Game Theory
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Lecture - 17
Zero-Sum Games: Matrix Game Properties

Welcome back to NPTEL course on game theory. In the previous session, we have seen some ways of computing the saddle point equilibrium for matrix games. Now, in fact, one of the interesting property that we have used there is that if Player 1 is let us say playing his optimal strategy x star, then we are checking essentially against the pure strategies of the Player 2. Now, the whole another interesting thing is that do we really need to check against all the pure strategies.

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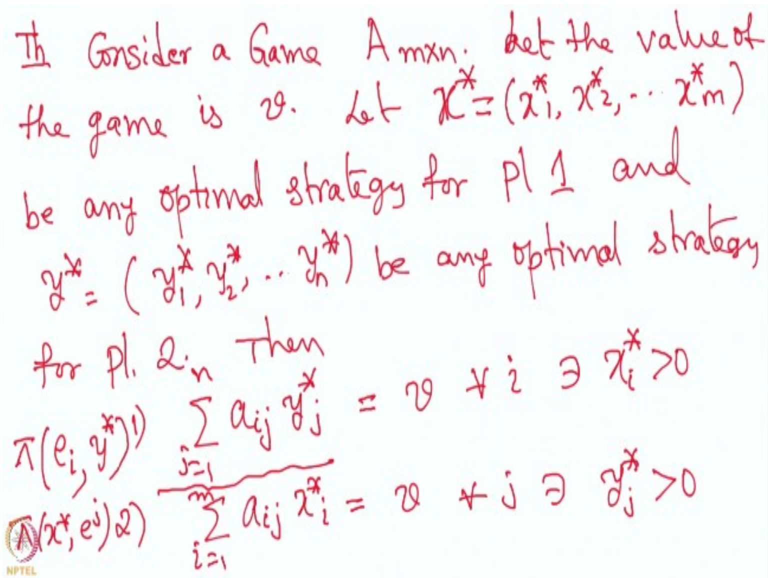

So here, we will make an interesting statement here and whose proof we will see is, let us say consider a game whose we take m by n size and let us assume the value of the game is v. Let x star, x star 1, x star 2, x star m be any optimal strategy for Player 1 and y star to be y1 star, y2 star, yn star be any optimal strategy for Player 2. Then, the following happens. Summation of aij y star j, j runs from 1 to n is nothing but v for all i such that xi star is greater than 0. Similarly, summation aij x star i, i runs from 1 to m is equals to be for all j such that y star j is greater than 0.

That means, when you take this, what is this? aij star yj star and you are summing over all j, so therefore, this is nothing but the payoff that player, okay, let me write it here, this is nothing but the pi of y star and ei. A player i is playing ith row and Player 2 is playing y star the pi eiy star is nothing but v for each i whenever xi star is greater than 0. Similarly, here it says is that pi x star ej, pi x star ej is equals to v for all j such that y star j is strictly greater than 0. So, in fact, this is not a difficult statement to prove because this in fact follows from the pi v is nothing but pi x star y star.

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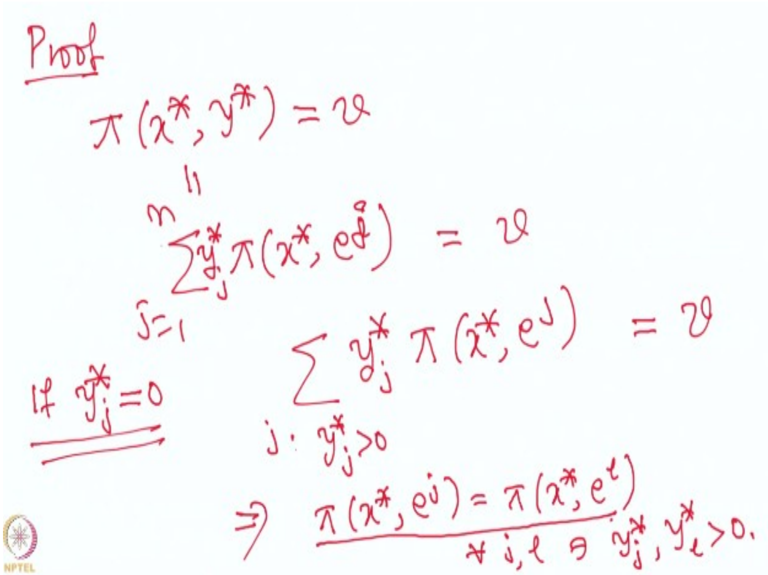

So, if you really look at proof, pi x star, y star is nothing but v, that is the value and what is pi x star, y star, this is nothing but summation pi x star, ej where j runs from 1 to n and if ej is 0, of course there should also be a y star j here, okay? If y star j is equal to 0 for some j, then that particular term can be removed here and what you are going to get here, this is nothing but summation y star j pi x star ej where j such that y star j is strictly greater than 0, this is equal to v. Now, this is a convex combination of pi x star ej's, a convex combination of pi x star ej is equals to v.

