Game Theory
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Lecture - 18
Non-Zero-Sum Games: Introduction and Examples

Welcome back to NPTEL course on game theory. In earlier sessions, we have discussed zero-sum games. Now, we will switch our focus to non-zero-sum games. So, in a zero-sum game, what we have is that the sum of the 2 payoffs, the Player 1 and Player 2 payoff is always 0. In non-zero-sum games, we do not take that and they can be different. In fact, one example that we have seen earlier is coordination game, which is a non-zero-sum game. Let me recall the coordination game first.

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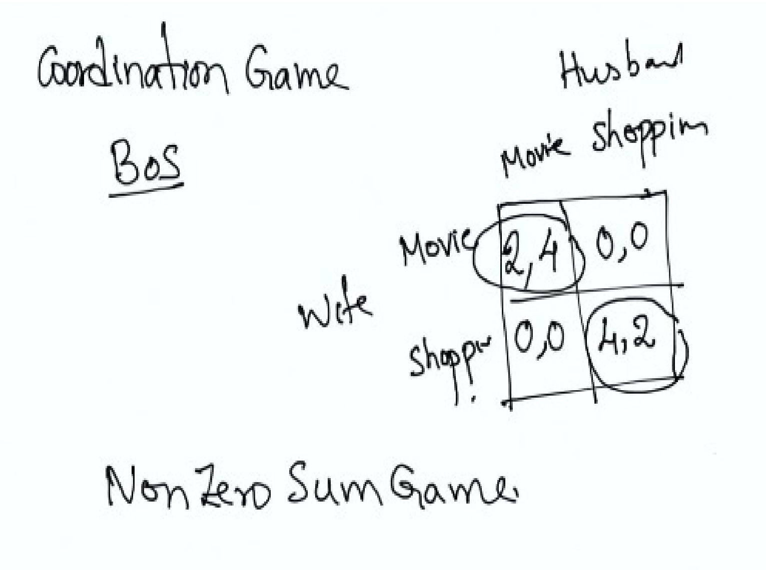

So what is this coordination game? In fact, the version that we are going to see is known as a battle of sexes. Let us say there are 2 players, let us assume wife and husband. So there are 2 choices, one is going to a movie or going to a shopping. Let me say the Player 2 is husband, Player 1 is wife okay. If the wife prefers shopping compared to movie and husband prefers movie to shopping. So, if they go to, husband goes to movie he gets a higher benefit whereas wife gets more benefit if she goes to shopping, but the most important thing is that the benefit they will get only when they go to, go together.

So when both of them go to movie, for example the husband is going to get 4 and let us say wife is going to get 2 and both of them go to shopping 4 and 2, this is their benefit and then the other things can be any numbers appropriately taken, but right now let us say I am taking zeroes. So we can make different numbers, but let us. So, this is a game where the sum of the payoffs is not zero, this is an example of a non-zero-sum game which we have seen earlier. So in fact, both going to movie is an equilibrium and both going to shopping is also an equilibrium.

So as I have been pointing out since earlier in the game theory, the most important thing is that the players are making their decisions simultaneously, they do not know what the other is going to do. So that is a very important thing. If both of them know what they are doing it, that becomes an optimization problem, but the game flavor will be lost in that situation. So, here, the most important thing once again I elaborate is, I stress on is that they are choosing their decisions simultaneously independent of others, that is very important. So, this is an example of a non-zero-sum game.

