Game Theory

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Lecture - 02
Combinatorial Games: N and P Positions

In this lecture, we come back to the Take Away game introduced in the previous lecture. We continue asking the question: Is there an optimal way of playing? An optimal way is a strategy which a player(say Player 1) can follow so that whatever the other player does, Player 1 can force a win.

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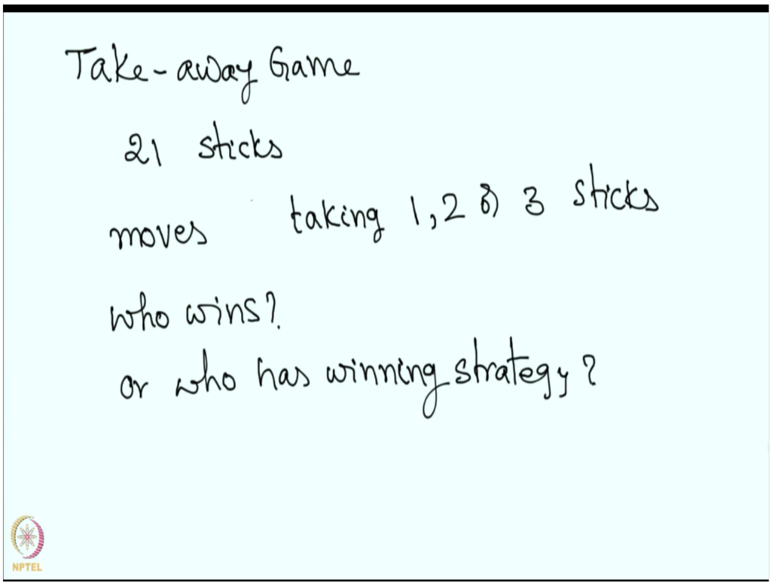

Consider a Take Away game with 21 sticks. The moves are choosing one, two or three sticks.
To give a motivation to the proof, let us think that there is only one stick. Then, obviously Player 1 wins. If there are 2 sticks, Player 1 wins again because he can remove both the sticks. Similarly, if we consider a game with 3 sticks, again Player 1 wins. Now, let’s consider four sticks. In this game, whatever Player 1 does, Player 2 has the upper hand. Player 1 may pick 1, 2 or 3 sticks which means that there is always 1 stick at the least and a maximum of 3 sticks remaining on the table. Hence, Player 2 can pick the remaining sticks and win the game.

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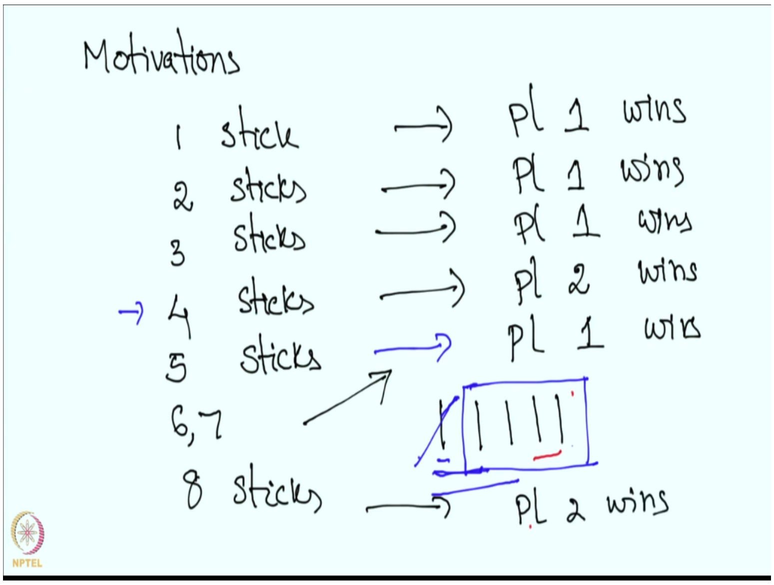

Now, consider a Take Away game with 5 sticks. If Player 1 picks more than 1 stick, Player 2 will pick the rest of the sticks and win the game. Hence, Player 1 picks one stick and reduces the game to a game with 4 sticks with Player 1 being Player 2 in the reduced game and thus enforcing a win. Note that, here the positions, rather than the players playing, are important. Hence, in a game of 5 sticks, Player 1 can enforce a win.

