**Game Theory**
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**Lecture - 19**
**Non-Zero-Sum Games:**
**Existence of Nash Equilibrium - I**

Welcome back to this NPTEL course on game theory. In the last lecture, we have seen some examples of non-zero-sum games, both finite as well as continuous games. Now, we will formally define the non-zero-sum game and the Nash equilibrium. So, let us start defining the non-zero-sum games.

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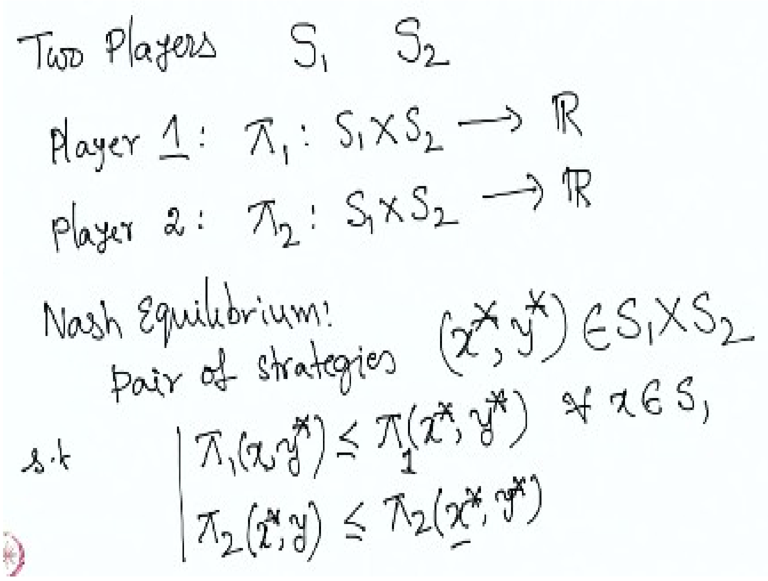


So, here there are 2 players. So they have their strategy spaces S1 and S2 and the payoff functions the player 1's payoff functions is given by pi1 which is a function from S1 cross S2 to R. Similarly, player 2 has a payoff function which is pi2 from S1 cross S2 to R. Now, the players are choosing their decisions in S1 which maximizes their payoff functions. Like in zero-sum games, the payoff of player 1 also depends on choice of player 2. So, therefore the issue comes. So here, what is the definition of a Nash equilibrium

So Nash equilibrium is a pair of strategies, let us say x star, y star they belongs to S1 cross S2 such that when player 2 has fixed at y star, x star should maximize player 1's payoff. This should happen for all x in S1. Similarly, when player 1 fixes x star, y star should maximize player 2's payoff. So these are the two conditions that define a Nash equilibrium. In fact, if you look back the examples, this is exactly what we have used. When one of the player fixes a strategy, the other player whatever he chooses should maximize it.

So in a sense, what you are really saying here is that when a player, let us say player 2 fixes y star, for player 1, deviating from x star is not profitable, and similarly player 1 fixes at x star, player 1 deviating from y star is not profitable to him. So these unilateral deviations are not profitable to the players. So this defines the notion of Nash equilibrium. This is introduced by John Nash and this has become a major tool in economics and many many other fields.

Okay, once we define this Nash equilibrium, now the whole question that comes here is the does there exist a Nash equilibrium and even before Nash equilibrium, how is this different from saddle-point equilibrium, a zero-sum game.

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Zero-Sum Game

pl 1 max $\pi(x,y)$

pl. 2 min $\pi(x,y)$

$\pi_1(x,y) = \pi(x,y)$

$\pi_2(x,y) = -\pi(x,y)$

$\pi_1(x,y^*) \le \pi_1(x^*,y^*)$

$\Rightarrow \pi(x,y^*) \le \pi(x^*,y^*)$

Now, let us look at a zero-sum game. In a zero-sum game, player 1 maximizes pi x, y, player 2 minimizes the same. In other words, what we have is that pi1 of x, y is nothing but pi x, y for pi2 it is simply minus pi x, y. So any zero-sum game is in that sense is a non-zero-sum game with a special property that the sum of the two payoffs is 0. So now, if I write down the definition of a Nash equilibrium, what we will get here is that pi1 x, y star is less than or equals to pi1 x star, y star, this is nothing but pi x, y star less than or equals to pi x star, y star. This is the first inequality that we have in the definition of saddle-point equilibrium.

