Game Theory

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Lecture - 20
Non-Zero-Sum Games:
Existence of Nash Equilibrium - II

Welcome back to NPTEL course on game theory. In the previous session, we have proved the existence of Nash equilibrium using Brouwer fixed point theorem and of course the proof is for a general class of games, that means the strategy spaces can be any convex and compact subsets of Euclidean space. As I mentioned, the convex compact subsets in Euclidean space that can be relaxed and you can go to any infinite dimensional spaces, you can take the convex and compact subsets of any infinite dimensional space.

The only issue that requires to be changed is instead of using the Brouwer fixed point theorem, we have to use for example Schauder fixed point theorem, that is mainly the place what we need to change. Now, we will consider a situation where the game is given by bi-matrices, 2 matrices, and we will again derive the proof but unlike in zero-sum games, in zero-sum games we have we could avoid using Brouwer fixed point theorem by using the convexity arguments, but in a non-zero-sum games, we cannot do that, but we will give another proof which is originally due to John Nash.

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$\pi_1(x,y) = x^T A y$

$\pi_2(x,y) = x^T B y$

mixed Payoff functions.

these are bilinear functions.

$\therefore$ The existence theorem proved in prev. session can be applied to guarantee the existence of mixed Nash equilibrium.

So we will consider the a bimatrix game where A is the payoff matrix corresponding to player 1, B corresponds to player 2 and of course we are looking that these m x n matrices okay. So, let me recall the pure strategies here. Pure strategies are basically for player 1, the pure strategies are exactly the rows, so the player 1 is going to choose one of the row, that is his pure strategies and similarly choosing a column is a pure strategy for player 2. Now, what are mixed strategies

It is exactly like in zero-sum games, there is no difference, you are choosing the rows with probabilities certain probabilities. So, it is like x1, x2, xm delta 1 is the corresponding simplex, these are all xi are greater than or equal to 0 and this sum to 1, so this is basically the delta1 is all points like this, that is delta 1. Similarly, this is for player 1, for player 2, y1, y2, yn in delta 2 where yj are nonnegative and they sum to 1. These are the mixed strategies for player 2 and this defines delta 2 okay.

So, once we have this pure and mixed strategies, the payoff extension is given by pi 1 x, y which is nothing but x transpose Ay pi 2 x, y is nothing but x transpose By, these are the mixed payoff functions. The most important thing is that these are bilinear functions. These are bilinear functions, therefore pi 1 is concave in x variable certainly and similarly pi 2 is concave in y variable, therefore, the min-max the existence theorem proved in previous session can be applied to guarantee the existence of mixed Nash equilibrium.

So of course before applying it, we need to realize that delta 1 and delta 2 both are convex, this is obvious, we have seen it previously also, delta 1 and delta 2 are convex and compact and pi 1, pi 2 satisfies the necessary assumption, so therefore, this theorem can be applied. So therefore, there is always a mixed Nash equilibrium.

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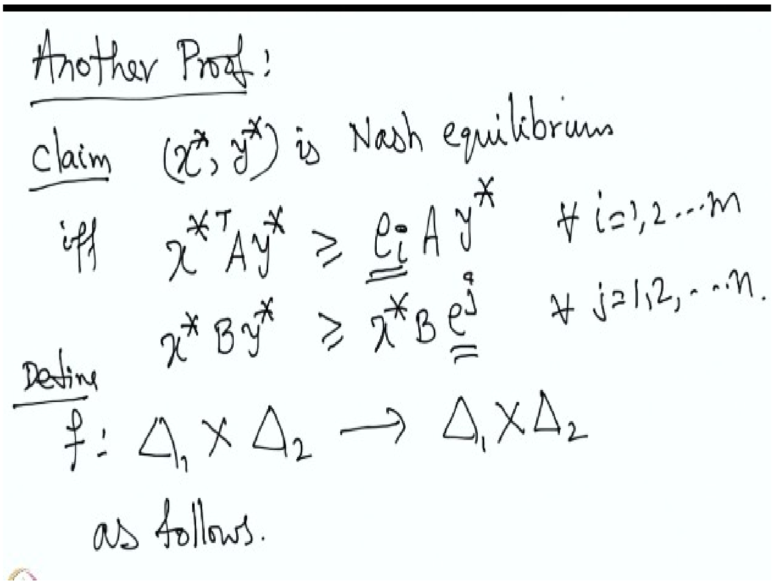

This is the theorem, but we will try to give you another proof, another proof for this result. So, first thing what I would like to say here is that when is want to make the following claim x star, y star is Nash equilibrium if and only if x star transpose A y star, this is greater than or equals to ei A y star this should be true for all i equal to 1 to m and similarly x star B, y star, this should be greater than or equals to x star B ej for all j is equals to 1 to n, remember ei and ej that we have this notation that we have been using throughout this course, ei are the pure strategies for player 1, ej are the pure strategies for player 2 okay.