We can similarly see that Player 1 has a winning strategy in games with 6 and 7 sticks. However, with 8 sticks, Player 2 can force a win. This gives the motivation for the following theorem:

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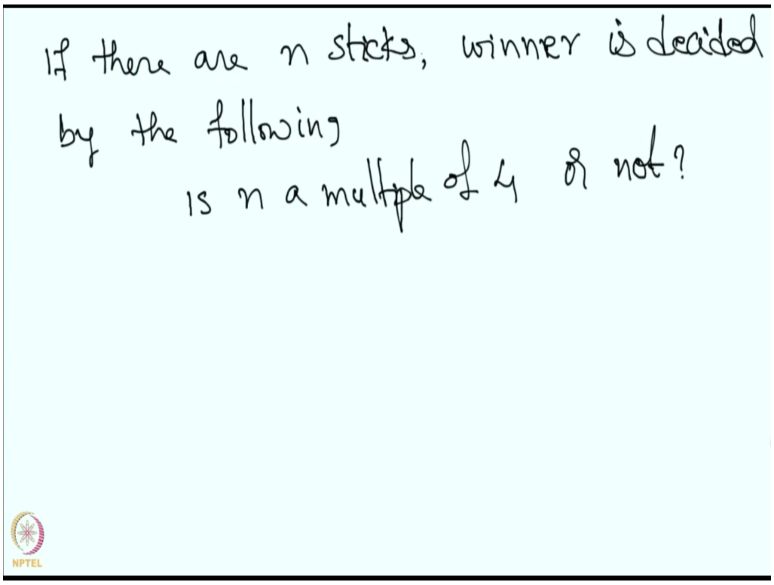

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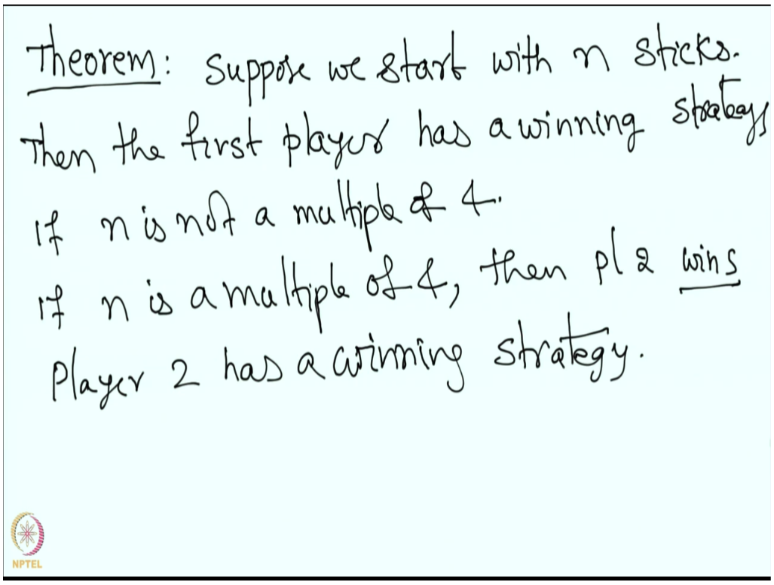

Theorem: *Consider a Take Away game with n sticks, where a player can pick one, two or three sticks during their turn. If n is not a multiple of 4, then the first player has a winning strategy. However, if n is not a multiple of 4, then the second player has a winning strategy.*

The proof of this requires a very important proof technique in mathematics called '*Principle of Induction*'.