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$$\pi_2(x^*, y) \leq \pi_2(x^*, y^*)$$

$$\parallel$$

$$-\pi(x^*, y) \leq -\pi(x^*, y^*)$$

$$\Rightarrow \pi(x^*, y^*) \leq \pi(x^*, y) \quad \forall y \in S_2$$

Now, look at this particular thing. If I look at that, what it says is that pi2 x, y, now pi2 x star, y is nothing but minus pi of x star, y, this is less than or equals to minus pi of x star, y star, this is nothing but pi x star, y star less than or equals to pi x star, y. this is true for every y in S2. So, if I combine this inequality and then the previous inequality, what we have is exactly the definition is a saddle-point equilibrium. So Nash equilibrium automatically encompasses the definition of saddle-point equilibrium.

Okay so once we know that the saddle-point equilibrium is exactly same as Nash equilibrium for zero-sum games, so now we will start seeing given a game whether a saddle-point equilibrium exist or not, not saddle point Nash equilibrium. Now, like in zero-sum games, we do know that we need some conditions to ensure the existence of a saddle-point equilibrium, the likewise, because non-zero-sum games includes zero-sum games, they do require some additional conditions. What are those additional conditions, we will look at them.

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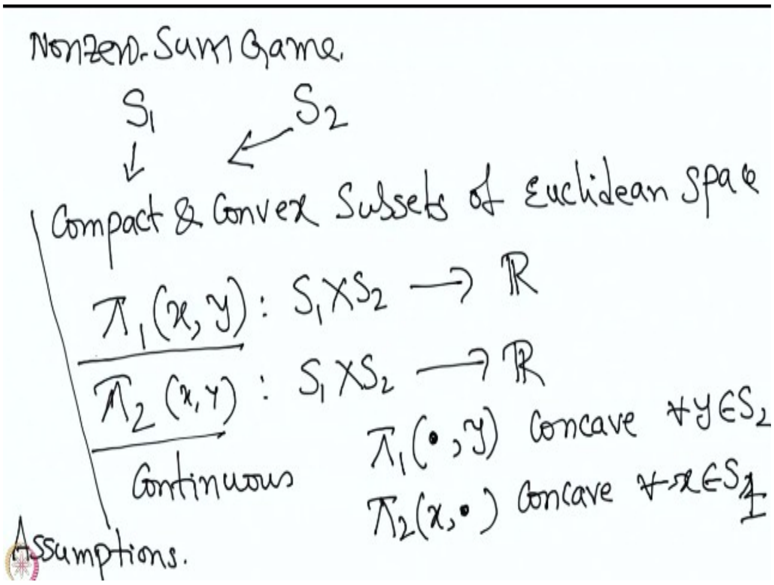


So let us see, so we will take the non-zero-sum game now. So, the payoff functions S1, S2 are the strategies of player 1. So we assume S1 is compact and convex, same with S2, compact and convex, subsets of Euclidean space. The payoff function pi1 x, y, this is a function from S1 cross S2 to R and similarly pi2, this is a function from S1 cross S2 to R, both have to be continuous okay. In the zero-sum games in the x variable, we assumed that to be concave, in y variable we assumed convex.

Now here, look at the pi1 is required only by player 1, this is only for player 1, it does not matter how it behaves with respect to y. So what we really want you that pi1 as a function of x should be concave for each fixed y. Similarly, pi2 as a function for second player should be again concave for each fixed x and S, x is in S1, y is in S2 okay. Pi1 should be concave in x variable, pi2 should be concave in y variable. So, these are the assumptions that we require. So, these are all the assumptions.