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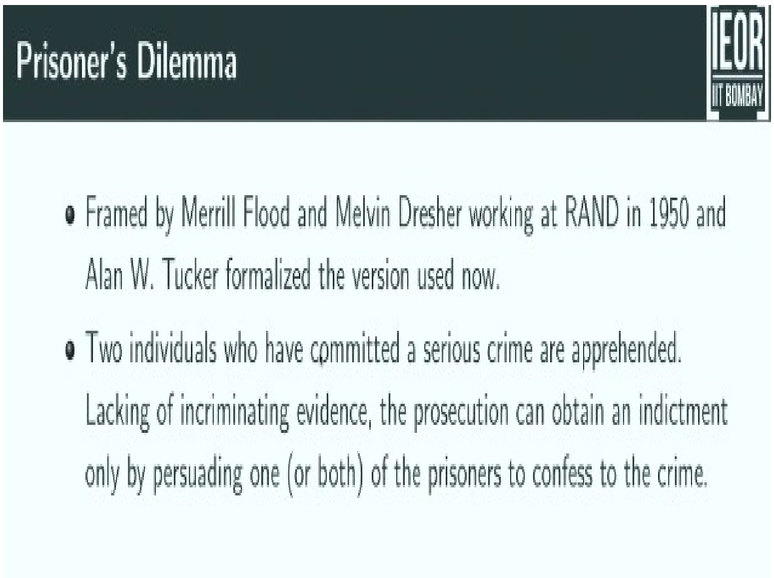

Now, we will see another very, very important example which is known as a prisoner's dilemma. The prisoner's dilemma, this example is framed by Merrill Flood and Melvin Dresher working at RAND Corporation in 50's and it is Alan Tucker who formalized the version that we are going to see now. So in this game, there are 2 individuals who have committed a serious crime, both of them are apprehended, but there is no criminal evidence.

So, the police has no evidence that they have done, but they believe strongly that they have done it. So, they can actually try persuading those guys asking them to confess against other. If they can confess, then that is going to help them. So, that is basically the situation here.

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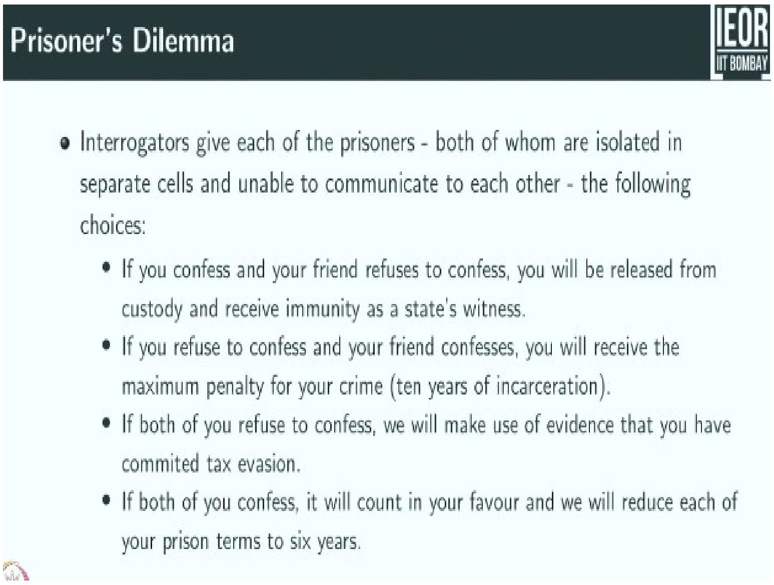

So, what the interrogators do is the following thing. They are isolated in separate cells and they cannot communicate each other and then they give the following choices. What the choices are? If you confess and your friend refuses to confess, you will be released from custody and receive immunity as a state's witness. If you confess and your friend refuses to confess, then you will be released and your friend will be prosecuted using your evidence.

If you refuse to confess and your friend confess, a symmetric situation, reversing the roles. Both refuses to confess, then the police has, they do not have a sufficient evidence, therefore they can only give a very little punishment, but if both of you confess, then that means we have evidence against each and then get a some reduced term of imprisonment. So that is basically the case here.