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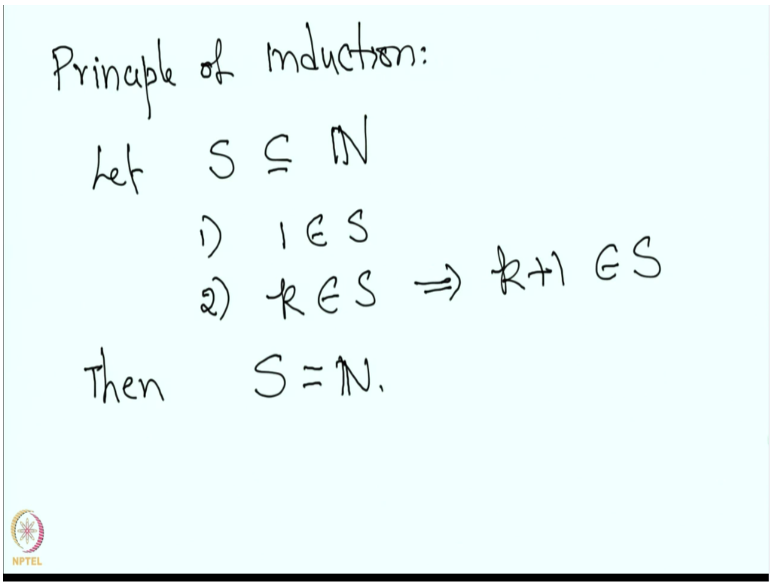

Principle of Induction: Let $S \subseteq \mathbb{N}$. Then, if $1 \in S$ and $k \in S \Rightarrow k+1 \in S$, then $S = \mathbb{N}$.

In fact, this is a very powerful proof technique and several beautiful results can be proved using this simple idea. We will use this technique in proving our result.

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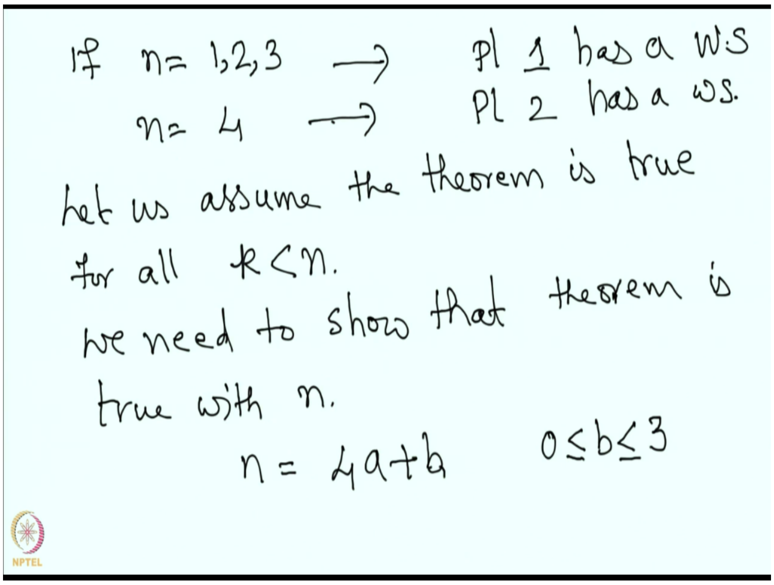

We start with $n=1,2,3$. We have already seen that the first player has a winning strategy. Moreover, for $n=4$, the second player has a winning strategy. So, our theorem holds for $n=1,2,3,4$. Let us assume that our theorem holds for all $k<n$. We need to show that the theorem is true for n. Now, as n is a positive integer, we can apply the *Division Algorithm* on it when divided by 4. This gives the following form of n.

$$n=4a+b \quad , \qquad 0\leq b\leq 3$$

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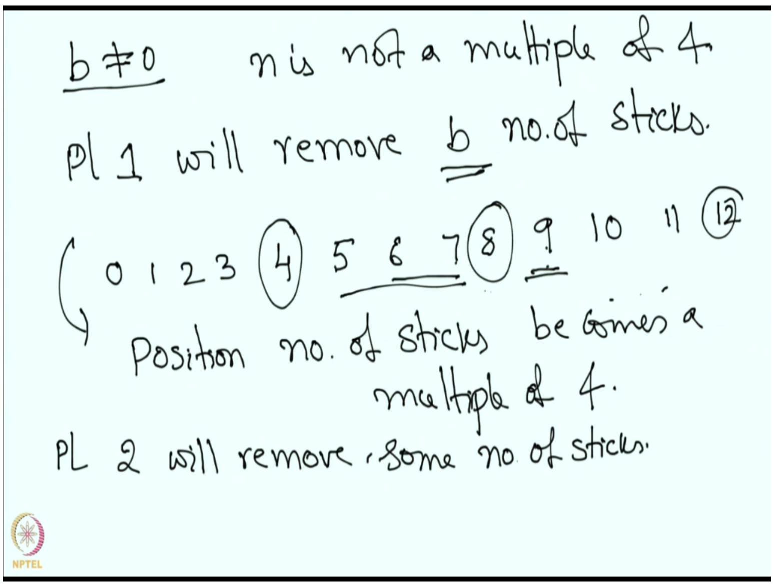

Now, if $b \neq 0$, then on dividing by 4, *n* leaves either 1,2 or 3. Hence, the first player removes *b* number of sticks to make it a reduced game of *4a* sticks. This, being a game of $4a < n$ sticks, by induction hypothesis, Player 1 who becomes the second player in the reduced game, wins.