This is, why is this true? Basically, this comes from the bilinearity of pi 1 and pi 2, so which the ideas we have been using again and again, so therefore, this does not require any further clarification, so this is a kind of on obvious statement once you recall all the arguments that we have been using okay. Now, we will define a function f. So, we will define a function f from the delta 1 cross delta 2 to delta 1 cross delta 2, which like the previously also we have done the same thing, we have used the best response structure there, but here we are using something different, so how we define.

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$f(x,y) = (x', y')$ where x', y' are defined as follows:

$$c_i(x,y) = \max\{0, e_i^T A y - x^T A y\} \rightarrow \text{Cont}$$

$$d_j(x,y) = \max\{0, x^T B e^j - x^T B y\} \leftarrow \text{Cont}$$

and $x_i' = \dfrac{x_i + c_i(x,y)}{1 + \sum_{i'=1}^{m} c_{i'}(x,y)}$, $y_j' = \dfrac{y_j + d_j(x,y)}{1 + \sum_{j'=1}^{n} d_{j'}(x,y)}$

I will define f x, y to be x prime, y prime where x prime, y prime are defined as follows. So, I define first ci x, y, this is nothing but max of 0, ei A y minus x transpose Ay this and dj x, y is max of 0, x B ej minus x B y and xi prime the ith coordinate of xi prime is given by xi plus ci x, y by 1 plus summation ci prime x, y where i prime 1 to m. Similarly yj prime is given by yj plus dj x, y by 1 plus summation j prime goes from 1 to n to and dj prime x, y okay. So, let us try to understand what these particular terms are giving.

So what is this term, ei prime ei transpose Ay minus x transpose Ay. Suppose if the player has played x player 2 has played y and player 1 has played x, so he is going to get x transpose ey. Instead of x, if the player 1 uses the pure strategy ei, how much is he going to get, that is how much extra he is going to get? Suppose if he is getting more, then I would like to increase the probability of the ei, the in x let us say I am playing with probability xi, and then if by playing the pure strategy ei, if I am going to get more than 0, then I would like to move towards ei.

So, that is exactly is captured here in this term, xi, originally xi is there and now I am moving towards the xy, I increased this probability because xy is bigger than 0, therefore xi. Now if eiAy is not, is less than x transpose Ay for example, then the means it is going to be negative value, so therefore this is 0, then I will not change xi. So this is the direction in which I am moving the x the probability with which I play ith pure strategy.

Now, of course, when I move like that, I do not know whether that is going to be a probability vector or not, so therefore, I am normalizing. So sum over all these things and sum over xi is nothing but 1 and sum over the ci is that is exactly this and similarly for the player 2 dj x, y is giving the excess pay the player 2 will get by deviating to pure strategy ej, if he instead of playing y, if he deviates to ej, the jth pure strategy, the excess payoff that he is going to get that is given by dj x, y.

Now if it is greater than 0, then the player 2 would like to move increase yj in the direction of the dj, however, that probability is this yj plus dj x, y and that normalizing and this is the normalizing thing. Now, this is. So now, f x, y is going to x prime and y prime. How is this x prime is defined, as xi plus ci x, y, ci x, y is nothing but this x's payoff function, this is clearly continuous function and this is also a continuous function.

So therefore, xi plus ci x, y is the continuous function, yj plus dj x, y this is also a continuous function and the term that is there in the denominator is a non-zero term because ci are always non-negative and 1 plus sum non-negative terms, so therefore this is always greater than or equals to 1. So therefore, these are all well-defined things and continuous function and bivaried because we have normalized by these factors

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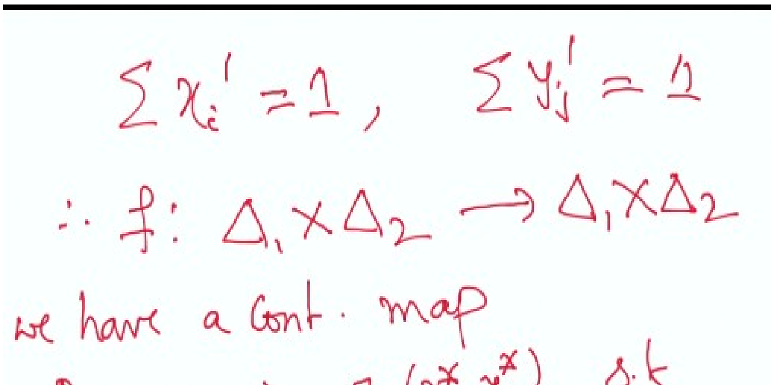