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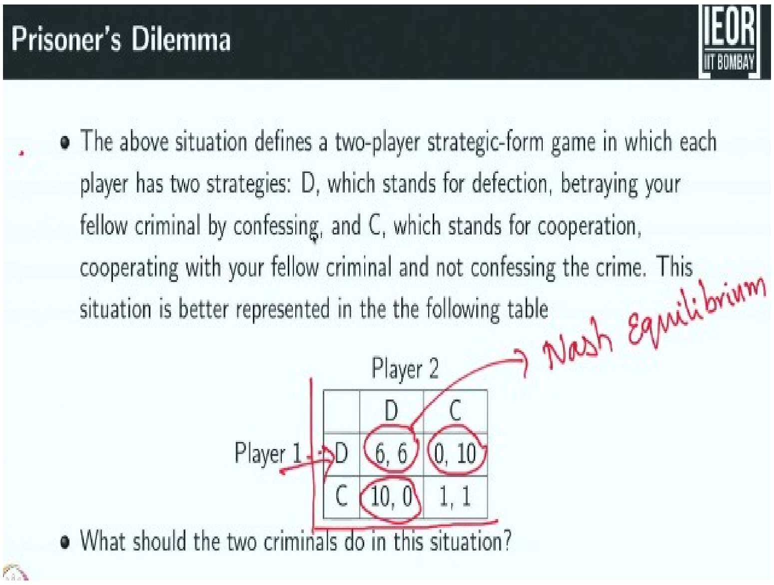

Now, the situation is a two-player strategic-form game, so there are 2 choices, D is for defection, that means betraying your fellow criminal by confessing, the D means you have confessed that this crime is done and then in a sense this is also betraying your fellow prisoner and C is here means cooperation, that means you are cooperating with the police. Here, C means cooperation, cooperation with the fellow criminal and you are not confessing the crime. So this situation you can see it as a picture here.

So there is a picture here in this you have 2 players which 2 by 2 matrix game, both are defecting, both the players, that means Player 1 is defecting against Player 2 and Player 2 is defecting against Player 1, that means both have confessed their crime. Therefore, the police has sufficient evidence against both. So in this case, 6 and 6 are going to be the years of imprisonment, and if Player 1 defects and Player 2 confesses, then Player 1 is immediately released and Player 2 gets 10 years punishment.

And similarly, if Player 1 confesses, here in this case, it is a cooperation, he is cooperating with the Player 2 and Player 2 defects the Player 1, then 10 and 0, that means Player 1 is getting 10 years of imprisonment and Player 2 is getting nothing and both are cooperating each other, both players are cooperating each other then, there is not much evidence for the police, therefore both of them get only 1 year of imprisonment. So, this is going to be the situation here. So if you look at it, so for what is going to be the equilibrium here in this setup?

Now, remember here people are minimizing because this is imprisonment, so this is a cost. So therefore so far in the zero-sum games, we are assuming the players are maximizers, but here we have a situation where both the players are minimizing their imprisonment. So, no player would like to get ten years imprisonment. So, therefore, clearly we can see that 10, 0 and 0, 10 these are not going to be a equilibrium here. We can easily verify it. What about D, D? Is this equilibrium? So, let us look at it.

The way to see is that like in zero-sum games, we have some, if let us say Player 1 has fixed to D, what is best for player C, the Player 2. Once I know that the Player 1 has used D, for Player 2, he has only 2 choices D and C, for D he is getting 6 and for C he is getting 10 years of imprisonment. So, therefore D is better. Similarly when Player 2 decides to play D, Player 1 the best is to play D. Certainly, therefore this is a Nash equilibrium, okay? So this is all thought process.

The way it happens is that the Player 1 will think, suppose if I do not defect, if I do not defect that means let me say I cooperative with my fellow criminal, then if I cooperate, what is going to happen to me? That means 10, 0 and 1, 1, I will get either 10 years or 1 year. Can I assume that the other player cooperates? If other player, why will he cooperate, if he cooperates he will get 1 year, if he does not cooperate, he is going to get zero, therefore not cooperating is better for him, therefore he will never cooperative.

So therefore, if I am cooperating, I assume that my fellow criminal will also he will never cooperate, he will only defects, so therefore, for me cooperating is not a good choice, therefore defecting is better. The same thought process will be with the other criminal and he will also think the same way and therefore defect happens. So therefore, this is essentially what I was telling earlier about the rationality, the people are rational, they maximize their benefits, the selfish behaviour is very important here.