If they are all not same, of course if they are not same, look if something is bigger, then that will be bigger than v. That is not true because all of the v is going to be there. x star is optimal, the y star is optimal. So, this immediately tells me that using convexity properties, the bilinearity of pi and this is a convex combination, what you have is that pi x star ej is going to be same as pi x star el for all j, l such that y star j, y star l greater than 0.

Whenever the probability with which those columns are chosen is strictly greater than 0, then the corresponding this is going to be same, that is exactly what we wanted to prove it. So, this is known as, such strategies are called equalizer strategies, sometimes this is quite useful in solving this problem, that what I mean is that computing an equilibrium. Suppose, you can always try to find some y star satisfying such properties, some y star, x star and v, then you can actually verify that the converse kind of result holds true.

In fact, this idea we have used in the early lectures where we showed that in the matching pennies game, the choosing with probability half, half, that is an equilibrium, we have used that fact. So, you can go back to that video and see that and try to relate to this theorem, in that way also, this theorem is also quite useful in solving zero-sum games.

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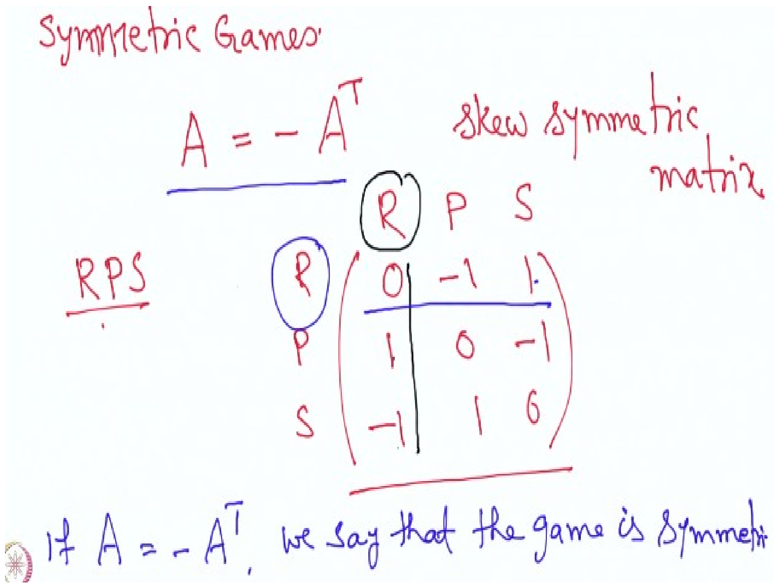

The next, we will look at another very interesting idea of these games, what is known as symmetric games. The symmetric games are a nice set of games where the players have a kind of a symmetric behaviour. So what exactly I mean is so whatever Player 1 is trying to do, the Player 2 is also trying to do the same thing. So, let us say Player 1 is maximizing certain utility and the same utility the Player 2 also has and therefore you wanted a zero-sum game.

That means, whatever you are trying to maximize, I am also trying to minimize in several this things, in particular in this setup what we have is that this A, the matrix A should be same as minus A transpose, that means this A is a skew symmetric. If the payoff matrix A happens to be a skew symmetry. So, let us look at an example. The best example is rock, paper, scissor's game. We will understand the symmetry better in this thing. So there are 3 strategies 0, -1, 1; 1, 0, -1; -1, 1, 0, so rock, paper, scissors.

So whatever Player 1 is doing, the same thing Player 2 is also trying here. So whatever Player 1 gets, the same amount he is also getting under the similar setup. So, both the players are trying to maximize the same pay off matrix. So let us look at it. The Player 1 is trying to maximize these entries into this thing. He chooses this one and Player 2 is minimizing this columns. If he is minimizing this column, let us say the column 1, Player 2 is minimizing this column that means 0, 1, -1, that means he is maximizing 0, -1, 1.

Minimization is nothing but minus of maximizing the minus of that, so minimizing 0, 1, -1 is nothing but 0, -1, 1 and Player 2 is choosing this column R, he is basically looking at minimizing this entry 0, 1, -1. Same thing if you look at Player 1, then Player 1 is choosing row 1, he is actually maximizing this entries 0, -1, 1 and 0, 1, -1 here, minimizing 0, -1, maximizing both are the same. So the symmetric game is exactly that. So, what is the condition for this one?