The sum xi prime is going to be 1, similarly sum yj prime is also going to be 1, therefore the function f that we have defined actually takes the values of delta 1 cross delta 2 into delta 1 cross delta 2. Therefore, we have a continuous map now and Brouwer fixed point theorem immediately implies there exist x star, y star such that f x star, y star is same as x star, y star. Brouwer fixed point theorem now gives a fixed point x star and y star such that f x star, y star is same as x star, y star.

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$$\therefore \quad x_i^* = \frac{x_i^* + c_i(x^*, y^*)}{1 + \sum_{i'} c_{i'}(x^*, y^*)}$$

$$\Rightarrow \quad x_i^* \sum_{i'} c_{i'}(x^*, y^*) = c_i(x^*, y^*)$$

Need to show

$\underline{c_i(x^*, y^*) = 0}$ whenever $x_i^* > 0$

Therefore, what we have is that xi star is same as xi star plus ci x star, y star by 1 plus summation ci prime x star, y star. So, let us look at this very careful. From here, we would like to get some contradictions or to say that xi x star and y star are Nash equilibrium, how do we prove it? This immediately implies as by cross multiplication, we have xi star summation i prime or summation over i prime ci prime x star, y star is same as ci x star, y star.

So, now the interesting thing here in this summation, for example i prime where ci is prime x star, y star is 0 that need not be considered if for some because the whole idea here for us is to show that this ci prime x star, y star has to be 0. So whenever we need to show ci x star, y star is 0 whenever xi star is greater than 0. So, why do we want to do this one. If x star is strictly greater than 0, that means the player 1 is playing the pure strategy ei with the positive probability.

Now, look at the definition of ci, if the player has played ei with a positive probability, that means ei transpose Ay star should be same as x star A, y star, so that should happen. Therefore, this ci x, y should be 0, so this has to happen whenever xi star is greater than 0. So, this is essentially the idea now. So, let us look at we need to show this fact okay.

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$$I = \{ i \mid x_i^* > 0 \}$$

$$J = \{ i \mid c_i(x^*, y^*) > 0 \}$$

<u>Claim</u> $\sum c_{i'}(x^*, y^*) = 0$

Suppose not $\sum c_{i'}(x^*, y^*) > 0$

$$= \sum_{i' \in J} c_{i'}(x^*, y^*) > 0$$

So let us take I to be set of all i such that xi star is greater than 0 okay. So, now therefore, let me also take J to be all i such that ci x star, y star is greater than 0 okay. So, let me take. Therefore, clearly from here, whenever xi star is greater than 0, look at those i's and here I am looking at i's where ci x star, y star is greater than 0. So now look at this. If xi star is greater than 0, okay, because we are interested in proving this one if xi star is greater than 0, I need to show that ci x star, y star should be 0.

Suppose if that does not happen, we are going via a contradiction, so in fact, our claim is going to be summation ci prime x star, y star this is going to be 0 okay. So, suppose this does not happen, that means summation ci prime x star, y star this is greater than 0. So, therefore, the summation in some sense I can take it to be i prime in j, ci prime x star, y star, this is greater than 0, of course, these two are same okay.

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$$i \in I \Rightarrow x_i^* > 0$$
$$\Rightarrow c_i(x^*, y^*) > 0$$
$$\Rightarrow e_i^T A y^* > x^{*T} A y^*$$
$$\underbrace{x^{*T} A y^*}_{u_1} = \sum_{i=1}^{m} x_i^* (e_i^T A y^*)$$
$$\geq \sum_{i \in I} x_i^* (e_i^T A y^*) > \sum x_i^* u_1 = u_1$$

So, now look at for all. Let us take i belongs to I. This implies xi star is greater than 0. This implies ci x star, y star is greater than 0 because this is by our contradiction, ci x star, y star is greater than 0, this is coming from this contradiction because we have assumed this. Therefore, this ci x star, y star is strictly greater than 0. What this implies by the definition of ci, ci is basically the excess pay that he is getting, what it says is that eiAy star is bigger than x star A y star, of course x star transposes okay, this happens okay.