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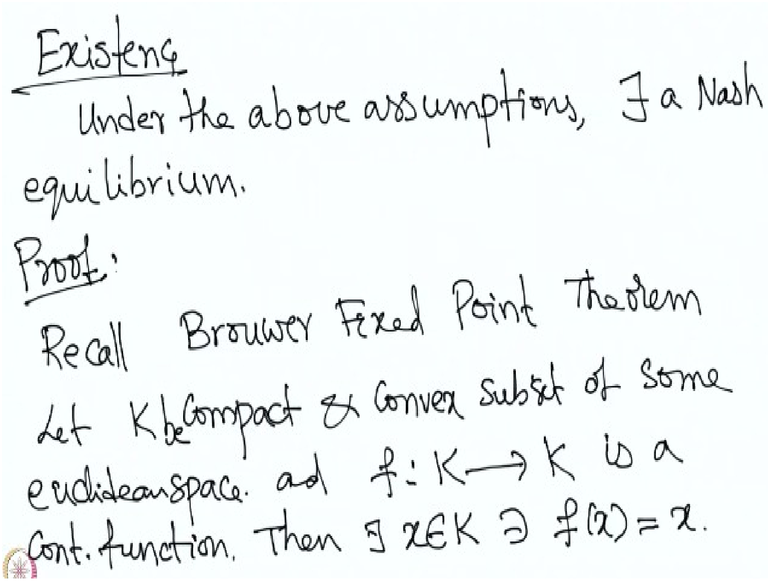


Now, we will prove our main theorem, existence. Under the above assumptions, there exist a Nash equilibrium. So, let us do this. So like in zero-sum games, we have to use the fixed point theorem, of course in zero-sum games, we did not use fixed point theorem, we have used the convexity properties, but in a non-zero-sum games, we are forced to use the fixed point theorem, the fixed point theorem that we are going to use is known as Brouwer fixed point theorem, which we have discussed already in the combinatorial games.

In fact, we have used the game of Hex to prove the Brouwer fixed point theorem in two dimensions, but we assume that result for n dimensions, we would not go into proving the n-dimensional theorem, in fact the hex can be extended to give a proof in n-dimensional as well. So let me recall Brouwer fixed point theorem. Let K be compact and convex subset of some Euclidean space and f from K to K is a continuous function. Then there exists x in K such that fx is exactly x.

So such a point x is known as the fixed point theorem, fixed point and Brouwer fixed point theorem is a very very important result, and of course, this Brouwer fixed point theorem assumes that the K to be a compact and convex subset of some Euclidean space. It is a finite dimension result. In infinite-dimensional analogous results do exist, we let us not worry about them.

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Fix $\bar{x} \in S_1$ , $\bar{y} \in S_2$

Consider $\underset{x \in S_1}{\operatorname{argmax}} \{\pi_1(x,\bar{y}) - |x-\bar{x}|^2\}$

Forces strict concavity

$\exists!\ x' \in S_1 \ni$

$$\pi_1(x',\bar{y}) - |x'-\bar{x}|^2 = \max_{x \in S_1} \{\pi_1(x,\bar{y}) - |x-\bar{x}|^2\}$$

So now let us go to the proof. Now, let us look at the payoff functions. So, fix x bar in S1, y bar in S2. Now, consider pi1 x, y x ,y bar plus mode x minus x bar. I can put a square here okay. So, now this is a continuous function. So, look at this function and look I would like at the argmax over x in S1 okay. Pi1 x, y bar plus mode x minus x bar square, and as a function of x, I already know that pi1 is a concave function in x and mode x minus x bar square, okay here is a small error here, it has to be minus to make it concave.

This is also a concave function, this is also a concave function,minus x minus x bar is a concave, so therefore this is a concave function, and in fact, this particular term forces strict concavity. We have already discussed about the strict concavity and convexity. So therefore, this function as a function of x, the maximum exist and the maxmizer is unique okay. Therefore, there exist unique x prime in S1 such that pi1 x prime, y bar minus mode x prime minus x bar square is nothing but maximum x in S1 pi1 x, y bar minus mode x minus x bar square, there exist a unique x prime.

