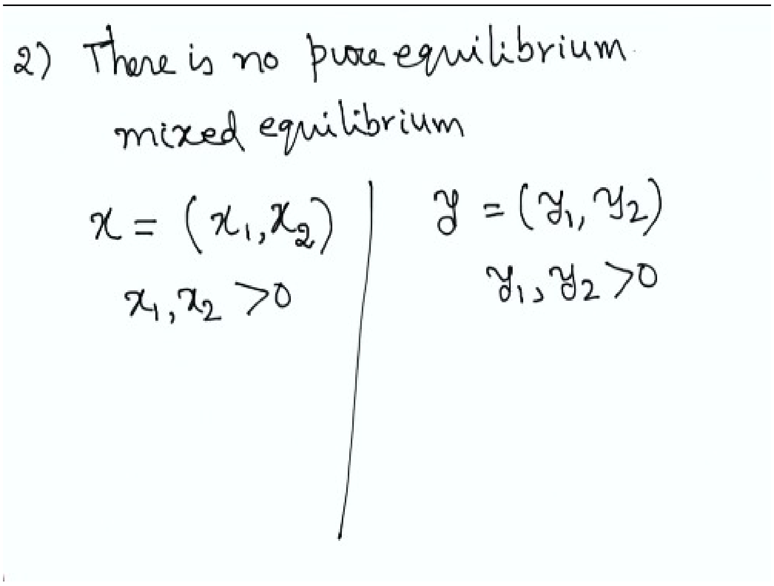

So, now we will look at the case where there is no pure equilibrium, so in which case how can we prove that there exists a saddle-point equilibrium. In fact, existence is known. We need to show in this case mixed equilibrium, we need to consider the mixed equilibrium, of course the existence is guaranteed by von Neumann minimax theorem, we need to calculate that. So let us look at it, because there is no pure equilibrium, there are only two mixed strategies to play, so there is only 2 pure strategies to play and no pure equilibrium that means the players are going to play both the strategies.

That means the player strategies you take x equals x1, x2 if you take if in a mixed strategy this is for Player 1 and for Player 2 if I take y1, y2, we know that x1, x2 both of them will be bigger than 0 and y1, y2 both will be bigger than 0. So remember so you can ask the following question, what if for example Player 2 has a pure equilibrium pure optimal strategy whereas Player 1 does not have pure strategy, does there exist such a situation?

So this is an interesting exercise to see. In fact if that happens, the Player 1 will also have a pure strategy because Player 2 is already playing a pure strategy that means he is choosing one of the column, if once he chooses one of the column and Player 1 knows which of the rows is giving him better payoff, if they are equal, then only he can choose any of them may be, otherwise he will always, there is always a pure equilibrium. So in that case, we can certainly need not consider. So, let us look at this situation where both the players have a pure equilibrium and how do we calculate this one. So, now let us look at it.

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$$\pi(\hat{x},\hat{y}) = \pi(x^*, e^1) = \pi(x^*, e^2)$$

$$\parallel$$

$$y^*_1\, \pi(x^*, e^1) + y^*_2\, \pi(x^*, e^2)$$

$$\pi(x^*, e^1) = \pi(x^*, e^2) \rightarrow x^*_1\, \pi(e_1, e^2) + x^*_2\, \pi(e_2, e^2)$$

$$\parallel$$

$$x^*_1\, \underbrace{\pi(e_1, e^1)}_{a} + x^*_2\, \underbrace{\pi(e_2, e^1)}_{c}$$

$$a x^*_1 + c x^*_2 = b x^*_1 + d x^*_2$$

Suppose, this is a pure strategy, so in this case what is the payoff that Player 1 is going to get? pi x, y, okay? Let me put x star, y star for the easiest this thing, remember this has to be same as pi x star, e 1 which has to be same as pi x star, e 2. Once again, the reason for this is the bilinearity. It is liner in this y variable. In fact if you look at this is nothing but y star 1 pi x star this is e1 plus y2 star pi x star e2. Now it is a convex combination of these two values.

If these two values are not the same, then if one of them is bigger, there is no reason for the second player to play that particular pure strategy in his optimal strategy. So therefore, we can assume in the said case that these two are, these two are same. So therefore, this condition comes. So if x star is an optimal strategy for Player 1, what we have is the pi x star e1 has to be same as pi x star e2.