Now, what is a x star transpose A y star, this is same as summation xi star of ei transpose A y star where i is equals to 1 to m, this is certainly greater than or equals to summation i prime i belongs to I of xi star ei transpose A y star because if i is not in i then x star is 0, therefore this happens, and then from this previous thing, we have this j is there here, so this is going to be greater than summation xi star into let me call this as a u1 and this is going to be u1 okay. So, what we have got is the u1 is strictly bigger than u1, so this is a contradiction, this contradiction happened because we have assumed this.

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$$\therefore \sum c_i'(x^*, y^*) = 0$$

$$\Rightarrow c_i(x^*, y^*) = 0 \quad \forall i$$

$$\Rightarrow e_i^T A y^* \leq x^{*T} A y^* \quad \forall i \in 1, .., m$$

$\Rightarrow$ x^* is optimal for player 1.

$|||^{ly}$ y^* is optimal for pl. 2.

$\therefore$ (x^*, y^*) is NE.

Therefore, summation ci prime x star, y star is going to be 0, therefore ci x star, y star is 0 for each i, this implies ei transpose A y star, this is certainly less than or equals to x star transpose A y, this is true for all i running from 1 to m. This implies okay, it is not y, y star, x star is optimal for player 1. In a similar fashion, y star is optimal for player 2. Therefore, x star, y star is Nash equilibrium okay

So this proof is again a simple proof which requires you to construct this specific function by this way, and then we show that this is a continuous function and this gives you a fixed point and we have to finally show that this fixed point is indeed a Nash equilibrium. So, this proves the existence of a Nash equilibrium.

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How do we Compute NE?

optimization Problem:

$$\max_{x, y, \alpha, \beta} \; x^T(A+B)y - \alpha - \beta$$

s.t.

$$Ay - \alpha \mathbb{1} \leq 0$$

$$B^T x - \beta \mathbb{1} \leq 0$$

$$x \in \Delta_1, \quad y \in \Delta_2.$$

Our next task now is to get some optimization problem. So, basically how do we compute Nash equilibrium? Are there ways to do, ways to compute the Nash equilibrium? So, we will now discuss one optimization problem now, and of course, the solving algorithms we will postpone to next session, but now we will please look at a optimization problem which gives you the reformulation of a Nash equilibrium. So the problem is the following thing, optimization problem. Let me write it first.

This is nothing but maximize x, y, alpha, beta so that x transpose A plus B y minus alpha minus beta subject to Ay minus alpha 1, I will put the bold one means it is a vector of ones less than or equal to 0. Similarly summation okay B transpose x minus beta again a vector of ones, this is less than or equals to 0 and of course x belongs to delta 1, y belongs to delta 2 okay. So this is a quadratic programming. So what we have is that in this there are 4 decision variables x, y, and alpha, beta and we need to choose the x, y, alpha, beta which maximizes this.

Whatever maximizes this one, the x star, y star, and then alpha star and beta star they correspond to the Nash equilibrium, x star is going to be the optimal strategy for player 1, y star is for player 2, and alpha star is the value of player 1 and beta star is going to be the value of player 2. So, we will try to prove this fact.

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Let (x^*, y^*) be NE.

$\alpha^* = x^{*T} A y^*$, $\quad \underline{\beta^* = x^{*T} . B y^*}$

$e_i A y^* \le \alpha^* \ \forall i$

$\Rightarrow | \ A y^* - \alpha^* \mathbb{1} \le 0$

$B^T x^* - \beta^* \mathbb{1} \le 0$

$\forall\ x \in \Delta_1,\quad x^T A y^* \le \alpha^*$

$y \in \Delta_2 \quad x^{*T} A y^* \le \beta^*$

So, first let us see let x star, y star be Nash equilibrium. Let alpha star to be x star A, y star and similarly beta star let me put it as x star transpose B y star, let us take this. Now, we know that in the previous itself we have seen that is that ei A y star is really less than or equals to alpha star, this is true for all i, that means ei A y star is let us suppose to alpha star for every i, that means every entry of ay star is going to be less than or equals to alpha star. This implies A y star minus alpha star, the vector of ones, this is less than or equals or 0, this is obvious. In a similar fashion from here, from beta star, the definition of beta star we can easily see that B transpose x star minus beta star ones less than or equals 0. This is an obvious thing from here and of course, x star, y star are all there and then the next thing is that for any x in delta 1, x transpose A y star is less than or equals to alpha star. Similarly, for any y in delta 2, x star transpose A y star is less than or equals to beta star.