So, the minus of this row should be this row, that is exactly this condition. So if A = –A transpose, we say that the game is symmetric, okay? The symmetric games have an interesting fact. So immediate thing is whatever in a symmetric game intuitively if you look at it, whatever I am maximizing, you are, the same thing, you are exactly the same position, you are also going to maximize the same thing, that means whatever I am getting you should also get the same.

So that essentially means that because it is a zero-sum game, whatever I am getting should be same as what you are getting and the sum of both is 0, therefore, the payoff that we get under an optimal play is 0. So, that is the interesting situation here.

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Theorem A symmetric game has value zero. whatever is optimal for Player 1 is also optimal for player 2.

Proof

$$\pi(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} x_i (-a_{ji}) y_j = -\pi(y,x).$$

So in fact, we can write this as a theorem. Symmetric game has value 0, the symmetric game will have a value 0. Not only that, whatever is optimal for Player 1 is also optimal for Player 2. So, let us try to prove this fact. So, what we have, pi x, y, is nothing but summation i is equals to 1 to m, summation j is equals to 1 to n, which is xi aij yj. This is same as summation i is equals to 1 to m, summation j is equals to 1 to n, xi minus aji by the symmetric property of the game. If the game is symmetric, the payoff matrix is going to be a skew symmetric matrix yj.

So if you rewrite this one, this is going to be pi y, x of course minus. Now here are few things. I have used m here, n here, remember because by the definition of symmetric game, that is the matrix A is same as minus A transpose, that means A the number of rows and number of columns are same, m and n are same. So that we have used explicitly here when I put y here and x here, we are using this one. So therefore, pi x, y is same as minus pi y, x okay? So that is basically the important fact here.

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$$\pi(x,y) = -\pi(y,x)$$

$$\Rightarrow \pi(x,x) = -\pi(x,x)$$

$$\Rightarrow \pi(x,x) = 0$$

$$\Rightarrow v(A) \leq 0.$$

$$\underbrace{\max_{x} \min_{y} \pi(x,y)} \leq \pi(x,x) = 0$$

$$\underbrace{\min_{y} \max_{x} \pi(x,y)} \geq \pi(y,y) = 0$$

$$\Rightarrow v(A) = 0.$$

Pi x, y is going to be same as minus pi y, x, okay? So, whatever you are getting x and y let us say x and y is what Player 1 and Player 2 are playing, then pi x, y is same as minus pi y, x. So this in fact, gives you the following fact pi x, x is minus pi x, x implies pi x, x is going to be 0. If both the players are playing the same strategies, we are getting 0. So, this immediately shows that the value of the game is less than or equal to 0.

Why, if this comes easily, what is the value of the game? Is nothing but max x min y pi x, y, minimum y pi x, y certainly less than or equals to pi x, y for each y, in particular pi x, x, this particular term is certainly less than or equals to pi x, x and this is 0, the maximum of that has to be less than or equals to 0 and hence v of A is less than or equals to 0, okay. Now, we have to use the symmetric argument now. In using a symmetric argument for example look at the minimum y max x of pi x, y.

Now, look at this, maximum of this, this has to be greater than or equals to pi y, y and this is 0 and this term for each y is greater than or equals to 0, therefore minimum of that is also greater than or equals to 0. Therefore, this says the lower value is less than or equals to 0, this says the upper value is greater than or equals to 0 and we know that the game admits the value, therefore lower and upper values are same and hence, value of the game is 0. So this is a very interesting argument which gives you that for a symmetric game, the value is always 0.

Now, you can check the rock, paper, scissors game and then see that game has a value 0. In fact, here is another interesting thing is that because you know that the rock, paper, scissors game is a symmetric game, so therefore the value is 0, therefore you can look for an equalizer's strategy assuming v is equals to 0 because you know value of, so v is 0 you put it and use the equalizing property of an equilibrium and then we can easily solve the this thing. So, that is an interesting simple exercise.

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Need to show that if x^* is optimal for pl 1,
then x^* is also optimal for Player 2.

(x^*, y^*) is optimal SPE

$$\pi(x^*, y^*) = 0 = \pi(x^*, x^*)$$

(x^*, x^*) is SPE.