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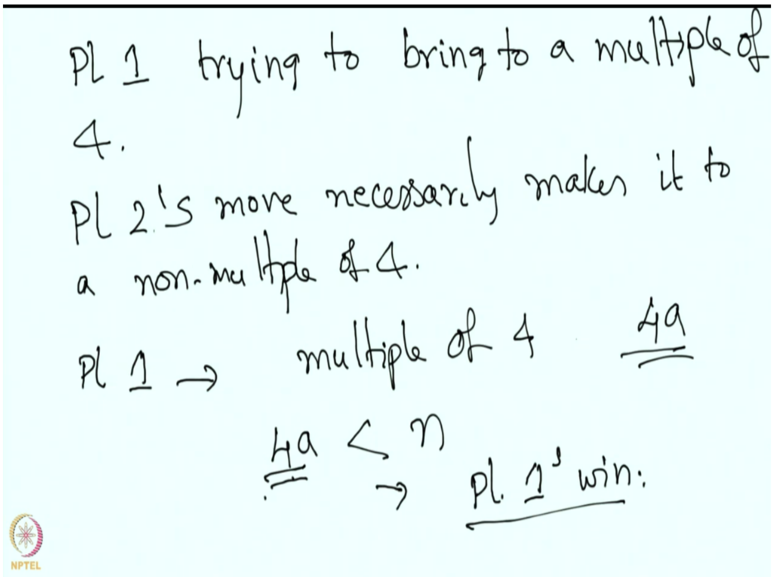

Further, if $b=0$, and let Player 1 pick $s \in \{1,2,3\}$ sticks. Then Player 2 picks $4-s \in \{1,2,3\}$ sticks so that the number of sticks remaining stays a multiple of 4. Hence, in this reduced game, there are $4(a-1)$ number of sticks, where $4(a-1) < n$. Hence, by induction hypothesis, in the reduced game, Player 2 who is also the second player in the reduced game, wins. Thus, our theorem holds in this case as well. This proves our theorem.

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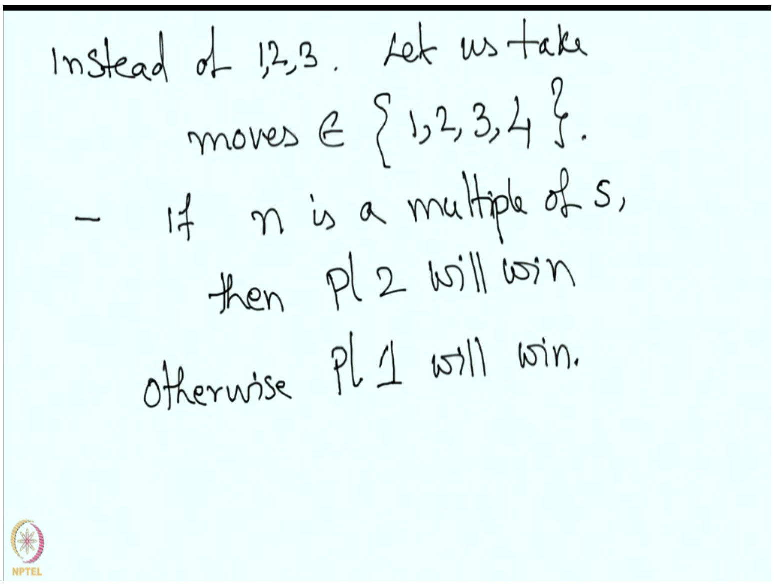

Next, consider the game where a player can move any number of sticks between 1 and 4. We can immediately see that the above theorem stays the same except the fact that now *n* should be a multiple of 5 for Player 2 to win. Moreover, if it is not a multiple of 5, then Player 1 wins.
This proof introduces some important ideas related to this game. Let us go back to the definition of a combinatorial game.