The uniqueness comes from the strict concavity and this is a convex function concave function and hence the maximizer do exist. So, everything is fine. So, for a fixed, I have fixed already x bar and y bar, I have picked this x prime.

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$$\max_{y\in S_2}\underbrace{\{\pi_2(\bar{x},y) - |y-\bar{y}|^2\}}_{\text{Strict Concave}}$$

$$\therefore \exists!\, y' \in S_2 \ni$$

$$\pi_2(\bar{x},y') - |y'-\bar{y}|^2 = \max_{y\in S_2}\{\pi_2(\bar{x},y) - |y-\bar{y}|^2\}$$

The next, I will similarly I look at this pi2 x bar, y minus y minus y bar square and I look at maximizing over y in S2 of this. Once again, in y, the function pi2 is a concave and mode y minus y bar is convex and minus of it means concave, so therefore, this is a concave function and minus mode y minus y bar square makes it strict. Therefore, this is strict concave function, therefore, there exist unique y prime in S2 such that pi2x bar, y prime minus mode y prime minus y bar square is nothing but max y in S2 of this quantity whatever is written there.

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Now

$$(\bar{x},\bar{y}) \overset{\varphi}{\longmapsto} (x',y')$$

$$\therefore\ \varphi: S_1\times S_2 \longrightarrow S_1\times S_2$$

Exercise $\varphi$ is a Continuous function

Brouwer $\Rightarrow \exists (x^*,y^*) \ni \varphi(x^*,y^*) = (x^*,y^*)$

Now, we have a function now, x bar, y bar going to x prime, y prime. Look at this, let me denote this by a function phi. So therefore, phi is a function from S1 cross S2 to S1 cross S2, we have a function from S1 cross S2 into S1 cross S2 defined by x bar, y bar going to x prime, y prime. What are x prime, y prime? The x prime is defined as the unique minimizer of this strict concave function and y prime is defined as the unique minimizer of this strict concave function and this maximizers, this as I change x bar and y bar, they move continuously.

So in fact, this is an exercise from analysis, real analysis, it says that phi is a continuous function okay. Once we know that the phi is a continuous function, what is going to happen? Because S1 one and S2 two are convex and compact and therefore S1 cross x S2 is also a convex and compact sets, therefore they are all unit to finite dimension spaces, the Brouwer fixed point theorem tells you that there exist a fixed point. Brouwer implies there exist x star, y star such that phi of x star, y star is nothing but x star, y star okay.

Now, let us rewrite what exactly it says. Recall this if x bar is x star, y bar is y star, then x prime is also x star, y prime is also x star, so I have to use that here, in this thing. So, let us write that.

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$$\pi_1(x^*, y^*) - |x^* - x^*|^2 = \max_{x \in S_1} \{\pi_1(x, y^*) - |x - x^*|^2\}$$

So pi x star, y star minus mode x star pi1 x star minus x star square. So, when I put x bar is and y bar is x star and y star, so this is y star, this is y star, this is x star and the x star should maximize it. That means, pi1 x star, y star minus x star minus x star square should be same as maximum of that. This should be same as maximum x in S1 of pi1 x, y star minus x minus x star square.

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$$\pi_1(x^*,y^*) = \max_{x\in S_1}\left\{\underline{\pi_1(x,y^*) - |x-x^*|^2}\right\}$$

$$\pi_2(x^*,y^*) = \max_{y\in S_2}\left\{\pi_2(x^*,y) - |y-y^*|^2\right\}$$

$$\frac{\partial \pi_1(x^*,y^*)}{\partial x} = 0$$

So, x star and y star satisfies the following thing. Pi1 x star, y star is nothing but max x in S1 of pi1 x, y star minus mode x minus x star square. Similarly, pi2 x star, y star is nothing but max y in S2 of pi2 of x star, y minus mode y minus y star square. Now, the interesting thing that I would like to point out here is that since x star is maximizing this quantity, let us assume everything is nice, that means the pi1 is a smooth function if we take it, pi1 is a continuously differentiable function.