So this is nothing but x star 1 of pi e1, e1 plus x star 2 of pi e2, e1, this value is nothing but pi e1, e1 is nothing but a, pi e2 e1 is nothing but c, therefore what we have is that a x1 star plus c x2 star, that is going to be pi x star e1. Similarly, we can calculate this value, this is going to be if you really write it here, this is nothing but x star 1 into x pi e1 e2 plus x2 star pi e2, e2. So therefore, this is nothing but x1 star x pi e1 e2, that is the first row second column that is value is b plus this is the second row second column that is d x2 star.

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If x^* is optimal mixed strategy for Player 1

then

$$a x_1^* + c x_2^* = b x_1^* + d x_2^*$$

$$x_1^* + x_2^* = 1 \Rightarrow x_2^* = 1 - x_1^*$$

$$a x_1^* + c(1-x_1^*) = b x_1^* + d(1-x_1^*)$$

$$x_1^* = \boxed{\frac{d-c}{a-c-b+d}} > 0$$

$$x_2^* = 1 - x_1^* = \frac{a-\cancel{c}-b+\cancel{d}-(\cancel{d}-\cancel{c})}{a-c-b+d} = \boxed{\frac{a-b}{a-c-b+d}} > 0$$

So, what we have here is if x star is optimal strategy for player optimal mixed strategy for Player 1, then we have the following condition that I rewrite this one ax1 star plus c x2 star is same as bx1 star plus d x2 star. Now, this is an equation, linear equation in 2 unknowns, but if you really look at it, it is not really 2 unknowns, we also know that x1 star plus x2 star is nothing but 1, therefore x2 star is 1 minus x1 star, we put it back here. What we are going to get here is ax1 star plus c into 1 minus x1 star is same as bx1 star plus d into 1 minus x1 star.

We regroup them and do this thing. What we are going to get here is x1 star is going to be the c d therefore d minus c by here x1 star as a here it is minus c. When it comes this side, it is minus b, this minus d it becomes plus d. So, it is going to be d minus c by a minus c minus b. So this computation led to this fact that x1 one star is nothing but d minus c by a minus c minus b plus d, this is going to be the probability with which the Player 1 is going to play the row 1 one. Similarly, x2 star is simply 1 minus of this. This is going to be a minus c minus b plus d minus of d minus c by a minus c minus b plus d. Just simplifying, this d minus d gets canceled and c this gets canceled.

This will be a minus b by a minus c minus b plus d. So therefore, finally we computed the optimal strategy for Player 1 which is playing the row 1 with this quantity and row 2 with this quantity. Note that this have to be nonnegative, so d minus c by a minus c minus b plus d that should be non-negative term and similarly of course the sum of these two you can easily see that they are 1, but are they nonnegative, that non-negativity comes from the fact that we have assumed there is no pure equilibrium, that means the conditions we have got here will be violated.

These conditions and of course all the other possibilities all of them will be violated and if you really work, check those conditions, from those conditions you can see that this is going to be greater than 0, this is also going to be greater than 0 and we can say that this pure equilibrium. Now this is a case where you can compute the equilibrium very easily. Okay, so in a 2 by 2 games, we can easily calculate the equilibrium. The way to do is the first verify whether there is a pure equilibrium or not.

If there is no pure equilibrium, you know the mixed strategy and the formula is given. Of course, this computation here, we have only done for Player 1, a similar analysis can be done for the Player 2 and it can be easily carried out.

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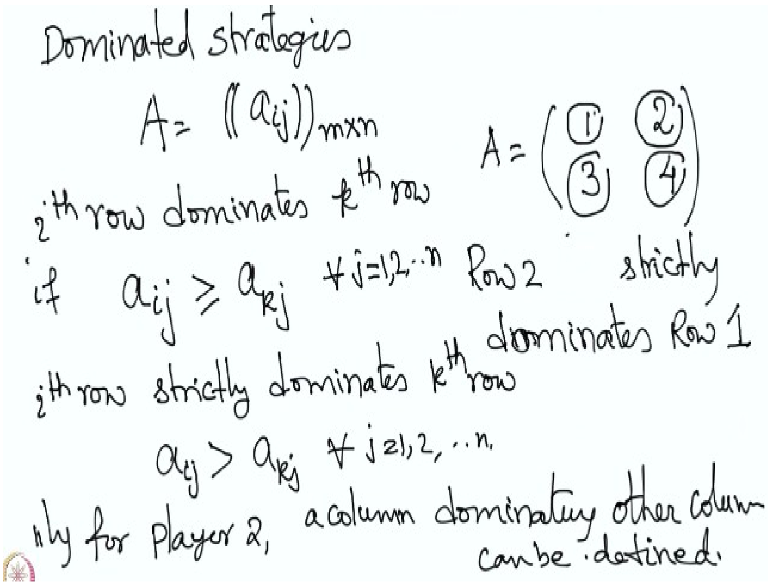