Now what else required? Now, we need to show that if x star is optimal for Player 1, then x star is also optimal for Player 2. So how do we prove this fact? This fact once again comes from this thing. Suppose x star, y star is optimal, what I mean is saddle-point equilibrium here. Then we know that pi x star, y star is 0, now this is also same as pi x star, x star. So, given x star, if Player 1 fixes x star, y star minimizes and similarly by this equality x star also minimizes, so therefore, you can use any properties like equalizing or you go back to the linearity, bilinearity of the payoffs and other things.

We can now conclude that x star is also optimal. In fact, x star, x star is saddle-point equilibrium. You can prove it without much difficulty. So in fact, this completes the proof, but the way to visualize this fact is that this is a symmetric game, that means whatever I am maximizing, the same thing you are also maximizing. Therefore whatever is good for me should also good for you because it is a symmetric environment and hence, such a result.

So this is quite a useful thing and in fact symmetric games play a very interesting class of games. In fact, there are ways to symmetrize a non-symmetric game. We will see that more later on when we go to the non-zero sum games.

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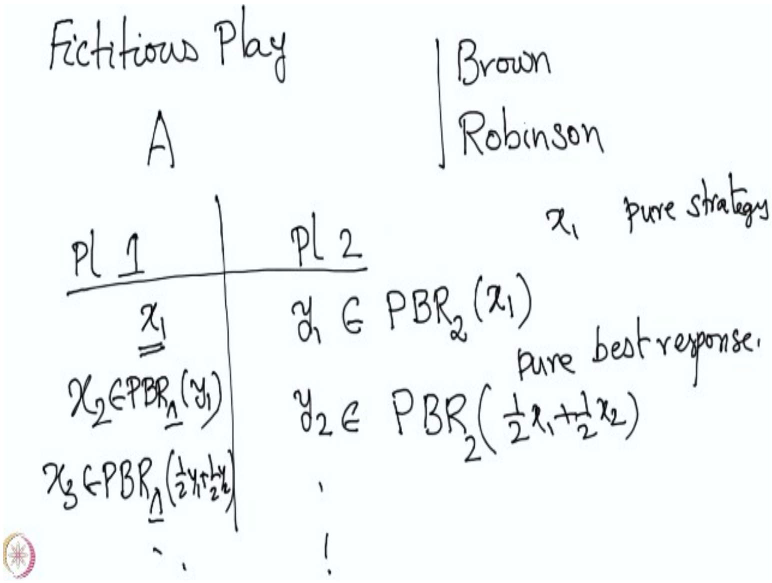

Now we introduce an iterative method to solve the zero-sum games. This method is known as a fictitious play. The method is introduced by Brown and the convergence proof is given by Robinson. So, fictitious play is an example of a method which is known as the learning methods, what it says is the following thing. So what is this learning particularly? So there is a game that people are playing it, so let us assume they cannot really compute the equilibria and other things, so the only thing is that they are playing it.

As they play again and again this game, can they infer, can they learn the equilibrium, what should be the optimal strategy for them. So this is basically the idea behind this learning algorithms. This fictitious play is one such algorithm. Let me describe how this method looks like? So, we have a zero-sum game A and there are 2 players. So, the most important thing here is that the players do not make mixed strategies, okay? So in other words, let us say in the first time when Player 1 is playing, he doesn't know anything about the game, he only knows what choices he has now.

So, he will choose some strategy, let us say the pure strategy, let me call x1, this is a pure strategy, x1 is a pure strategy, okay? So, Player 2 will actually play a best response to x1. Once again, he will choose only pure, so he will look at y1 which is basically a pure best response to x1. He is going to choose a pure best response, this is pure best response which is pi 1. Now, in the next round, Player 1 observes that Player 2 has played y1. So, if he has played y1, what should he play.

So, he will look at it, x2 which is pure best response to y1. Now, in the next setup, next iteration, the Player 2 observes that Player 1 has played x1 and x2. Therefore, he thinks that Player 1 will play x1 or x2 with equal probabilities. Therefore, he will look at the best response to half x1 plus half x2. Again, he will only look at the pure best response, the pure strategy which is a best response to half x1 plus half x2, that means he choosing x1 with probability half, x2 with probability half.

Then if the Player 2 thinks that Player 1 one is going to play x1 one with half probability, x2 with half probability. Therefore, he will look at the best response corresponding to that, let me call that as y2. Then, once Player 2 chooses this y2 in the next round, Player 1 will look at the Player 2 and he has played y1 and y2 and therefore he thinks Player 2 will play y1 one with half probability, y2 with half probability. Therefore, he will look at x3 which is basically a pure best response to half y1 plus half y2 and it goes on like this.