So, using that rationality behaviour, rationality we know that D, D is going to be a Nash equilibrium. So, this is essentially what happens here. So D, D is a Nash equilibrium, but here is an interesting situation, the C, C when both of them cooperate, they are getting only 1 year imprisonment but that is not a Nash equilibrium, they will never play C, C. So, this is exactly what I have been saying, the thought process of the player, he will always think that if I cooperate with my fellow criminal, the other player is going to defect.

Therefore, defecting is better and a symmetric nature, this happens and therefore the pair of strategies D, D is going to be a Nash equilibrium.

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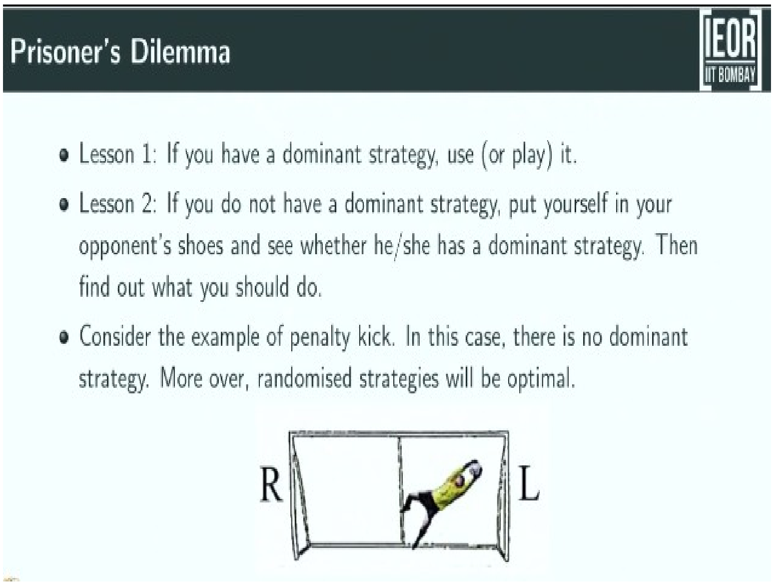

Okay, so here are the few lesions that we need to see. So, there is no dominant strategy. We have used the domination earlier in zero-sum games if a specific column is dominated by some other column, you do not want to use that column. So if a domination is possible, you will play it, but if there is no domination what we are going to look at is the thought process. If I play this, what my opponent will think about and how he will react to it and then based on that we do, that is exactly the way we did it.

For example if you look at this penalty kick, the penalty kick, there are 2 players, now if the one player knows that I am very good on a left side, then other guy will certainly going to wait for me at the left corner. So, for me it is not good to play the left corner, so this is essentially the thought process that we have seen earlier. So this is a good example which illustrates that aspect okay. So, now the most important question that comes here is that is there a way to make the prisoners cooperate?

Now this situation arrives at many instances, so I would like you to think about various situations, one example, for example is when two neighbouring houses when you look at it, when they are cleaning it, they try to through the dirt on the other side thinking that their side is cleaner, but other side will also do the same thing and result is both are getting dirt outside their house. This is a common phenomena that observe in many situations, now in such situations, the most important question is how can we get the cooperation.

So this is a nontrivial question which we may not discuss in this course much, but this is a question that economists, biologists, behavioural economists and several people have been studying.

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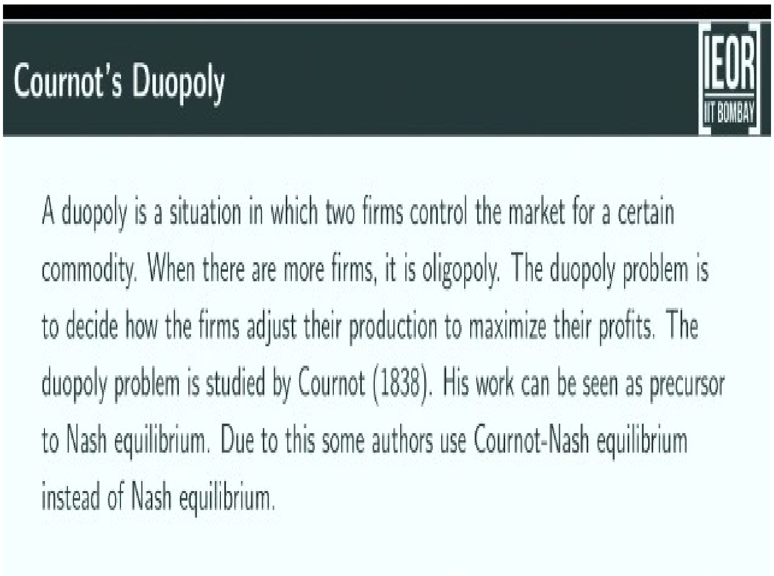