Now, there is another method to solve some of these games, zero-sum games, in fact as we go to non-zero sum, we can use the same thing is by dominated strategies. So, what it means is that suppose you take A to be a matrix with entries a i j, so this is our zero-sum game. Suppose let us consider an example. A to be 1 2 3 4 okay. When you take this game, does Player 1 choose row 1. If you really observe it, in the row 1 one, either he will get 1 payoff as opposed to 3 in row 2, he will get 2 opposed to 4 when he plays row 2.

In either case, choosing row 2 is better for him. So therefore, row 1 he will never play in this game, so row 2 is his automatically optimal strategy. So, this is basically the domination. So here in this case, row 2 strictly dominates row 1, therefore, the domination is helping here to simply compute easily. So in a sense let us write it, ith row dominates kth row if aij is bigger than akj for all j, 1 to n, so this is a m by n matrix. This is known as a domination, but what if ith row strictly dominating means?

A strict domination means, aij is strictly greater than akj for all j, okay? This is for a player one's perspective. Similarly for Player 2, a column dominating other column can be defined, okay? So, this is a useful thing when this game posses this domination property. So, let us see an example and then we will see. So let us check.

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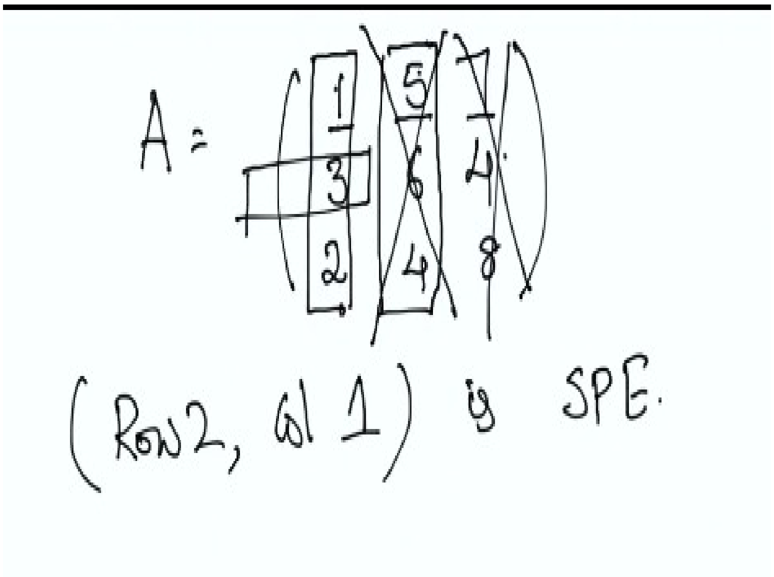

So let us take an example. 1, 5, 7; 3, 6, 4; 2, 4, 8. So what about, does a row dominates another row here? So here if you look at in the first row, you have 1, 5, 7 and 3, 6, 4 in the second row and 2, 4, 8, no row is dominating no other row, but if you really look at it, for the Player 2, the column 1, 3, 2 and the column 2 if you look at it 5, 6, 4. So in the second column he has to pay higher than column 1, so in the column 1, he pays 1, in the column 2, he pays 5, 3 versus 6, 2 versus 4.

So column 2, he is incurring a larger loss, therefore column 1 one dominates column 2. Remember here domination for Player 2 is in terms of a minimization because Player 2 is a minimizer here. So, therefore, he will never play column 2. So, because the Player 2 is never playing column 2, therefore Player 1 will never count this column 2 because we are assuming that the both players are equally intelligent. So this is actually known as a rationality assumption in game theory.