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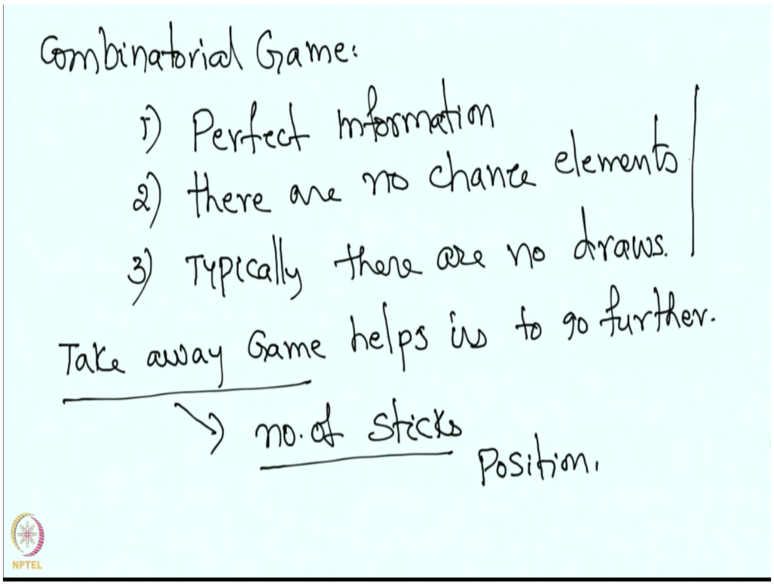

In a Combinatorial Game:

- There is perfect information
- There are no chance elements
- Typically, there are no draws

Combinatorial games have perfect information. This is an important fact. This means that both the players know what is happening in the game and there is no incompleteness of information. Next, there are no chance elements. It means that there are no moves which take certain randomness i.e., every move is fixed. Also, there are no draws.

The Take Away game helps us to go further. The Take away game is determined by the number of sticks. This is the *position* of the game. The position of this game is the number of sticks in the Take away game.

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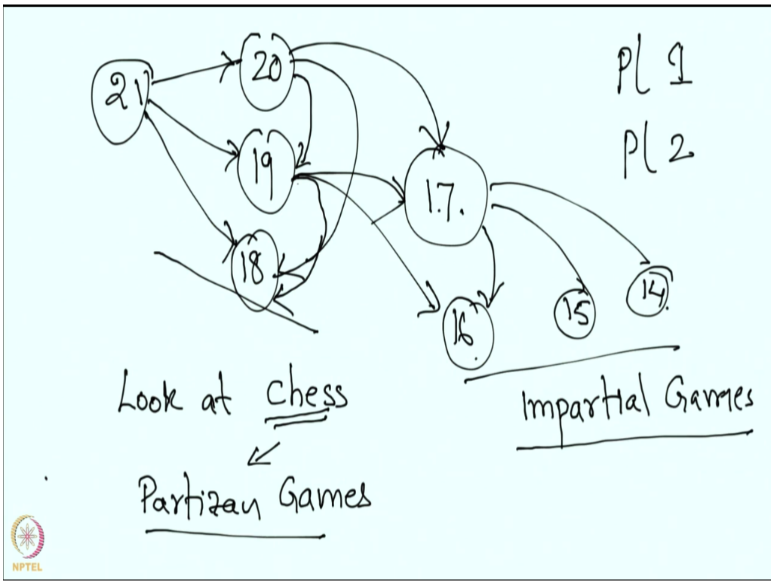

Initially, if we start with 21 sticks, then Player 1 can go to 20 or 19 or 18 sticks. From 20 sticks, the game can go to 19, 18 or 17 sticks. So, in all these positions, whether Player 1 plays or Player 2, it makes no difference. So, in that sense, there is no distinction between the moves for players.

Contrary to this, if we look at chess, the player playing white may not be able to occupy the places the player playing black occupies, with the same pieces. Hence, chess is very different from the Take Away game. We call games like the Take Away game, *Impartial* games. Where as, games like chess where both the players need not have the same set of moves at each position, are called *Partisan* games.

