Game Theory
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Lecture 27
Evolutionarily Stable Strategies - III

In the previous lecture, we have introduced replicator dynamics and also stated a theorem. We will now complete the proof of that theorem. Let's recall what the statement of the theorem is.
Theorem. Let $x \in \Delta$. If x is a symmetric Nash Equilibrium, then it is a stationary point of the Replicator Equation. The converse is true when one of the following holds:

1. x is in the interior of Δ.

2. x is a limit state of trajectory lying in the interior of Δ.

3. x is a Lyapunov stable state of the Replicator Equation.

Proof. First part: The replicator dynamics is characterized by,

$$\dot{x}_i = x_i\big(f(e^i,x) - f(x,x)\big)$$

where $f(x,y) =< x,Ay >$ is the payoff function and A is the corresponding symmetric matrix. Let us assume that x be a symmetric Nash equilibrium. Then, clearly,

$$f(e^i,x) = f(x,x) \;\; \forall e^i \in S(x)$$

where $S(x)$ is the support of x. Therefore, for all i with $x_i > 0$, we have

$$\begin{aligned} &f(e^i,x) - f(x,x) = 0 \\ \Rightarrow &x_i(f(e^i,x) - f(x,x)) = 0 \\ \Rightarrow &RHS = 0 \end{aligned}$$

for every i with $x_i > 0$. For all i with $x^i = 0$, it is true that

$$x^i(f(e^i,x) - f(x,x)) = 0$$

Therefore, x is a stationary point of the Replicator Equation.

Conversely, suppose x is interior to Δ. x is also a stationary point. This implies,

$$\begin{aligned} x^i(f(e^i,x) - f(x,x)) &= 0 \\ x^i > 0 \Rightarrow f(e^i,x) - f(x,x) &= 0 \end{aligned}$$

Hence, this says that each e^i is the best response to x and as x is a convex combination of e^i's, x is a best response to itself. Hence, $x \in BR(x)$. Therefore, x is a symmetric Nash Equilibrium.

Second part: Assume that the stationary point x is a limit state, that is,

$$x = \lim_{t\to\infty} x(t)$$

where $x(t)$ satisfies the Replicator Equation and it lies in the interior of Δ for all $t \geq 0$.

Note that,

$$x_i(t) = x_i(t_0) \exp \int_{t_0}^{t} [f(e^i, x(s)) - f(x(s), x(s))] ds$$

for all i.

If possible, let x be not a symmetric Nash Equilibrium. Then there is a pure strategy $e^j \in S(x)$ satisfying

$$\begin{aligned} &f(e^j, x) > f(x,x) \\ \Rightarrow &2\delta = f(e^j, x) - f(x,x) > 0 \end{aligned}$$

This implies that there exists a neighbourhood U such that

$$f(e^j, y) - f(y,y) \geq \delta \quad \forall \; y \in U, y \neq x$$

Since, $x = \lim_{t\to\infty} x(t)$, there is t_0 such that $x(t) \in U$ for all $t \geq t_0$. Therefore,

$$f(e^j, x(t)) - f(x(t), x(t)) \geq \delta \quad \forall t \geq t_0$$

Therefore, $x_j(t) \geq x_j(0) \exp \delta(t - t_0)$ for all $t \geq t_0$. This is a contradiction as $x = \lim_{t\to\infty} x(t)$ should converge but the above says that it diverges to infinity. Therefore, x is symmetric Nash Equilibrium.

Third part: Let x be a Lyapunov stable stationary point of Replicator equation. If x is not symmetric Nash Equilibrium, then there will be pure strategy e^j and t_0 such that

$$x_j(t) \geq x_j(0) \exp^{\delta(t-t_0)} \quad \forall t \geq t_0$$

Therefore, x has to be symmetric Nash equilibrium(follows from the previous argument). This completes the proof.

