Game Theory
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Lecture 23
Non-Zero-Sum Games: Lemke-Howson Algorithm - II

In the previous lecture, we looked at the support of mixed strategies and using support enumeration we computed a Nash equilibrium. The algorithm actually asks to verify the subset of inequalities for every k sized subsets.

It is more complex in terms of computing because it requires exponential time. Is there a better way of doing this? That is the question the Lemke-Howson algorithm answers.

Recall the non-degenerate conditions. For (A,B), $\forall (x,y) \in \Delta_1 \times \Delta_2$, $|supp(x)| \geq |supp(y)|$ if y is the best response to x and $|supp(y)| \geq |supp(x)|$ if x is the best response to y. We assume that A and B have no zero column. This makes sense as for A' and B' given by

$$A' = (a_{ij} + \alpha)$$
$$B' = (b_{ij} + \alpha)$$

the equilibrium structure does not change. Now let us define certain sets given by:

$$\overline{P}_1 = \{(x,v) \in \mathbb{R}^m \times \mathbb{R} | x \geq 0, B^T x \leq v\mathbb{1}, x^T \mathbb{1} = 1\}$$
$$\overline{P}_2 = \{(y,u) \in \mathbb{R}^n \times \mathbb{R} | y \geq 0, Ay \leq u\mathbb{1}, y^T \mathbb{1} = 1\}$$

Now consider $< x, By >$. This equals $< B^T x, y > \leq v$. Hence, when Player 1 plays mixed strategy x, Player 2 cannot get more than v utility. By a similar argument we can see that Player 1 does not get more than u utility. These are known as *Best Response Polyhedrons*. Moreover, we can see that this is not a polytope as v is only an upper bound and anything bigger than $\mathbb{1}v$ would work.

Consider $\overline{P}_1$. We have that $x_k \geq 0$ for all k. Let $x_k = 0$ for some k. Hence, $k \in S_1$ is not played in x. Moreover, let equality hold in the constraint $(B^T x)_l \leq u$, i.e. $(B^T x)_l = u$. This implies that $l \in S_2$ is best response of Player 2 to x.

The polyhedron $\overline{P}_1$ lives in $\mathbb{R}^{m+1}$. It has one equality constraint. We can also verify that it is a m-dimensional polyhedron. Now, any extreme point of $\overline{P}_1$ will satisfy the inequality constraints with equality. That is, an extreme point (x,v) corresponds to a situation where $supp(x) = k$ for some $k \leq m$ and at least k pure strategies of Player 2 are best responses of x. Since we are assuming non-degeneracy, exactly k pure best responses of Player 2 exist.

We proceed further to simply normalize the sets $\overline{P}_1$ and $\overline{P}_2$. We get,

$$P_1 = \{x \in \mathbb{R}^m | x \geq 0, B^T x \leq 1\}$$
$$P_2 = \{y \in \mathbb{R}^n | y \geq 0, Ay \leq 1\}$$

Note that these are normalized vectors and not mixed strategies. Now, $(x,v) \in \overline{P}_1$ implies $\frac{x}{v} \in P_1$. Hence, $x \in P_1$ is an extreme point of P_1 implies $\left(\frac{x}{x^T \mathbb{1}}, \frac{1}{x^T \mathbb{1}}\right)$ is an extreme point of $\overline{P}_1$.

Therefore, for all $x \in P_1$ and $yinP_2$,

$$L(x) = \{k \in S_1 | x_k = 0\} \cup \{j \in S_2 | (B^T x)_j = 1\}$$
$$L(x) = \{l \in S_2 | y_l = 0\} \cup \{i \in S_1 | (Ay)_i = 1\}$$

called *labels* of x and y.

We say that $(x, y) \in P_1 \times P_2$ is fully labeled if

$$L(x) \cup L(y) = S_1 \cup S_2$$

Lemma: In a non-degenerate game, for a pair of extreme points (x, y) we have $|L(x)| = m$, $|L(y)| = n$.