Okay, we will now see another example which is known as a Cournot's duopoly. What is a duopoly? Duopoly is a situation where there are two firms who wants to control the market for a certain commodity. So, we are considering a market and there is a commodity that 2 firms are selling and they want to control the situation. When there are more than 2 firms, they call it as an oligopoly. So the duopoly is actually to decide how firms adjust their production to maximize their profits.

The duopoly problem is studied by Cournot very, very long back, 1838. His work can be seen as precursor to Nash equilibrium, so sometimes in economics, the Nash equilibrium, particularly in this oligopoly's framework they call Cournot-Nash equilibrium. So let us look at how this is.

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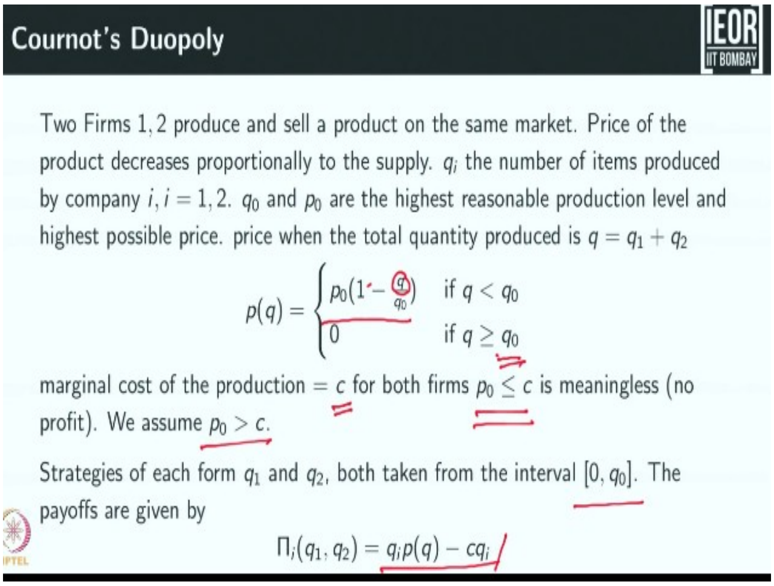

There are two firms, 1 and 2. They producing some product and they sell the product on the same market. Both of them are operating in the same market. The price of the product decreases proportionally to the supply. So let us assume qi is the number of items produced by company i, q0 and p0 are the highest reasonable production level and highest possible price. Price when the total quantity produced is q that is q1 plus q2.

So when firm 1 is producing q1 and firm 2 is producing q2, the total quantity available in the market is q1 plus q2 which is let me say this is q, written by q. Now, the price when the total product available in the market is q, that is given by p of q which is given by p0 into 1minus q by q0 that is this. So, if the quantity q increases, then this reduces, therefore price reduces and if quantity available q is smaller, the price increases. Now if the product available, the quantity available in the market is bigger than q0, the price is going to be 0.

This is one of the very simple example of a price quantity. We assume the cost of producing a product is c and we assume that, in fact it is not assume, the price of the product can never be less than the marginal cost. So therefore, p0 less than or equal to c is a meaningless thing. So therefore, we always make sure that p0 is bigger than c. Now strategies of each firms, now they are q1 and q2 both can be taken from the interval 0 and q0, 0 is the least that they can produce, q0 is the maximum they would like to produce.