If we take it, what is going to happen intuitively is that the derivative of pi1 at x star, y star in the variable x star minus the derivative of x minus x star square that is going to be 2 into x minus x star, but at x star that is going to be zero, therefore, this is going to 0. This immediately tells you that when you fix y star, pi1 also has a maximizer at x star because pi1 is a concave function in x star, the first order condition is also sufficient, but this is all assuming several conditions, for example, x star has to be an interior point and other kind of issues, but this is an intuition.

So, the essence of this intuition is that x star not only maximizes this pi1 x, y star minus mode x minus x star square, it also maximizes pi1 x, y star. So, how do we prove this fact? So, let us prove this.

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Goal: $x^*$ maximizes $\pi_1(x, y^*)$.

Fix $x \in S_1$, $\lambda \in (0,1)$

$$\lambda x + (1-\lambda) x^* \in S_1$$

$$\underbrace{\pi_1(\lambda x + (1-\lambda) x^*, y^*)}_{1} - |\lambda x + (1-\lambda) x^* - x^*|^2 \le \pi_1(x^*, y^*) - \underbrace{|x^* - x^*|^2}_{0}$$

$$\lambda^2 |x - x^*|^2$$

So, goal is to prove x star maximizes pi1 x, y star. So how do we prove it? So, let us take some point x from S1 and also take a number lambda in this open interval 0, 1 and look at lambda x plus 1 minus  lambda x star, so this is a convex combination of x and x star and both x and x star are in S1, therefore, this belongs to S1. Now, x star is by the definition of x star, x star maximizes this entire quantity, so we use that now to say that pi1 lambda x plus 1 minus lambda x star, y star minus mode lambda x plus 1 minus lambda x star square.

This is less than or equals to pi1 of x star, y star minus of course mode x star minus x star square that is going to be 0, this is 0, so this is what we have it. Now, this particular term, we know that this function pi1 is convex in this variable, and similarly, let us look at what this particular term is going to be. If we look at this term, that is nothing but lambda x plus x star minus lambda x star, there is a missing term here, minus x star square that is a missing term there. So using that, what we have is that lambda x plus x star minus lambda x star minus x star.

If you use all that, what we are going to get here is plus x star, minus x star gets cancelled. What you have is lambda square mode x minus x star square.

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$$\lambda \pi_1(x, y^*) + (1-\lambda)\pi_1(x^*, y^*) \leq \pi_1(\lambda x + (1-\lambda)x^*, y^*)$$

$$\therefore \lambda \pi_1(x, y^*) + (1-\lambda)\underline{\pi_1(x^*, y^*)} - \lambda^2 |x - x^*|^2 \leq \underline{\pi_1(x^*, y^*)}$$

$$\Rightarrow \underline{\underline{\lambda}} \pi_1(x, y^*) - \underline{\underline{\lambda}}^2 |x - x^*|^2 \leq \underline{\underline{\lambda}} \pi_1(x^*, y^*)$$

Divide by $\lambda$, to get

Now, in the first term here, we use the concavity and to say that lambda pi1 x, y star plus 1 minus lambda pi1 x star, y star, this is less than or equals to pi1 lambda x plus 1 minus lambda x star, y star, the concavity used. So, therefore, using that concavity and this term, this entire inequality can be rewritten as follows. Therefore, this lambda pi1 x, y star plus 1 minus lambda pi1 x star, y star that is smaller than this one and then the minus of that, that is minus lambda square mode x minus x star square, this is less than or equals to pi1 x star, y star.

So, now once this equation is there, so there is a pi1 x star, y star, there is again pi1 x star with one here, so these two get canceled. So, if we remove that particular thing, what we have is lambda pi1 x, y star minus lambda square mode x minus x star square which is less than or equals to this minus lambda pi1 x star, y star comes to the right hand side, that becomes pi1 x star, y star.

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$$\pi_1(x, y^*) - \lambda |x - x^*|^2 \leq \pi_1(x^*, y^*)$$

Now $\lambda \in (0,1)$ arbitrary

let $\lambda \searrow 0$

$\Rightarrow \pi_1(x, y^*) \leq \pi_1(x^*, y^*)$

Again $x \in S_1$ is arbitrary

$\Rightarrow$ $x^*$ maximizes $\pi_1(x, y^*)$.

