Game Theory
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Lecture 24
Non-Zero-Sum Games: Lemke-Howson Algorithm - III

In the previous lecture we introduced labels for the vertices of the best response polyhedron and the corresponding normalized polytope. We also stated a lemma without a proof. We will go back to that lemma and continue discussing the Lemke-Howson algorithm.

Lemma. In a non degenerate game for a pair of extreme points (x,y) we have $|L(x)| = m$ and $|L(y)| = n$.

Proof: Game is non-degenerate. Therefore, at most $Supp(\bar{x})$ pure strategies can be best responses to $\bar{x}$, where $\bar{x}$ is the normalized vector of x. And similarly, at most $Supp(\bar{y})$ pure strategies can be best responses to $\bar{y}$.

$$|L(x)| \leq |S_1 \setminus Supp(x)| + |Supp(x)| = m$$
$$|L(y)| \leq |S_2 \setminus Supp(y)| + |Supp(y)| = n$$

These polytopes are full dimension and (x,y) is an extreme point. Therefore,

$$|L(x)| \geq m$$
$$|L(y)| \geq n$$

Hence, this proves that $|L(x)| = m$ and $|L(y) = n|$. This proves the lemma.

Now, let us look at the following theorem:
Theorem. A pair of extreme points $(x,y) \in P_1 \times P_2 \setminus \{(0,0)\}$ is fully labelled iff the corresponding normalized vector $(\bar{x},\bar{y})$ is a Nash equilibrium.

Proof. Let $(x,y) \in P_1 \times P_2 \setminus \{(0,0)\}$ be fully labelled. Let $T_1 = Supp(x)$ and $T_2 = Supp(y)$. For all $k \in T_1$, x does not have label k as $x_k > 0$. Therefore, y must have label k.

$$\Rightarrow (Ay)_k = 1$$
$$\Rightarrow (A\bar{y})_{k'} = \frac{1}{y^T \mathbb{1}}$$

while

$$(A\bar{y})_{k'} \leq \frac{1}{y^T \mathbb{1}} \ \forall k' \in S_1$$

This implies that k is the best response to $\bar{y}$. Further, for $k \notin T_1$, x does have label k. A label cannot appear twice in a fully labelled pair. This implies that y does not have label k, which in turn implies that k is not the best response to $\bar{y}$. Hence, $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Conversely, let $(\bar{x}, \bar{y})$ be a Nash equilibrium. This implies,

$$S_1 \setminus Supp(x) \cup Supp(y) \subseteq L(x)$$
$$S_2 \setminus Supp(y) \cup Supp(x) \subseteq L(y)$$

Therefore, $L(x) \cup L(y) = S_1 \cup S_2$. This is the same as saying that they are fully labelled. This proves the theorem.

Lemke-Howson Algorithm

The idea is to start from the origin and pivot alternatingly in P_1 and P_2 until a labelled pair is found. Let V_1 be the set of extreme points of P_1 and V_2 denote the set of extreme points of P_2.

Let E_i be the set of edges between adjacent extreme points in V_i.

$$E_1 = \{(x, x') \in V_1 \times V_1 : |L(x) \cap L(x')| = m - 1\}$$
$$E_2 = \{(y, y') \in V_2 \times V_2 : |L(y) \cap L(y')| = n - 1\}$$

Let $V = V_1 \times V_2$ and E be given by the following:

$$\Big\{\big((x,y),(x',y)\big) \in V \times V | (x,x') \in E_1\Big\} \cup \Big\{\big((x,y),(x,y')\big) \in V \times V | (y,y') \in E_2\Big\}$$

The whole idea here is that if we restrict our attention to extreme points that are almost fully labelled with the possible exception of a particular label i, then there is always a unique way in which we can proceed.

We introduce another notation. For $i \in S_1 \cup S_2$, let

$$V^i = \{(x,y) \in V : L(x) \cup L(y) \supseteq S_1 \cup S_2 \setminus \{i\}$$
$$E^i = E \cap (V^i \times V^i)$$

Theorem. Let $i \in S_1 \cup S_2$. Then, V^i contains $(0,0)$ as well as every $(x,y) \in V$ such that $(\bar{x}, \bar{y})$ is Nash Equilibrium. Assuming non-degeneracy, the point $(0,0)$ and the elements of V^i corresponding to an equilibrium have degree one in the graph (V^i, E^i) and all other nodes in V^i have degree two.

Proof. $(0,0)$ and all pairs corresponding to Nash Equilibria are fully labelled. Therefore, the first part is obvious.
For the second part, consider $(x,y) \in V^i$ and let $(\bar{x}, \bar{y})$ be the corresponding normalized strategies. Now, as x, y are extreme points, from an earlier lemma that we have seen, $|L(x)| = m$ and $|L(y)| = n$.

If $(x,y) = (0,0)$ or $(\bar{x}, \bar{y})$ is Nash Equilibrium, then (x,y) is fully labelled and $L(x) \cap L(y) = \phi$. The neighbours of (x,y) are those elements in V^i that replace i with some other label, those where the constraints hold with equality instead of one corresponding to i. Since only x or y has label i but not both, we may only replace it from one of them.

Dropping the label i, we obtain a new label and by non-degeneracy this label is unique otherwise, if $L(x) \cap L(y) = \{j\}$ for some duplicate label $j \in S_1 \cup S_2$, then the neighbours of (x, y) are obtained by replacing j with another label. This replacement can be done either in P_1 or in P_2 and for each of them, there is exactly one neighbour by the same reasoning as before.

$(V^i, E^i) \forall i \in S_1 \cup S_2$ consists of paths and cycles that are pairwise disjoint and the ends of paths correspond to the pair $(0, 0)$ and to the equilibrium of the underlying game. This is the Lemke-Howson algorithm. So, the Lemke-Howson algorithm essentially starts with $(0, 0)$ and it looks for a path. From $(0, 0)$ it goes to another extreme point along this graph which we have introduced. It eventually stops at a node with degree one, a Nash equilibrium.

In fact, this also provides a proof of existence of a Nash equilibrium. This is a very important algorithm for solving bimatrix games. Moreover, this algorithm belongs to the class of path following algorithms or Homotopy based algorithms. In the worst case, however, this algorithm can take upto exponentially many steps.