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We can easily verify that

$(x^*, y^*, \alpha^*, \beta^*)$ is optimal solution

for the optimization problem.

Next

let $(x^*, y^*, \alpha^*, \beta^*)$ be optimal solution

of the optimization problem

Putting all these things together and combining with these facts, we can easily verify that x star, y star, alpha star, beta star is optimal solution for the optimization problem. So x star, y star, alpha star, beta star will satisfy. So, this here, when you are trying to prove form here, we have to use these facts as well okay. This is not a very hard thing, it is straightforward, but it requires just little effort, so one should try proving it. Here, we also have to understand the following fact.

In this optimization problem using these facts, we need to show that this is always less than or equal to 0, because Ay minus alpha is less than or equal to 0, therefore x transpose Ay minus alpha is less than or equals to 0 that has to be used here. Therefore, this maximum value is always non-positive and at Nash equilibrium, this is equal to 0, therefore x star, y star, alpha star, beta star is going to be the optimal solution. So the details have to be furnished here, but they are a simple exercise.

Next, let x star, y star, alpha star, beta star be optimal solution of the optimization problem. So, once we take this one, we need to show that this x star, y star corresponds to saddle Nash equilibrium. So let us see how we prove.

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$$Ay^* - \alpha \underline{\mathbb{1}} \le 0$$

$$\Rightarrow \quad x^{*T} A y^* \le \alpha.$$

$$\text{Illy} \quad x^{*T} B y^* \le \beta.$$

Take any $x \in \Delta_1$

$$\Rightarrow \quad x(A+B)y^* - \alpha^* - \beta^*$$
$$\le x^*(A+B)y^* - \alpha^* - \beta^*$$

First thing is because it is a solution of this optimization problem, we have the following this thing conditions Ay star minus alpha 1 this is less than or equal to 0, this immediately tells me that x star A y star is less than or equals to alpha that is there and similarly x star transpose B y star is less than or equals to beta that is also there, so these are there. Now if this is always true, so that is the first fact that we need to show here. Then we will try to show that x star, y star satisfies the equilibrium condition, how do we prove that?

So, take any x in delta 1. This implies x A plus B y star minus alpha star minus beta star is less than or equals to x star A plus B y star minus alpha star minus beta star. As I said, we have to verify that x star, y star, alpha star, beta star is an optimal, is corresponds to the Nash equilibrium, how do we do this? Let us look at it.

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$$x^*(A+B)y^* - \alpha^* - \beta^* = \max_{x,y,\alpha,\beta} \underbrace{\left(x(A+B)y - \alpha - \beta\right)}_{\leq 0}$$

$$\leq 0.$$

$$\therefore\ x^*(A+B)y^* - \alpha^* - \beta^* = 0$$

$$\underbrace{\left(x^*Ay^* - \alpha^*\right)} + \left(x^*By^* - \beta^*\right) = 0$$

$$\Rightarrow\ x^*Ay^* = \alpha^*, \quad x^*By^* = 0$$

First thing is that we know that x star into x A plus B y star minus alpha star minus beta star, this is nothing but the maximum over x, y, alpha, beta of x A plus B y minus alpha minus beta, this is true. Now, what we really need to understand here is that this is always less than or equals to 0. Therefore, this is less than or equal to 0. Now, here is another important thing that we need to show here is that there is we already proved the existence of Nash equilibrium, so therefore there exist a Nash equilibrium.

So let me call that as x hat and y hat. For the Nash equilibrium, x hat and y hat, this value is going to be 0 because x hat, y hat corresponding values satisfy the constraints of this optimization problem, therefore this value is going to be 0, therefore this is going to be 0, that is the first important thing. Once this is there, now we use the constraints. So by using the constraints, we can easily see that because we can write it as x star A y star minus alpha star this is one term plus another term is x star B y star minus beta star.

The sum of these 2 terms is equal to 0, but by the constraints, this is x star A y star minus alpha star should be less than or equals to 0, this is also less than or equal to 0, both of them are non-negative numbers, therefore what we get is that they must be 0.

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$\Rightarrow (x^*, y^*)$ is NE.

Once these are 0s, then now using the constraint this thing immediately we can prove that x star y star is Nash equilibrium. So this proves the equivalence between these 2 problems. So if you have an existence of a Nash equilibrium, that immediately says that the Nash equilibrium is the solution corresponding to this optimization problem. Similarly, a solution of this optimization problem is a Nash equilibrium. So in the second part, we have used the fact that the equilibrium existence happens. Okay, with this, we conclude this session. We will meet again in the next session.