Theorem. If x is a Evolutionarily Stable Strategy, then it is asymptotically stable state of the Replicator Equation.
Proof. Let x be ESS.

$$\sigma(x,y) = f(x,y) - f(y,y)$$

Since x is ESS, we have shown that there exists a neighbourhood U of x such that

$$\sigma(x,y) > 0 \;\; \forall y \in U \setminus \{x\}$$

Consider

$$O = \{y \in \Delta : S(x) \subseteq S(y)\}$$

where O is a relative neighbourhood of x. Define $V : O \to \mathbb{R}$ by

$$V(y) = \sum_{i \in S(x)} x_i log\left(\frac{x_i}{y_i}\right)$$

This is an entropy function defined on all the pure strategies on the support of x. The verification of V to be continuous and $V(x) = 0$ is left to the reader. For $y \in O \setminus \{x\}$,

$$\begin{aligned}
V(y) &= -\sum_{i \in S(x)} x_i \log\left(\frac{y_i}{x_i}\right) \\
&> -\sum_{i \in S(x)} x_i\left(\frac{y_i}{x_i} - 1\right) \quad \textit{since } \log r < r - 1 \textit{ for all } r \neq 1. \\
&= 1 - \sum_{i \in S(x)} y_i = 0
\end{aligned}$$

Therefore, $V(y) \geq 0$.

Time derivative of $V(y(t))$ is strictly negative.

$$\begin{aligned}
\frac{d}{dt}V(y(t)) &= \sum_{i=1}^{k} \frac{\partial V}{y_i}(y(t)).\dot{y}(t) \\
&= \sum_{i=1}^{k} \frac{-x_i}{y_i(t)} y_i(t) \sigma(e^i, y(t)) \\
&= -\sigma(x, y(t))
\end{aligned}$$

Therefore, $\frac{d}{dt}V(y(t)) < 0$. This comes from the fact that x is ESS. Hence, x is an asymptotically stable state. This completes the proof.

Let us see some examples to see that the converse need not be true.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 5 & 0 & 4 \end{bmatrix}$$

We can see that none of the pure strategies e^1, e^2, e^3 are symmetric Nash Equilibrium. Let us calculate $f(e^i, y)$:

$$\begin{aligned}
f(e^1, y) &= y_1 + 5y_2 \\
f(e^2, y) &= y_2 + 5y_3 \\
f(e^3, y) &= 5y_1 + 4y_3
\end{aligned}$$

It follows that

$$
\begin{aligned}
f(e^1,y) = f(e^2,y) &\text{ iff } 6y_1 + 9y_2 = 5 \\
f(e^2,y) = f(e^3,y) &\text{ iff } 6y_1 = 1 \\
f(e^3,y) = f(e^1,y) &\text{ iff } 9y_2 = 4
\end{aligned}
$$

Hence, the game has one symmetric Nash Equilibrium, $x = \left(\frac{1}{6}, \frac{4}{9}, \frac{7}{18}\right)$. We can also verify that x is not ESS. For example,

$$
\begin{aligned}
f(x,e^3) &= 4x_3 = \frac{28}{18} \\
f(e^3,e^3) &= 4 > \frac{28}{18}
\end{aligned}
$$

Hence, this says that x is not an ESS as one of the conditions of the previous theorem is violated. Nonetheless, we can show that x is asymptotically stable point of the Replicator Equation. The Replicator Equations in this game are given by,

$$
\begin{aligned}
\dot{y}_1 &= -y_1(y_1^2 + y_2^2 + 4y_3^2 + 5y_1y_2 + 5y_2y_3 + 5y_3y_1 - y_1 - 5y_2) \\
\dot{y}_2 &= -y_2(y_1^2 + y_2^2 + 4y_3^2 + 5y_1y_2 + 5y_2y_3 + 5y_3y_1 - y_2 - 5y_3) \\
\dot{y}_3 &= -y_3(y_1^2 + y_2^2 + 4y_3^2 + 5y_1y_2 + 5y_2y_3 + 5y_3y_1 - 5y_1 - 5y_3)
\end{aligned}
$$

Take the RHS of the Replicator Equation. This is a map from $\mathbb{R}^3 \to \mathbb{R}^3$. The gradient matrix at x is given by

$$
\begin{bmatrix}
-\frac{7}{12} & \frac{2}{9} & -\frac{37}{36} \\
-2 & -\frac{32}{27} & \frac{14}{27} \\
\frac{7}{36} & -\frac{77}{54} & -\frac{91}{108}
\end{bmatrix}
$$

Look at the eigenvalues. From the theory of differential equations, we know that if the real part of the eigenvalue is negative then the corresponding point is going to be an asymptotically stable point. We can easily verify that the real part of the eigenvalue is less than 0. Hence, x is an asymptotically stable point. This concludes the theory of Evolutionary Stable Strategies.