So, the players are assumed to be rational, that means they know that they are going to play their best, okay? So, there is lot more to rationality we will come, we will talk about this whenever stated. So, therefore Player 2 is not going to play second column. What about third column? So, even third column is, all the corresponding entries are higher than corresponding entries in column 1, so therefore, the Player 2 is never going to play column 3 as well.

So this Player 1 can immediately infer that Player 2 is going to play only column 1. Therefore, Player 1 simply maximizes among this column 1 only. So then, he knows that he is going to play this one. Therefore, here the row 2, column 1 is going to be saddle point equilibrium. So, the domination actually helps him this way. We will see this domination more when we go to the nonzero-sum games and in fact we will try to prove some interesting results on this thing, but right now, we will leave at this place okay.

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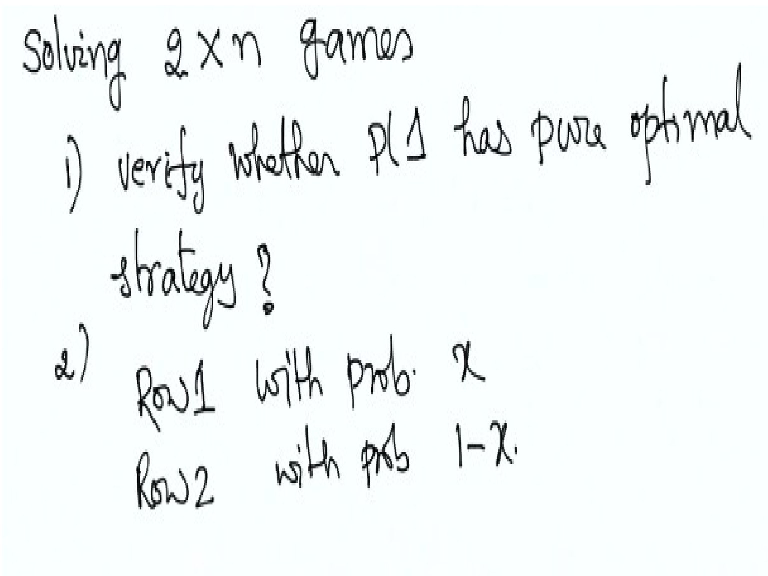

The next thing, how do we solve? So let us assume Player 1 has only 2 strategies. So we are actually going to use the previous idea itself in some way, but player one has only 2 strategies whereas Player 2 has n choices here. So now, because there are only player this thing, if player; you can verify whether Player 1 is going to have a pure optimal strategy. So first question is verify whether Player 1 has pure optimal strategy.

Okay, so this is not going to be hard task because he has only 2 choices, whether he should choose one row or the other, so he can actually verify like in the previous case we verified similarly one can verify the conditions and see what is good. If this is not there, then he is going to play mixed strategy. So therefore, row 1 let us say with a probability x, row 2 with probability 1 minus x. So, let us work out this thing.

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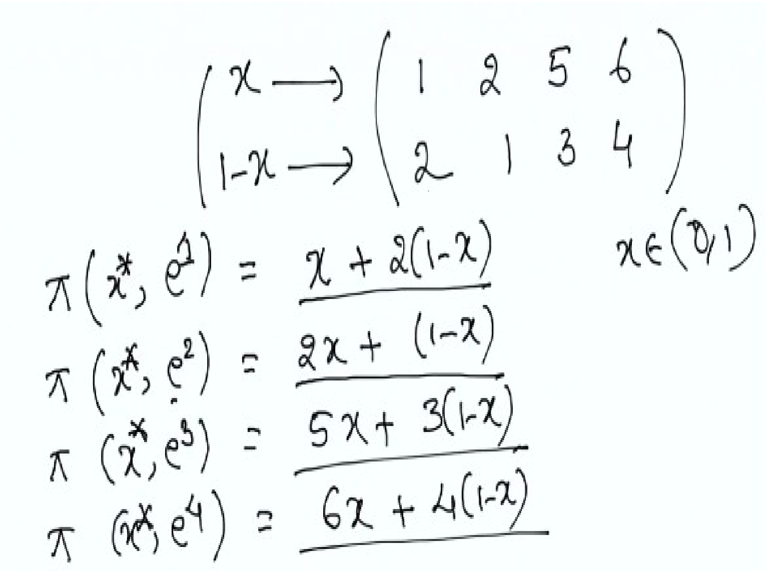

So basically, there is, let us take one example 1, 2, 5, 6; 2, 1, 3, 4, so let us take. So therefore he is going to play with this with probability x, this with probability 1 minus x. So, now what are his payoffs. So, let us take the pure strategy here, that we put x star or rather let me take against with the pure strategy of Player 1 how much he is going to get, okay? So, he is going to play with probability x this one and with 1 minus x, so therefore x plus 2 into 1 minus x is going to be his pay off function here and of course remember x is in 0, 1.