So therefore, both the firms have same possibilities, so q1 and q2, they will choose simultaneously and what are the payoffs? If firm 1 is choosing q1, firm 2 chooses q2, the price of the product is p of q and firm is producing qi therefore his profit is going to be qi into p of q and he also incurs a cost that c into qi, so we reduce that. So therefore, this is going to be the payoff function of the firm i. Once we know this one, now we are in a game setup. So there are 2 firms, they are making their decisions and they have a payoff functions, now each firm's objective is to maximize their profit. So, what exactly will they do?

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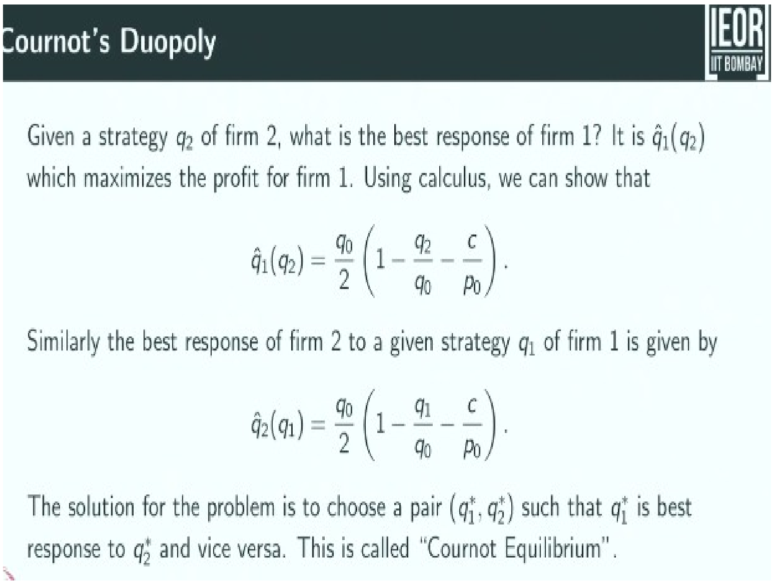

So, let us do this one. So as we have been doing it earlier, we look at what is known as a best response. When Player 2, here Player 2 means firm 2, let us say he decides a strategy, q2, then what should firm 1 do. So let us assume that when q2 is produced by firm 2, let us say q1 hat is going to be the quantity that firm 1 is going to produce and this q1 hat should maximize the profit. So what it means is that this q1 hat should be maximizing the firm 1's profit.

So when firm 2 is producing q2, let us say q2 is fixed, then firm 1 should produce q1 which maximizes this pi1 q1, q2. Now if I look back this, the profit function is given by this one, so I will write it here again. pi1 q1, q2 is nothing but p1 of q into q1 minus c into q1. So p1q is a price curve that is given in the previous slide and this is going to be the profit that Player 1 is getting. So player, we are looking at the maximization. So we use the first ordered derivative with respect to q1 and equate it to 0.

If you equate it to 0, so what you are going to get is p1q plus q1 into the derivative of p1 with respect to q1 minus c, you calculate that, we do and then equate it with 0, then you are going to get this much. So this is a simple calculus which I haven't done and in a similar fashion, if you look at the firm 1's decision and let us say it is fixed at q1, if firm 1 fixes at q1, firm 2 will produce q2 hat which should maximize his profit, and then if you do the same analysis with pi2 q2 hat is going to be this much.

Now, another very important point we have done only the first order analysis, that means the gradient is equal to 0 and how do we know that the differential is 0 and it need not be maximizer. The corresponding critical point need not be maximizer, but in this case, what is going to happen is that if you look at the price curve, price curve is linear in q and the profit if you look at it, it has the product terms q1 into p1q minus cq1. You can actually verify that this is a convex in the variable q1.

Similarly the profit of Player 2 is also a concave in the profit in the decision variable of Player 2. Therefore, the concavity together with the first order conditions will ensure that the q1 hat and q2 hat that we have got here is going to be the maximizers. Therefore, when Player 2 fixes to q2, q1 hat is the maximizer of the firm 1's profit and similarly when q1 is fixed by firm 1, q2 hat is going to be the maximizer of the firm 2. Now, what is the equilibrium?