Now, if we look at it, in all the terms there is a lambda factor, so divided by lambda to get pi1 x, y star minus lambda mode x minus x star square, this is less than or equals to pi1 x star, y star. Now, lambda in open 0, 1, this is arbitrary, therefore, let lambda decrease to 0, lambda go to 0, this immediately gives us that pi1 x, y star less than or equals to pi1 x star, y star and now again x in S1 is arbitrary. So, recall x is in arbitrary, this immediately tells you that x star maximizes pi1 x, y star. Therefore, x star lays now maximizing pi1 x, y star.

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$$\pi_1(x^*, y^*) \geq \pi_1(x, y^*) \quad \forall x \in S_1$$

$$\pi_2(x^*, y^*) \geq \pi_2(x^*, y) \quad \forall y \in S_2.$$

$\Rightarrow (x^*, y^*)$ is Nash equilibrium. $\square$

Therefore, what we have is pi1 x star, y star is greater than or equals to pi1 x, y star for all x in S1. A similar procedure with pi2, similar procedure with pi 2 we will actually get pi2 x star, y star is bigger than or equals to pi2 x star, y, this will happen for any y in S2. This is exactly the definition of Nash equilibrium. This implies x star, y star is Nash equilibrium. This proves the existence of Nash equilibrium of this game where the payoffs are given by pi1, pi2 and pi1 is concave in x, pi2 is concave in y, and of course, we have to assume that they are jointly continuous okay.

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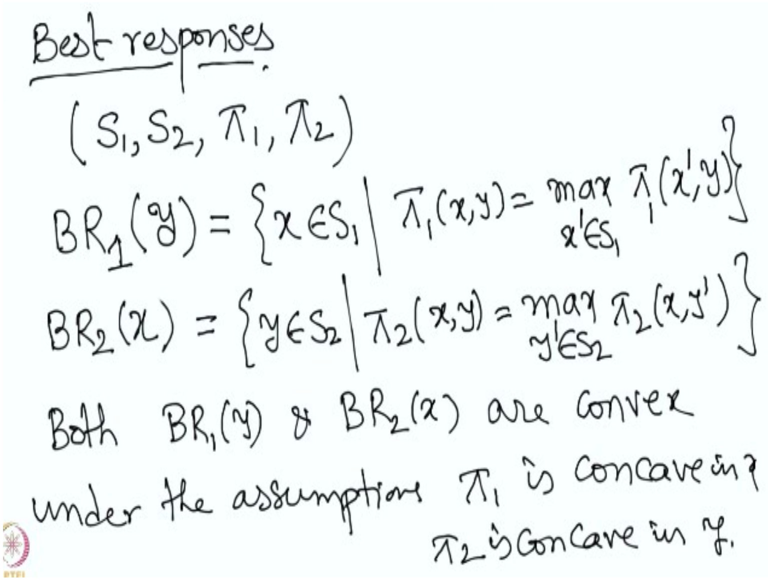


Now, I would like to point out the most common method that people use here is using what is known as best responses. So, let me introduce that. So we have this game with strategies at S1, S2, pi1, pi2 are the payoff functions. This is a non-zero-sum game that we have. The best response of player 1 when player 2 fixes y is nothing but set of all x in S1 such that pi1 x, y is nothing but max x prime in S1 pi1 x prime, y.

Similarly the best response of second player when player 1 fixes x is all y in S2 such that pi2 x, y is max y prime in S2 of pi2 x, y prime. Now, because of the concavity assumption, all these best response sets are convex, both BR1 y and BR2 x are concave are convex under the assumptions pi1 is concave in x, pi2 two is concave in y okay.