So, every time a player makes a decision, he will look at the empirical behaviour of other player what he has played, how often he has played a pure strategy, he looks at that, then he forms an opinion and he plays a best response, a pure best response to that opinion. So, this is basically known as a fictitious play. So, recall in a fictitious play, players make an opinion about the other players' strategies. So, this opinion is formed based on the empirical average.
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$$\frac{x_1+x_2+\cdots+x_n}{n} \to x^*$$

$$\frac{y_1+y_2+\cdots+y_n}{n} \to y^*$$

(x^*, y^*) is Saddle point equilibrium

Now, they choose the pure best response to those empirical averages and then the convergence is that the x1 plus x2 plus xn these are the, this thing, by n. This is basically the empirical average of Player 1's choices. This converges to some x star. Similarly, the choices made by Player 2, look at its empirical average, this converges to y star. The theorem that is proved by Robinson is that x star, y star is saddle-point equilibrium. This convergence, the proof is actually a length proof, at this moment, we will not go into the proof, but we will see an example illustrating this result.

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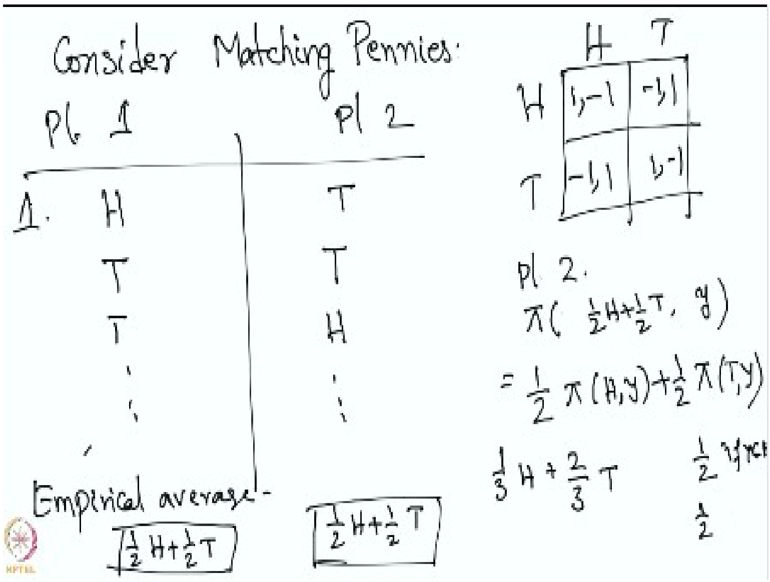

So, let us consider matching pennies. So, this is a very simple example. There are only 2 pure strategies to make and if both match, Player 1 gets 1 unit. If they do not match, Player 2 gets 1 unit. This is a zero-sum game of course. So, how does this go in the fictitious play? So Player 1, Player 2. Let say in the first round, Player 1 has H. If initially if Player 1 plays H, Player 2 will play T. And in the next round, Player 1 because Player 2 has played T, so he will play T because he has played T, now Player 2 will see that Player 1 has played H and T.

So therefore, with half probability H, with half probability T, therefore he can play anything that maximizes that. Let us look at what is the payoff the Player 2 will get? So, Player 2's payoff is basically pi half H plus half T, whatever he chooses, y. So, this is nothing but half probability if he chooses H, this is T, y. So, if y happens to be H, then he will get pi H, H will be 1 and then half and this is 0, therefore, he will get half, if y is H. If y is T, then also he will get half, so he is going to get, whatever he chooses H or T, whatever he chooses, he will get half, so therefore he will pick one of them.

Let us say he has picked T. T and H both are best responses to half H plus half T okay. So now, once you know that he picks T, now the Player 1 has picked T, so therefore Player 2 will continue to pick T because that is going to be the best response to this because both times he has picked T, so therefore T, the best response to T is T, now Player 2 will see that Player 1 has played H one time and T two times. Therefore, he will play 1 by 3 H plus 2 by 3 T. Now, Player 1 has played H once in three times whereas T in two times.

So that means, T he has played more often than H, therefore for the Player 2, the best response is going to play something not equal to T, that is H, so he will play H. Now, you go on like this and in fact, it is an interesting exercise to see that this empirical average converges to half H plus half T here, here also half H plus half T. So, one can actually do this iterations for several times and then get some opinion about how much it is. One of the very important point here is that this convergence rate is actually very slow.

This is not a very fast converging method, but nevertheless, this is a good method to show the convergence of this saddle-point equilibrium and in fact, for any zero-sum games, this convergence automatically happens, this is a proof due to Robinson. In the next sessions, we explore further properties of this fictitious play and then study some of its convergence properties. With this, we will stop this session, we will continue in the next session.