Equilibrium is such that q1 star and q2 star is the equilibrium such that when Player 2 fixes a q2 star, q1 star should be the best response to q2 star. Similarly, if Player 1 fixes at q1 star, q2 star should be the best response. So, such a thing is Cournot equilibrium it is called, and of course in the modern language, this is exactly a Nash equilibrium and this is the reason why it is also called Cournot-Nash equilibrium.

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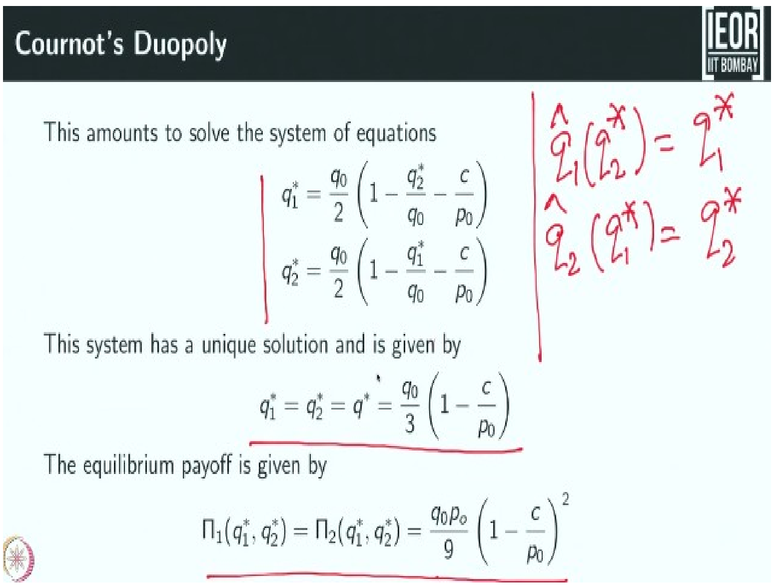

So, what we get here is that we need to consider the following thing. The equations are the following thing q1 hat of q2 star should be q1 star. Similarly q2 hat of q1 star this should be q2 star. These are the conditions that we have and if you rewrite those conditions, if you put the q1 hat functions thing, you are going to get these equations. Okay, these are 2 equations and q1 and q2 are unknowns here, and if you solve these things, what we get is exactly this, the q1 star and q2 star are given by q0 by 3 into 1 minus c by p0.

Under this, the payoffs are also going to be exactly same pi1 q1 star, q2 star is same as pi2 q1 star, q2 star which is given by this. So, this is a Cournot's duopoly. Of course, the same problem one can actually do with instead of 2 firms, some arbitrary number of firms, multiple firms, then it is an oligopoly, we can start doing it. Now, this is an example where the firms are deciding the quantity to produce, but exactly similar model where the firms are going to determine the price rather than the quantity, so that is known as Bertrand model.

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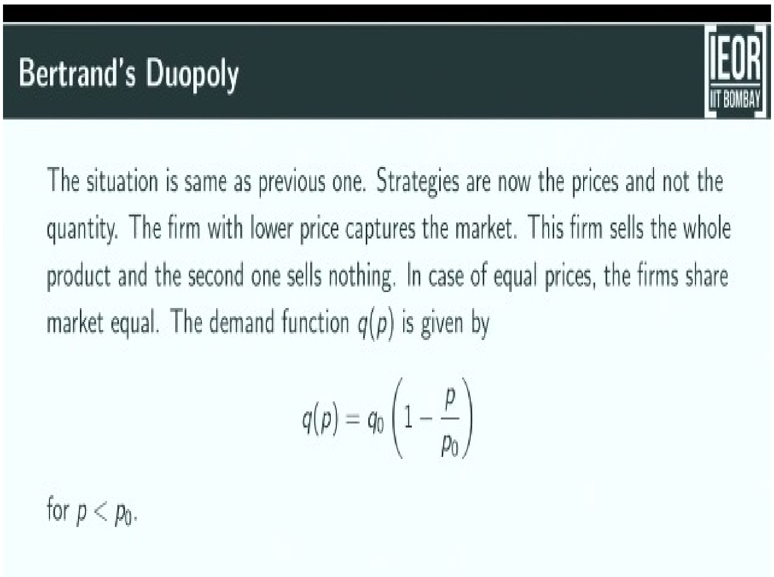

