Game Theory
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Lecture 25
Evolutionarily Stable Strategies - I

In this lecture, we discuss about evolutionary stability by first considering non zero-sum games, in particular, a symmetric game.

- We consider a symmetric game $A_{m\times n}$.
- The pure strategies are given by $e_1, e_2, .., e_m$.
- Mixed strategies are given by $\Delta = \{x \in \mathbb{R}^m : x_1 + x_2 + ..x_m = 1; x_1, x_2, .., x_m \geq 0\}$.
- The mixed payoff is given by $\pi(x,y) = x^T Ay = \sum_{i,j=1}^{m} a_{ij} x_i y_j$.

Now, we define the Evolutionary Game. Let us consider a large population. All people decide how to choose one of these choices available to them. Let us assume that they are all playing pure strategies. Each person is associated with some pure strategy. A random pair of people is picked and they will play the above game with these strategies.

A population state is the fraction of people who are playing pure strategy 1, fraction of people playing pure strategy 2, and so on. So the state of the population tells the fraction of people who are playing the pure strategy e_1, fraction of people who are playing e_2, and likewise.

The notation for the same is as follows:

- Incumbent strategy $x = (x_1, x_2, .., x_m)$
- ε fraction of the population are mutants. They play y.
- In the new state, $1-\varepsilon$ of the population plays x and ε fraction of the population plays y.
- For the individuals playing x, their payoff is now $\pi(x, \varepsilon y + (1-\varepsilon)x)$ and for the individuals playing y, their payoff is $\pi(y, \varepsilon y + (1-\varepsilon)x)$.

Definition. x is said to be an evolutionarily stable strategy(ESS) if $\forall y \neq x$, there exists $\in (0,1)$ such that

$$\pi(x, \varepsilon y + (1-\varepsilon)x) > \pi(y, \varepsilon y + (1-\varepsilon)x)$$

for all $0 < \varepsilon < \overline{\varepsilon}$.

$\overline{\varepsilon}$ is called the *invasion barrier*. If $\overline{\varepsilon}$ is independent of y, then this is called *uniform invasion barrier*.

$$\pi(x,\varepsilon y+(1-)x) > \pi(y,\varepsilon y+(1-)x)\ \forall 0<\varepsilon<\overline{\varepsilon}$$
$$\pi(x,y)+(1-\varepsilon)\pi(x,x) > \pi(y,y)+(1-)\pi(y,x)$$
$$\Rightarrow\left[\pi(x,y)-\pi(y,y)\right]+(1-)\left[\pi(x,x)-\pi(y,x)\right] > 0\ \forall 0<\varepsilon<\overline{\varepsilon}$$

Let $\varepsilon \to 0$. Then,

$$\pi(x,x)-\pi(y,x) \geq 0$$
$$\Rightarrow \pi(x,x) \geq \pi(y,x) \forall y \in \Delta$$

This implies that (x,x) is a Nash equilibrium. In fact, this is a symmetric Nash equilibrium.
Theorem. The following are equivalent

- x is ESS.
- x is symmetric NE & if $y \in BR(x)$, then $\pi(x,y) > \pi(y,y) \forall \pi(x,x) = \pi(y,x)$.

Proof: x is ESS implies

$$\left[\pi(x,y)-\pi(y,y)\right]+(1-)\left[\pi(x,x)-\pi(y,x)\right] > 0$$

Letting $\varepsilon \to 0$,

$$\pi(x,x)-\pi(y,x) > 0$$

for all $y \in \Delta$. Hence, (x,x) is a symmetric Nash equilibrium.

Now, let us prove the other way. Let x be a symmetric Nash equilibrium. This implies that $\pi(x,y) > \pi(y,y)$ whenever $\pi(x,x) = \pi(y,x)$. Let $y \in \Delta$, $y \neq x$.

$$\pi(x,\varepsilon y+(1-)x)-\pi(y,\varepsilon y+(1-)x)$$
$$=\varepsilon(\pi(x,y)-\pi(y,y))+(1-)\left[\pi(x,x)-\pi(y,x)\right]$$

As (x,x) is a symmetric equilibrium, the second part of the above equation is always greater than or equal to zero. Moreover, if $y \in BR(x)$, by the hypothesis we have taken to hold, the first part is strictly greater than 0. Hence, the whole is strictly greater than 0.

Now, let $y \notin BR(x)$. Then, the second part of the equation is now strictly greater than 0. However, the first part can take any value, whether it be positive, negative or zero. But, as $\varepsilon \to 0$, the first part goes to 0. Hence, for sufficiently small ε, the entire sum will be strictly greater than 0.

Therefore, we can conclude that, for any y, there exists an $\overline{\varepsilon}$ such that for any $< \overline{\varepsilon}$,

$$\pi(x,\varepsilon y+(1-\varepsilon)x)-\pi(y,\varepsilon y+(1-\varepsilon)x) > 0$$

This proves the theorem.

Let us see an interesting example of Evolutionary Game Theory. This is known as the *Hawk-Dove* game. So, in a Hawk-Dove game there is a population of birds and two behaviors. One is the Hawk behavior and other is the Dove behavior. When two species having Hawk behavior encounter, they fight for a resource.

When both of them fight for a resource, one of them will lose the resource to the other and only one of them will get it. Whereas, when Hawk and Dove species encounter, the Dove immediately gives the resource to the Hawk. Moreover, when two Doves encounter, they split the resource. So, let us assume that the resource is worth V and cost of fighting is C. The game matrix of the above is given by:

		P1	
		H	D
P2	H	$\frac{V-C}{2}, \frac{V-C}{2}$	$0, V$
	D	$V, 0$	$\frac{V}{2}, \frac{V}{2}$