Game Theory

Prof. K. S. Mallikarjuna Rao
Department of Industrial Engineering & Operations Research
Indian Institute of Technology - Bombay

Lecture 21
Iterated Elimination of Strictly Dominated Strategies

In the previous lecture, we have discussed bimatrix games, existence of Nash equilibrium and a nonlinear programming method. In this lecture, we will see an intuitive way of solving certain games. This method is known as solving by dominance, which we have seen already in the context of zero-sum games. Now we will do this for bimatrix games and, in particular, we introduce what is called *iterated elimination*.

We start with an example of a simple game. Consider the following bimatrix game:

P1

		C	D
P2	C	2,2	0,3
	D	3,0	1,1

In this game, D strictly dominates C for Player 1. Similarly, for Player 2, as this is a symmetric game, D strictly dominates C. Hence, as both are utility maximizing players, none will play C and both end up playing D. This will lead to (D,D) being an equilibrium. This is known as solving the games by dominance. Let us define this concept more formally:

- $G = (S_1, S_2, \pi_1, \pi_2)$
- Player i's strategies: $s_i', s_i'' \in S_i$
- Then, s_i' strictly dominates s_i'' iff $u_i(s_i', s_{-i}) > u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$.

We illustrate this by the following bimatrix game example:

P1

		L	C	R
P2	U	4,3	5,1	6,2
	M	2,1	8,4	3,6
	D	5,9	9,6	2,8

Let us try solving this game by dominance. For Player 2, column R dominates C. Hence, he will never play C as column R strictly dominates column C. This leads to the following matrix:

In the new matrix, Player 1's strategy U strictly dominates strategy M. Hence, Player 1 never plays column M. Continuing to do this, we get the profile (L,R) as the strict Nash equilibrium which we get by iterated elimination of dominated strategies.

		P1 L	*P1* R
P2	U	4,3	6,2
	M	2,1	3,6
	D	5,9	2,8

Theorem. *If iterated elimination of strictly dominated strategies leads to a single pair of strategies, then this single pair is Nash equilibrium.*

Proof. We prove this by contradiction. Let iterated elimination of dominated strategies lead to a single pair of strategies, given by (x^*, y^*). Suppose x^* is not the best response to y^*.

Let $X = \{x \in S_i | \pi_1(x, y^*) > \pi_1(x^*, y^*)\}$. Therefore, X is non-empty. All the strategies in X must have been eliminated. Look at the last stage where a strategy $x \in X$ is eliminated.

For x to be eliminated, there must be $x' \in S_1$ such that x_1 dominates x. Therefore, $\pi_1(x_1, y^*) > \pi_1(x, y^*) > \pi_1(x^*, y^*)$. This means that the strategy x^* is dominated by x_1. But this is a contradiction as x^* is the last remaining strategy after iterated elimination of dominated strategies. Therefore, (x^*, y^*) is a Nash Equilibrium. ■

Moreover, when we look at iterated elimination it always leads to a unique solution. Next, we define weakly dominated strategies:

Definition. s' is said to be *weakly dominated* by s'' for player i if

$$\pi_i(s', s_{-i}) \leq \pi_i(s'', s_{-i})$$

for all $s_{-i} \in S_{-i}$ and there must exist at least one s_{-i} where this inequality is strict.

Let us see what happens if we do an iterated elimination of weakly dominated strategies. Consider the following game:

		P1 L	*P1* R
P2	T	1,1	0,0
	M	3,2	2,2
	B	0,0	1,1

As we can clearly see, row T and row B are strictly dominated by column M. Removing it, we get

		P1 L	*P1* R
P2	M	3,2	2,2

Now, as R weakly dominates L, (M, L) is eliminated. But (M, R) is not the only Nash equilibrium. (M, L) is also a Nash equilibrium. Hence, the order in which you are eliminating the strategies affects which equilibrium we are arriving at. An interesting exercise would be to construct an example, where the iterated elimination of weakly dominated strategies need not lead to a Nash equilibrium. The iterated elimination of strictly dominated strategies however, if leading to a single strategy profile, leads to the only Nash in the game. The same argument can be easily extended to the mixed strategy space.

However, this method does not apply to all sorts of games. There are games that cannot be solved by dominance. So, for that we require other kinds of algorithms. one algorithm that we have seen already in the previous lectures is nonlinear programming. Another algorithm that we are going to see in the next set of lectures, is known as the Lemke-Howson Algorithm which is more of a combinatorial algorithm.

In fact, this Lemke-Howson algorithm also proves the existence of a Nash equilibrium using fairly simple arguments with no fixed point argument required.