So, in a Bertrand model, the strategies are not the quantities, but prices, and the firm with a lower price captures the market. This firm sells the whole product and second one sells nothing. In case of equal prices, the firms share market equal. So the demand function now is if the price is fixed, the demand function is given by q of p which is nothing but q0 into 1 minus p by p0. If the price is higher, the quantity that they are going to produce will be small. If the price is small, they will produce higher, so that is exactly captured by this demand function, okay?

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Bertrand's Duopoly

The payoffs are given by

$$\Pi_1(p_1, p_2) = \begin{cases} (p_1 - c)q(p_1) & \text{if } p_1 < p_2 \\ (p_1 - c)q(p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{if } p+1 > p_2 \end{cases}$$

$$\Pi_2(p_1, p_2) = \begin{cases} (p_2 - c)q(p_2) & \text{if } p_2 < p_1 \\ (p_2 - c)q(p_2)/2 & \text{if } p_2 = p_1 \\ 0 & \text{if } p_2 > p_1 \end{cases}$$

$p_0 > c$ denote the highest reasonable price for the product. It is a symmetric game with strategy spaces $P_1 = P_2 = [c, p_o]$.

The payoff functions will be simply if p1 and p2 are the prices chosen by the firms, it will be p1 minus c into q p1 is going to be the profit when firm 1 has a smaller price. If both the prices are same, they share. If the price is higher, so there is a small mistake here, this is not p plus 1 this is p1. If the price is going to be higher, then of course the firm 1 is not getting anything. Similarly, pi2. So the firm 2's profit is also there and exactly similar kind of conditions when the price has to be bigger than c, that is a reasonable price, c is the marginal cost.

So therefore, p0 has to be bigger than c. So therefore, what is p0, p0 is the maximum price that they would like to consider, if beyond p0, they cannot set anything. Now, therefore, the prices will be between c and p0.

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Bertrand's Duopoly

- $p_1 > p_2$ can not be best response to p_2, whenever $p_2 > c$. Similarly $p_2 > p_1$ can not be best response to p_1, whenever $p_1 > c$.
- $c = p_2 < p_1 \leq p_0$: In this case p_2 is not the best response to p_1, as choosing anything between c and p_1 yields a better payoff for firm 2. So this case can not give a Nash Equilibrium.
- Similarly the case $c = p_1 < p_2 \leq p_0$ can not give Nash equilibrium.
- $c < p_1 = p_2 \leq p_0$. A slight decrease in p_i give firm i better pay than p_i. so this case also can not give Nash equilibrium.
- Remaining case: $c = p_1 = p_2$. In this case the payoff to both firms is zero and this is the Nash equilibrium.

Then if you go through it, then we can actually solve the case and one by one, we can verify the Nash equilibrium conditions. Then actually what happens the p1 greater than p2 cannot be best response to p2. If a player is choosing p2, the firm 2 has decided a price p2, then firm 1 can never go beyond p2 okay. Similarly, for p if firm 1 one decides p1, firm 2 will never go beyond p2, okay? So c is equals to p2 less than p1 less than or equals to p0. In this case, p2 two is not the best response to p1 as using anything between c and p1 is a better payoff for firm 2. So this case cannot give a Nash equilibrium.

So, likewise, you can analyze all the situations and would like you to work out the details of this Nash equilibrium and if you compute the Nash equilibrium in the setup, you will get a different framework than the Cournot. Of course, there are lot of differences between this Cournot and Bertrand, we will not go into those aspects, we only give these as examples of non-zero-sum games. So with this, I will stop this session, and in the next session, we will formally define the Nash equilibrium and by non-zero-sum games and we proceed to prove the existence of Nash equilibrium.