Game Theory
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Lecture 26
Evolutionarily Stable Strategies - II

In the previous session we introduced the notion of evolutionary stable strategies. In fact, we ended with an example of a Hawk Dove game. The value of the resource is V and the cost is given by C. The game matrix is given as follows:

P1

P2	H	D
H	$\frac{V-C}{2}, \frac{V-C}{2}$	$0, V$
D	$V, 0$	$\frac{V}{2}, \frac{V}{2}$

Computing the Nash Equilibrium itself is an interesting exercise which should be tried by the reader. Now Evolutionary Stable Strategies follow the following equations:

$$\pi(x, \varepsilon y + (1-\varepsilon)x) > \pi(y, \varepsilon y + (1-\varepsilon)x) \ \forall y \in BR(x)$$
$$\Longleftrightarrow x \text{ is symmetric Nash Equilibrium}$$
$$\& \ \pi(x,y) > \pi(y,y) \ \forall y \in BR(x)$$

Let us look at an example. Take

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Consider the mixed strategy $\left(\frac{1}{2}, \frac{1}{2}\right)$. Let us calculate $\pi(x,y)$.

$$\pi(y,x) = -\frac{y_1}{2} + y_1 + \frac{1-y_1}{2} = \frac{1}{2}$$

$\forall y \in \Delta$. Moreover,

$$\pi(x,y) - \pi(y,y) = 2(y_1 - \frac{1}{2})^2 \geq 0$$

Equality holds only if $y_1 = \frac{1}{2}$ i.e. $y = x$. We need to check if $(\frac{1}{2}, \frac{1}{2})$ is a symmetric Nash Equilibrium which would imply that $(\frac{1}{2}, \frac{1}{2})$ is ESS. This is left as an exercise for the reader.

Now, let us look at another game called *Rock-Paper-Scissors*. Recall the game matrix for this game. We can easily verify that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a symmetric Nash equilibrium. We need to check whether this is an Evolutionary Stable Strategy. Let us calculate $\pi(x, e')$.

$$\pi(x, e^1) = 0 = \pi(e^1, e^1)$$

Therefore, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not ESS as $\pi(x, e^1)$ should be strictly greater than $\pi(e^1, e^1)$. Next, the support of mixed strategies is given by,

$$S(x) = \{e_1 | x_i > 0\}$$

Moreover, let $\Lambda(x)$ be the convex hull of $S(x)$.
Theorem. If $x \in \Delta$ is ESS, then there is no other symmetric Nash Equilibrium in $\Lambda(x)$.
Proof. x is an ESS implies that x is a symmetric Nash Equilibrium and hence all pure strategies in $S(x)$ are going to be best responses to x. Hence,

$$\Lambda(x) \subseteq BR(x)$$
$$\therefore\ \pi(x,y) > \pi(y,y) \ \ \forall y \in \Lambda(x)$$

Therefore, y cannot be a symmetric Nash equilibrium.

Theorem. The following are equivalent:

1. x is ESS.

2. $\exists$ a neighbourhood U of x such that $\pi(x,y) > \pi(y,y)\ \forall y \in U, y \neq x$.

Replicator Dynamics

So, Evolutionary Stable Strategies are strategies of the incumbents. If some mutants enter the population, then ESS talks about whether the incumbent strategy survives or not. It however, does not talk about how that ESS is arrived at in the society. Replicator dynamics gives a way of dealing with this.

Consider a large population playing pure strategies from $\{e_1, e_2, .., e_m\}$. Let the population state be given by,

$$x(t) = (x_1(t), x_2(t), ..., x_m(t))$$

where $x_i(t)$ is the fraction of population playing e_i, given by

$$x_i(t) = \frac{n_i(t)}{n(t)}$$

At time t, the average payoff to an individual adopting e_i in a random match is given by $\pi(e^i, x(t))$. The population average is given by, $\pi(x(t), x(t))$. We get this if we pick people randomly and calculate this average.

Whereas, if a random individual adapts he is going to get $\pi(e^i, x(t))$. We denote the difference in payoffs by σ, given by

$$\sigma(e^i, x) = \pi(e^i, x(t)) - \pi(x(t), x(t))$$

Let us now look at the relative rate of change, given by,

$$\frac{\dot{x}_i(t)}{x_i(t)} = \sigma(e^i, x)$$

where,

$$\dot{x}_i(t) = x_i(t)\left[\pi(e^i, x(t)) - \pi(x(t), x(t))\right]$$

This is known as the *Replicator Equation*. On taking the summation of $\dot{x}_i(t)$ over all i, we get

$$\begin{aligned}
\sum_{i=1}^{m} \dot{x}_i(t) &= \sum_{i=1}^{m} \left[x^i \left(\pi(e^i, x(t)) - \pi(x(t), x(t)) \right) \right] \\
&= 0 \\
\therefore \ \sum x_i(t) &= \sum x_i(0) \\
\therefore \ x(t) &\in \Delta \ \forall t
\end{aligned}$$

The next question we ask is whether there exists a solution to the replicator equations. As this is a bi-linear function, Lipschitz continuity holds which makes the right side of the replicator equation a Lipschitz continuous function. Therefore, the replicator equation admits a unique solution in Δ whenever $x(0) \in \Delta$.

Consider $\Delta^0 = \{x \in \Delta | \sigma(e^i, x) = 0 \forall i \in S(x)\}$. This is the set of stationary points of the replicator equation.

Theorem. Let $x \in \Delta$. If x is a symmetric Nash Equilibrium, then it is a stationary point of the Replicator Equation. The converse is true when one of the following holds:

1. x is in the interior of Δ.

2. x is a limit state of trajectory lying in the interior of Δ.

3. x is a Lyapunov stable state of the Replicator Equation.

We look at the proof of this theorem in the next lecture.