Similarly, if the other player is playing e2 two, this is going to be 2x plus 1 minus x, pi x star e3 is going to be 5x plus 3 into 1 minus x, pi x star e4 is going to be 6x plus 4 into 1 minus x, okay? So as x varies, we know that the Player 1 is going to get this much payoff and if Player 2 is playing e2, Player 1 is getting this much, and likewise for the column 3, column 4. Now as x varies, this is the payoff that he is going to receive.

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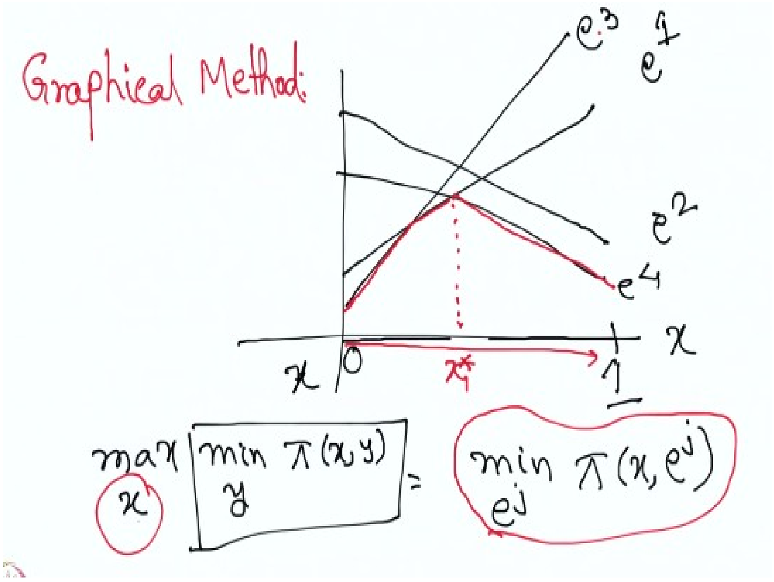

For example when you try to draw a graph, so let us take, so this is x-axis, so 0, say this is 1. When Player 1 is playing e1, let us say, so there is some linear equation that is coming something like this and this is the payoff let us say corresponding to e1. For e2, let us say this is, this is corresponding to e2. Let us say this is corresponding to e3. Let us say this is corresponding to e4. So you can draw the graphs. Now Player 1 is a maximizer, so which gives them maximum among these things.

So player, so recall the Player 1 is going to be maximize overall his strategies x, Player 2 is minimizing over all his strategies y, then pi x,y and this particular term is nothing but minimum over his pure strategies and then Player 1 is maximizing. So these are basically the payoffs that he is going to get. So, if the Player 1 is choosing let us say x as varies from here to here, as it varies from here to here, the Player 1 is going to receive things like this, but he knows that Player 2 is going to play rationally.

So therefore, Player 2 will always choose one of the e's which minimizes that, that means for example as this is the minimum up to this, then here, then here, this is going to be the this function. Now, among this, Player 1 maximizes. Which x maximizes this red graph here? So, this is going to be the x star or x1 star. So, this way, we can calculate when there are only, one of the player has only 2 strategies and other player can have multiple.

So here, we have taken example where the 4 pure strategies for Player 2 and only 2 strategies, 2 choices, for the Player 1. Now, the same thing can be done with Player 1 one having many choices, Player 2 having exactly 2 choices that can be done. So this is known as a graphical method, okay? So, using this example, when one of the player has exactly 2 pure choices, we can compute easily and of course, when the people have multiple choices, these methods would not work, and in such a case, we have to use other methods.

One method which we have already discussed is linear programming method, so we already derived the linear programming formulation for the optimal strategies of Player 1 and Player 2, we use that. In the next session, we explore few more properties of this zero-sum games and check what we can infer from them. Okay, good day!