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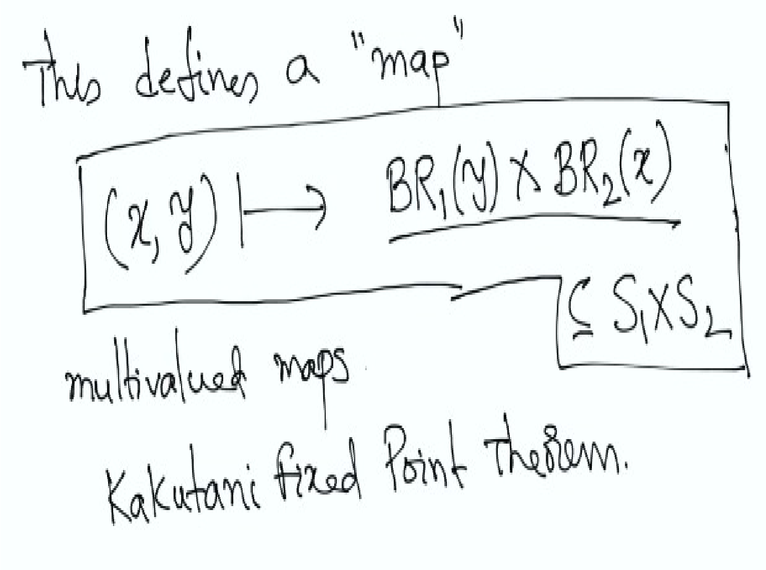


Therefore, this defines a map, let me put it this thing x, y going to BR1 of y cross BR2 of x. So what kind of map, any point x, y the pair of points they are taken to a sets here, these are subsets of S1 cross S2. So, these are the multivalued maps okay. So here, the common approach the people follow here is to show that this particular map satisfies certain assumptions of a theorem called Kakutani fixed point theorem okay. We need this certain continuity properties of this set valued map which I will not go through.

So, but our proof has an advantage that we do not need to worry about the set valued maps, this is just simply a normal continuous functions and Brouwer fixed point theorem is sufficient. In fact, the same ideas can be extended to infinite-dimensional spaces where the Brouwer fixed point theorem has to be replaced with appropriate fixed point theorem infinite dimensions. Here, what helps us is Schauder fixed point theorem, which we will not go into the details. So, we will only stop here.

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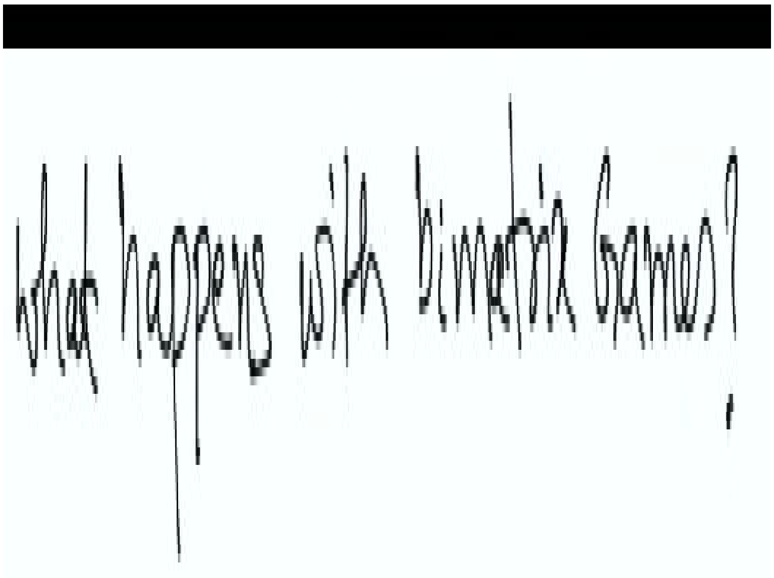


The next thing would like to ask is what happens with bimatrix games? Now, in the zero-sum games, we have seen that the matrix games you can actually prove the minimax theorem, the Von Neumann minimax theorem using nice convexity ideas. Can we really prove something like that here? The proof using just fair convexity ideas is not that easy, but we will prove a proof will provide a proof which is due to Nash in the next session where we also discuss some other properties of this Nash equilibrium. With this, we stop this session. We will continue in the next session.