An impartial game is a combinatorial game where there is no difference in the set of moves available at a position between the players. That is, whoever occupies that particular position, the moves available to them are the same.

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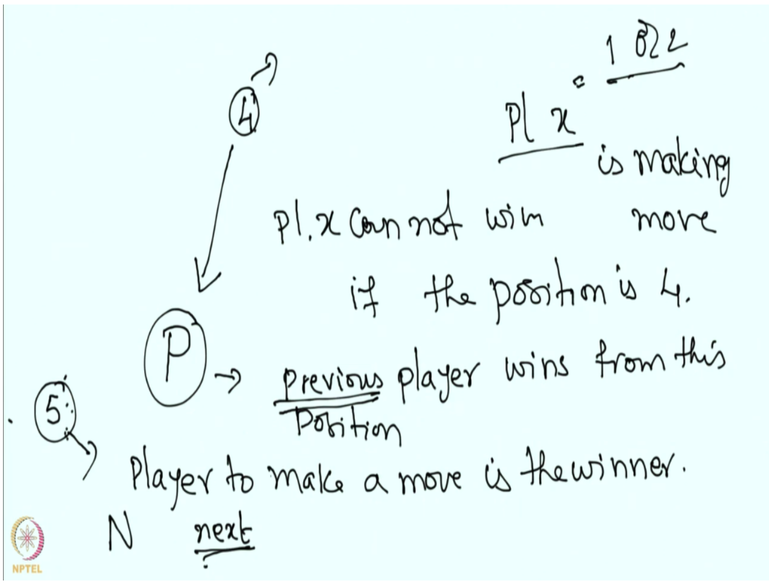

Next we come to what are called N positions and P positions in a game. Consider the Take Away game with a player able to pick 1, 2, or 3 sticks. Consider a point in the game where there are 4 sticks left.

Then, irrespective of whether it is Player 1 or Player 2 who plays next, the player playing next cannot win. This is so because, no matter what number of sticks s they pick, the other player picks $4-s$ and wins the game. Now, at this point of the game, because the previous player who played brought the game to this point, it is the previous player's winning position. This is known as a *P position* in the game.

Now, suppose there are 5 sticks left. Then the next player to play picks 1 stick and becomes the player who brings the game to the position of 4 sticks left. Hence, this player can force a win. Therefore, this position is a winning position for the player playing next in the game. Hence, 5 is a winning position for the next player who plays, irrespective of whether it is Player 1 or Player 2. This is known as an *N position* in the game.

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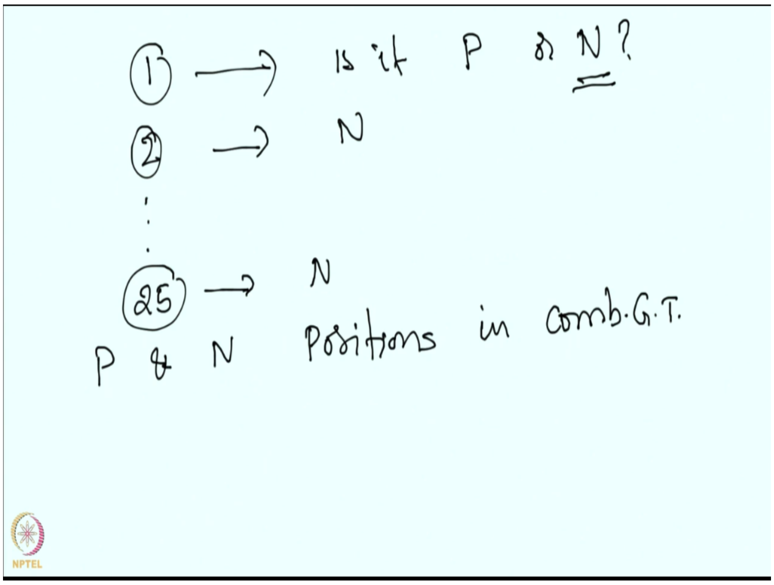

We can easily classify the positions in the game into N or P positions, as can be seen in the slide above. In this particular game, any position which is divisible by 4 is a P position and others are N positions. So, the concept of P and N positions is very important in combinatorial games. We will look at this more in the following lectures.